Quantum World

Lecture notes for course BMETE15MF77
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Lecture 1

Copenhagen interpretation: what we teach at QM courses

Quantum theory Microscopic physics A.k.a. the Quantum World	**Heisenberg cut**	Classical physics Macroscopic objects

Experiments now probe the boundary between microscopic and macroscopic physics
No sign of any clearly defined boundary: maybe the Quantum World is the whole reality?
But then how does classical behaviour emerge?

Fundamentals of quantum theory

- States: vectors $|\Psi\rangle$ in a Hilbert space $\mathcal{H}$
- Observables: linear operators $O: \mathcal{H} \to \mathcal{H}$, hermitian $O^\dagger = O$
- Dynamics: $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle$

Eigenstates and eigenvectors:
$O|\psi_i\rangle = \lambda_i|\psi_i\rangle \qquad \langle\psi_i|\psi_j\rangle = \delta_{ij}$

Spectral decomposition:

$$O = \sum_i \lambda_i\,|\psi_i\rangle\langle\psi_i|$$

Some details:

- Multiplicities (degenerate eigenvalues): to each distinct eigenvalue
  $\lambda_i \to |\psi_i^{\alpha_i}\rangle \quad \alpha_i = 1 \ldots d_i$

  $$P_i = \sum_{\alpha_i=1}^{d_i} |\psi_i^{\alpha_i}\rangle\langle\psi_i^{\alpha_i}| \qquad P_i^2 = P_i \quad P_i^\dagger = P_i \;\text{ and }\; P_iP_j = 0 \;\; i \neq j$$

  $$O = \sum_i \lambda_i P_i \qquad \sum_i P_i = I$$

- Continuous spectrum: projector valued measure $dP(\lambda)$

  $$O = \int_{-\infty}^{+\infty} \lambda dP(\lambda) \qquad \int_{-\infty}^{+\infty} dP(\lambda) = I$$

  $$P(\lambda) = \int_{-\infty}^{\lambda} dP(\mu)$$: projector to subspace where value of observable is $\leq \lambda$

  Special case of discrete spectrum

$$dP(\lambda) = \sum_i P_i \delta(\lambda - \lambda_i) d\lambda$$

Example: position operator

State of particle in 1D: wave function

$\Psi(x)$: $\int_{-\infty}^{\infty} dx\, |\Psi(x)|^2 = 1$

Position operator: $x \cdot$ multiplication by x

$P(\lambda) = \theta(\lambda - x)\cdot$ (multiplication operator by $\theta(\lambda - x)$)

$$dP(\lambda) = \delta(\lambda - x) d\lambda$$

$$x = \int_{-\infty}^{+\infty} \lambda dP(\lambda) = \int_{-\infty}^{+\infty} \lambda \delta(\lambda - x) d\lambda$$

Problem for the devoted: how to do the same for the momentum operator?
(hint: use Fourier decomposition!)

Connection to observations

Assume we perform a measurement of O at time t_M

Measurement postulate:
the possible outcomes are the eigenvalues λ_i of the operator O with probabilities

$$p_i = |\langle \Psi(t_M)|\psi_i\rangle|^2 \qquad \textbf{Born's rule}$$

Note: $|\langle \Psi(t_M)|\psi_i\rangle|^2 = \langle \Psi(t_M)|\psi_i\rangle\langle \psi_i|\Psi(t_M)\rangle = \langle \Psi(t_M)|P_i|\Psi(t_M)\rangle$

In case with degeneracies:

$$p_i = \langle \Psi(t_M)|P_i|\Psi(t_M)\rangle$$

For continuous spectrum: we can give probability that the value is in an interval $[a, b]$

$$P([a,b]) = \int_a^b dP(\lambda) = P(b) - P(a)$$

$$p_{[a,b]} = \langle \Psi(t_M)|P([a,b])|\Psi(t_M)\rangle$$

E.g. for position operator

$$P([a,b]) = \theta(b - x) - \theta(a - x)$$

$$p_{[a,b]} = \int_{-\infty}^{\infty} dx\, \Psi(x, t_M)^* \big(\theta(b - x) - \theta(a - x)\big)\Psi(x, t_M) = \int_a^b dx\, |\Psi(x, t_M)|^2$$

Collapse postulate

After measurement with outcome λ_i, the state of the system abruptly changes to

$$|\Psi(t_M)\rangle_{post} = |\psi_i\rangle = \frac{P_i|\Psi(t_M)\rangle}{||P_i|\Psi(t_M)\rangle||} \qquad ||P_i|\Psi(t_M)\rangle|| = \sqrt{\langle\Psi(t_M)|P_i^\dagger P_i|\Psi(t_M)\rangle}$$

Also known as **wave-function reduction**.

