Problem 1

Consider the famous "twin-paradox". Let's call the twins Bobby and George. Bobby travels to one of the exo-planets of the Alpha Centauri system, whose distance is 4.5lightyears from the Earth, and it is almost in rest in the reference frame of the Earth. The maximal speed of the spacecraft is $0.75c$, and the accelaration and braking times are negligeble.

After reaching his destination, Bobby studies the exo-planet for 1year time and then he travels back to Earth.

a.) Draw the world lines of Bobby and George in the Minkowski plane.

b.) How many years does George age, who stayed on Earth, until Bobby gets back to the Earth?

c.) How many years does Bobby age at the same time?

A (wrong) explanation of the twin paradox states, that the different aging is caused by the acceleration of Bobby. Indeed, Bobby needs to have nonzero acceleration, if he wants to come home. However, using the following thought experiment we can exclude that explanation.

Let's suppose that Bobby and George both travel on the spacecraft, but at half distance George decides to stop − using the spacecrafts rescure cabin − and takes a long holiday at a space-motel that rests in the frame of Earth. When Bobby is traveling back, George accelerates his cabin to 0.75 c, joins Bobby in the spacecraft, and they arrive together back to Earth. We can see, that in this case George and Bobby can have exactly the same acceleration processes.

d.) Draw the modified world line of George in the figure!

e.) How much time does George spend in the motel?

f.) What is George's total aging during the travel?

Problem 2

In the upper athmosphere μ particles or muons are produced by cosmic rays colliding with molecules, and then these unstable particles are moving with almost constant velocity towards the Earth's surface. The half time of decay for resting μ is $T_{1/2} = 2.2\mu\text{s} = 2.2 \cdot 10^{-6}s$.

a.) Assuming Newtonian mechanics to be correct, what distance would a μ travel (with having velocity $V_\mu \approx c$) until it is expected to decay?

b.) Assuming that muons are produced in an altitude of 10km, what fraction of them would reach the Earth's surface?

We know that Newtonian mechanics fails to describe the above questions. We want to measure the velocity of μ, therefore we perform the following experiment. We have created two identical μ detectors. One is attached to a wheather balloon and is lifted up to $h = 3$km altitude. The other one remains on the surface of Earth. We measure $n_b = 700$ counts at the balloon while only $n_s = 500$ counts on the surface in an hour.

c.) Assuming we know the V_μ velocity of the muons, what is the connection between n_b and n_s?

d.) According to the measured velues, determine the velocity of the μ particles.

Problem 3

At time $t = 0$ two spacecrafts depart from Earth in perpendicular directions with velocities $3/5c$.

a.) Determine the position vectors $r_1(t)$ and $r_2(t)$ of the two spacecrafts as a function of time. (Use a convenient coordinate-system in the reference frame of Earth.)

b.) Let's sit in the reference frame of the spacraft "1". Determine the position vector $r_2'(t')$ of the spacraft "2" in this reference frame.

c.) What is the velocity vector of the 2nd spacecraft in that reference frame? Determine also the direction of this velocity vector.

Problem 4

There are given two 4-vectors with their contravariant coordinates in some inertial system, $a^\mu = (a^0\, a^1\, a^2\, a^3)$ and $b^\mu = (b^0\, b^1\, b^2\, b^3)$. The metric tensor of the Minkowskian spacetime is simply $g_{\mu\nu}$.

a.) Using Einstein's convention, and the metric tensor, express the Minkowski length square of a^μ and the Minkowskian scalar product of a^μ and b^μ.

b.) How we define the covariant coordinates of these 4-vectors? Determine the $a_\mu = (a_0\, a_1\, a_2\, a_3)$ and $b_\mu = (b_0\, b_1\, b_2\, b_3)$ "lower index" coordinates. With the help of these covariant coordinates, express again the Minkowski length square of a_μ and the Minkowskian scalar product of a_μ and b_μ.

c.) As we see, the indices can be lowered by multiplication with the $g_{\mu\nu}$ tensor. The inverse of this manipulation is the "raising" of indices. What tensor $g^{\mu\nu}$ can be used to raise the indices?

Problem 5

Consider the following transformation,

$$\Lambda^{\mu}_{\cdot\,\nu} = \begin{pmatrix} 5/3 & 0 & -4/3 & 0 \\ -4/3 & 0 & 5/3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

a.) Consider the 4-vector $a^\mu = (1, 1, 0, 0)$. What is its Minkowski length square? Apply the above transformation on this vector. Show that its Minkowski length square is invariant.

b.) Consider the 4-vector $b^\mu = (6, 1, 3, 1)$, and show that its Minkowski lenght square is also invariant.

c.) Show in general, that the transformation $\Lambda^{\mu}_{\cdot\,\nu}$ is a Lorentz transformation.

d.) Express the components b_μ. Express also the transformed b'_μ components.

e.) Determine the appropriate form of Λ that transforms the lower-index vectors, $b'_\mu = \Lambda_{\mu}^{\cdot\,\nu} b_\nu$.

f.) Show that $a^\mu b_\mu$ remains invariant.

g.) Show directly that $\Lambda_{\rho}^{\cdot\,\mu} \Lambda^{\rho}_{\cdot\,\nu} = \delta^{\mu}_{\cdot\,\nu}$, where $\delta^{\mu}_{\cdot\,\nu}$ stands for the Kronecker-delta.