The second form also holds for degenerate and/or continuous spectrum:

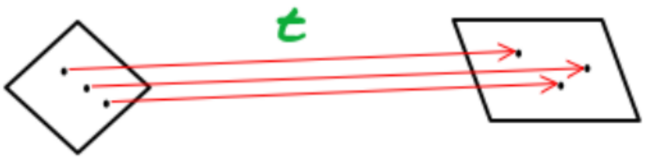
if the outcome is described by projector P then the post-measurement state is given by

$$|\Psi(t_M)\rangle_{post} = \frac{P|\Psi(t_M)\rangle}{||P|\Psi(t_M)\rangle||}$$

Classical Hamiltonian dynamics

- States: points (q, p) in phase space Γ
- Observables: real functions $F: \Gamma \to \mathbf{R}$
- Dynamics:
$$\dot{q}(t) = \frac{\partial H}{\partial p} \qquad \dot{p}(t) = -\frac{\partial H}{\partial q}$$
- Observed values:
$$F\big(q(t), p(t)\big)$$

Liouville theorem: phase space volume is conserved during Hamiltonian evolution.



$\Omega(t_1)$ $\qquad$ $\Omega(t_2) = \{\big(q(t_2), p(t_2)\big)|q(t_1) = q_0, p(t_1) = p_0$ where $(q_0, p_0) \in \Omega(t_1)\}$

$$\int_{\Omega(t_1)} dpdq = \int_{\Omega(t_2)} dpdq$$

Time evolution map

$R_t(q, p) = \big(q(t), p(t)\big)$ where the trajectory has initial condition $q(0) = q,\ p(0) = p$

$\Omega(t_2) = R_{t_2-t_1}\big(\Omega(t_1)\big)$

Probabilistic formulation

Assume we do not know the exact trajectory of the system, which can be characterised by a probability distribution $\Pi(p,q)$ on the phase space Γ satisfying

$$\int_\Gamma dpdq\,\Pi(q,p) = 1$$

Then the expectation value of observable $F(q,p)$ at time t is given by

$$\langle F(t)\rangle_\Pi = \int_\Gamma dpdq\,\Pi(q,p)F\big(q(t),p(t)\big)$$

Classical Heisenberg picture: time evolution is carried by observable:

$$F\big(q(t),p(t)\big):\ F_t = F\circ R_t$$

Classical Schrödinger picture: change integration variable to
$(q',p') = \big(q(t),p(t)\big) = R_t(q,p)$

$$\langle F(t)\rangle_\Pi = \int_\Gamma dp'dq'\,\Pi_t(q',p')F(q,p) \quad \Pi_t(q',p') = \Pi\big(R_t^{-1}(q',p')\big) \text{ i.e. } \Pi_t = \Pi\circ R_t^{-1}$$

It follows immediately that total probability is conserved in time (show it!):

$$\int_\Gamma dpdq\,\Pi_t(q,p) = 1$$

Classical probability theory on phase space

Event space: phase space Γ
Elementary events: phase space points (q,p)
Composite events: (Borel) subsets of Γ

Events form a Boolean algebra:

Impossible event: $\emptyset$ Certain event: Γ
Negation of event A: $\bar{A}$
AND of two events A and B: $A\cap B$ OR of two events A and B: $A\cup B$

Probability of event A:

$$\mathrm{Prob}(A) = \int_A \Pi(q,p)\,dqdp$$

Events as projections

A: $\chi_A(q,p)$ characteristic function of set A
$\chi_A(q,p) = 1$ if $(q,p)\in A$ and zero otherwise.

$$\chi_A(q,p)^2 = \chi_A(q,p)$$

$$\chi_{A\cap B}(q,p) = \chi_A(q,p)\chi_B(q,p) \qquad \chi_{\bar{A}}(q,p) = 1-\chi_A(q,p)$$

$\chi_{A\cup B}(q,p) = \chi_A(q,p) + \chi_B(q,p) - \chi_{A\cap B}(q,p)$

Note that

$$\text{Prob}(A) = \int_\Gamma \chi_A(q,p)\Pi(q,p)dqdp = \langle\chi_A\rangle_\Pi$$

Classical collapse postulate

If we measure an event A at time t_M, we then know for sure that the phase space point corresponding to the system is inside the set A → we can update the probability distribution as

$$\Pi_{t_M}(q,p) \;\to \Pi_{t_M}(q,p)_{post} = \frac{\chi_A(q,p)\Pi_{t_M}(q,p)}{\int_\Gamma \chi_A(q,p)\Pi_{t_M}(q,p)dqdp}$$

Quantum probability theory

Event: Hermitian projector P $\quad P^2 = P,\ P^\dagger = P$

Note: it is an observable with possible values 0 or 1 just like its classical counterpart!!!

Take a self-adjoint operator

$$O = \sum_i \lambda_i P_i \qquad \sum_i P_i = I \qquad P_i P_j = 0 \text{ for } i \neq j$$

Elementary event: O takes some value λ_i
Composite events: O takes one value in a set $\{\lambda_{i_1}, \dots \lambda_{i_k}\}$

$$P = P_{i_1} + \dots + P_{i_k}$$

Negation: $\bar{P} = I - P$ AND: PQ OR: $P + Q - PQ$

Born's rule: $\text{Prob}(P) = \langle P\rangle_\Psi = \langle\Psi|P|\Psi\rangle$

Example: particle in 1D

Take the event that the particle is found in interval $x \in [a,b]$

Classically: corresponding projection is $\chi_{[a,b]}(x) = \theta(b-x) - \theta(a-x)$

$$\text{Prob}(x\in[a,b]) = \langle\chi_{[a,b]}\rangle_\Psi = \int_{-\infty}^{+\infty} dx\, \chi_{[a,b]}(x)\Pi(x) = \int_a^b dx\, \Pi(x)$$

Quantum: the projection operator is multiplication by $\chi_{[a,b]}(x)$

$$(P_{[a,b]}\Psi)(x) = \chi_{[a,b]}(x)\Psi(x)$$

$$\text{Prob}(x\in[a,b]) = \langle P_{[a,b]}\rangle_\Psi = \int_{-\infty}^{+\infty} dx\, \Psi(x) * \chi_{[a,b]}(x)\Psi(x) = \int_a^b dx\, |\Psi(x)|^2$$

Quantum-classical dictionary

Concept	**Classical**	**Quantum**
Event space	Phase space Γ	Hilbert space $\mathcal{H}$
Elementary event	Point (q,p)	1D subspace (projection $\vert\Psi\rangle\langle\Psi\vert$)
Event	Subset $A \subset \Gamma$	Subspace A in $\mathcal{H}$
Event projector	$\chi_A(q,p)$	$P_A: P_A^2 = P_A,\ P_A^\dagger = P_A\ \mathrm{Im}(P_A) = A$
Observable	$F(q,p)$	$O\colon O^\dagger = O$
Expectation value	$\langle F\rangle_\Pi = \int_\Gamma dpdq\, \Pi(q,p)F(q,p)$	$\langle O\rangle_\Psi = \langle\Psi\vert O\vert\Psi\rangle$
Probability of event	$\mathrm{Prob}(A) = \langle\chi_A\rangle_\Pi$	$\mathrm{Prob}(A) = \langle P_A\rangle_\Psi$
Time evolution	$\dot{q}(t) = \partial_p H\ \ \dot{p}(t) = -\partial_q H$ $\big(q(t),p(t)\big) = R_t\big(q(0),p(0)\big)$	$i\hbar\partial_t\vert\Psi(t)\rangle = H\vert\Psi(t)\rangle$ $\vert\Psi(t)\rangle = e^{-iHt}\vert\Psi(0)\rangle$
Probability conservation	R_t conserves phase space volume	e^{-iHt} is unitary
Collapse postulate	$\Pi_{t_M}(q,p)_{post} = \dfrac{\chi_A(q,p)\Pi_{t_M}(q,p)}{\langle\chi_A\rangle_{\Pi_{t_M}}}$	$\vert\Psi(t_M)\rangle_{post} = \dfrac{P\vert\Psi(t_M)\rangle}{\vert\vert P\vert\Psi(t_M)\rangle\vert\vert}$

Is this all that quantum theory is about?
Simply replacing the phase space by Hilbert space?

Lecture 2

The qubit a.k.a. spin 1/2

Hilbert space: $\mathcal{H} = \boldsymbol{C}^2$

General Hermitian operators:

$O = \alpha_0 \boldsymbol{I} + \alpha_i \sigma_i$

$\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\sigma_k \sigma_l = \delta_{kl} \boldsymbol{I} + i\epsilon_{klm}\sigma_m$

$[\sigma_k, \sigma_l] = 2i\epsilon_{klm}\sigma_m \qquad \{\sigma_k, \sigma_l\} = 2\delta_{kl}\boldsymbol{I} \qquad \mathrm{Tr}\sigma_k\sigma_l = 2\delta_{kl}$

Spin operators:

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma} \qquad \vec{S}^2 = \vec{S}\cdot\vec{S} = \frac{3}{4}\hbar^2\boldsymbol{I}$$

Trivial basis:

$|\uparrow_z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\downarrow_z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\sigma_3|\uparrow_z\rangle = +1\cdot|\uparrow_z\rangle \qquad \sigma_3|\downarrow_z\rangle = -1\cdot|\downarrow_z\rangle$

Basis aligned in direction $\vec{w}$ $(|\vec{w}\,| = 1)$

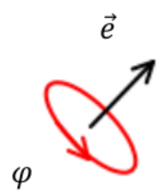
$(\vec{w}\cdot\vec{\sigma})|\uparrow_w\rangle = +1\cdot|\uparrow_w\rangle \quad (\vec{w}\cdot\vec{\sigma})|\downarrow_w\rangle = -1\cdot|\downarrow_w\rangle$

$\vec{w} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$|\uparrow_w\rangle = e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}|\downarrow_z\rangle \qquad |\downarrow_w\rangle = -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}|\uparrow_z\rangle + e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}|\downarrow_z\rangle$

Spatial rotation around axis $\vec{e}$ by angle φ is represented by the unitary operator

$R(\varphi\vec{e}) = e^{-\frac{i}{2}\varphi\vec{e}\cdot\vec{\sigma}}$



Note that

$|\uparrow_w\rangle = R(\phi\vec{e}_3)R(\theta\vec{e}_2)|\uparrow_z\rangle \qquad |\downarrow_w\rangle = R(\phi\vec{e}_3)R(\theta\vec{e}_2)|\downarrow_z\rangle$

Incompatible observables

We can measure the spin in any direction $\vec{w}$:

$$P_w^{\uparrow} = |\uparrow_w\rangle\langle\uparrow_w| = \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}$$

$$P_w^{\downarrow} = |\downarrow_w\rangle\langle\downarrow_w| = \begin{pmatrix} \sin^2\frac{\theta}{2} & -e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ -e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \cos^2\frac{\theta}{2} \end{pmatrix} = I - P_w^{\uparrow} = P_{-w}^{\uparrow}$$

Note that projectors to different orientations generally do not commute:

$$\left[P_w^{\uparrow}, P_{w'}^{\uparrow}\right] = 0 \Rightarrow \vec{w} = \pm\vec{w}'$$

This is a consequence of the non-commutativity of different spin components:

$$[S_k, S_l] = i\hbar\epsilon_{klm}S_m$$

Examples:

z: $\theta = 0$, $\phi = 0$ $\quad |\uparrow_z\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad |\downarrow_z\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$P_z^{\uparrow} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \quad P_z^{\downarrow} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

x: $\theta = \frac{\pi}{2}$, $\phi = 0$

$$|\uparrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle + |\downarrow_z\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$|\downarrow_x\rangle = \frac{1}{\sqrt{2}}(-|\uparrow_z\rangle + |\downarrow_z\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$$

$$P_x^{\uparrow} = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} \quad P_x^{\downarrow} = \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$$

Does it make sense to talk about the spin pointing in direction *x* OR *z*?

By the rules of event projectors

$$P\left(x \bigvee z\right) = P_x^{\uparrow} + P_z^{\uparrow} - P_x^{\uparrow}P_z^{\uparrow} = \begin{pmatrix}1 & 1/2\\0 & 1/2\end{pmatrix}$$

This is not a projector, not even a Hermitian operator - i.e. not an observable! In addition

$$P\left(z \bigvee x\right) = P_z^{\uparrow} + P_x^{\uparrow} - P_z^{\uparrow}P_x^{\uparrow} = \begin{pmatrix}1 & 0\\1/2 & 1/2\end{pmatrix} = P\left(x \bigvee z\right)^{\dagger}$$

When do logical operations make sense?

AND: $P\left(A \bigwedge B\right) = P_A P_B$

$P\left(A \bigwedge B\right)^{\dagger} = P_B^{\dagger} P_A^{\dagger} = P_B P_A = P\left(A \bigwedge B\right)$ only when $P_A P_B = P_B P_A$

So, the projections must commute! Then we also have

$P(A \wedge B)^2 = P_A P_B P_A P_B = P_A P_A P_B P_B = P_A P_B = P(A \wedge B)$

and $P(A \wedge B) = P(B \wedge A)$

Homework: show similarly that operation OR

$P\left(A \bigvee B\right) = P_A + P_B - P_A P_B$

is only sensible when the projectors commute.

Logical operations only make sense for commuting projectors, and for commuting projectors all identities of logic (Boolean algebra) hold.

Quantum logic was proposed by Birkhoff and Neumann - has not proven to be a fruitful direction.

There is a one-to-one map between projectors and subspaces of the Hilbert space:

$P_A \leftrightarrow \mathcal{H}_A$

One can then attempt to define the following logical operations:

$AND: \mathcal{H}_{A \wedge B} = \mathcal{H}_A \cap \mathcal{H}_B$
$OR: \quad \mathcal{H}_{A \vee B} = \mathrm{span}(\mathcal{H}_A \cup \mathcal{H}_B)$
$NOT: \mathcal{H}_{\bar{A}} = \mathcal{H}_A^{\perp}$

Problems:

- For a nontrivial event *A*, there are events which are neither in *A* nor its negation - law of excluded middle does not work.

- Distributivity of logic: normal event logic satisfies
-

  $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$
  $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$

  But for quantum events

  $\uparrow_y \wedge (\uparrow_x \vee \uparrow_z) = \uparrow_y$

$$(\uparrow_y \wedge \uparrow_x) \vee (\uparrow_y \wedge \uparrow_z) = 0 \text{ (impossible event)}$$

Under these operations, quantum events do not obey the rules of logic! There is no way under which the full "quantum logic" can be thought of as the logical structure of actual physical events.

Quantum events (subspaces of Hilbert space) form a so-called orthomodular σ-lattice, which is not distributive!

Boolean lattices (event spaces) are orthomodular σ-lattice which satisfy distributivity.

If one takes appropriate (algebraically complete) subsets of quantum events for which the corresponding operators commute, then they obey the rules of logic.

Consequence: simultaneous measurements only make sense for commuting observables, since we must be able to say that we obtained λ_1 for O_1 AND/OR another value for λ_2 for O_2 .

Density operators

What is a state? It must assign probabilities to events in a sensible way.

Gleason's theorem

Assume that we have a map *f* from projectors on a Hilbert space with dimension ≥ 3 to the unit interval *[0,1]* such that for any decomposition of the identity into orthogonal projectors P_i :

$$\sum_i P_i = I \qquad P_i P_j = P_i \delta_{ij} \quad {P_i}^\dagger = P_i$$

it satisfies the sum rule of probabilities

$$\sum_i f(P_i) = 1$$

Then there exists a positive semidefinite operator ρ such that

$$f(P_i) = \mathrm{Tr}\, \rho P_i$$

which has a unit trace: $\mathrm{Tr}\, \rho = 1$

Remarks:

i. A is positive semidefinite if for any vector ψ in the Hilbert space $(\psi, A\psi) \geq 0$.

ii. Positive semidefinite operators are automatically Hermitian (more precisely self-adjoint).

iii. If two positive semidefinite operators commute, then their product is positive semidefinite.

iv. The probability assignment f is required to be non-contextual: it does not matter what is the experiment and what observable is measured, probability only depends on the mathematical representation of the outcome (i.e. the projection operator).

For states corresponding to vectors in Hilbert space: $\rho = |\Psi\rangle\langle\Psi|$

v. Invariant under $|\Psi\rangle \to e^{i\alpha}|\Psi\rangle$

vi. $\mathrm{Tr}\,\rho = 1 \Rightarrow \langle\Psi|\Psi\rangle = 1$

These states really correspond to rays (1D subspaces) of the Hilbert space:

$\{c|\Psi\rangle : c \in \mathbf{C}\}$ with some fixed nonzero $|\Psi\rangle$

Pure and mixed states, quantum and classical

Expectation value of an observable

$$O = \sum_i \lambda_i P_i : \langle O\rangle_\rho = \sum_i \lambda_i \,\mathrm{Prob}(P_i) = \sum_i \lambda_i \,\mathrm{Tr}(\rho P_i) = \mathrm{Tr}\left(\rho \sum_i \lambda_i P_i\right) = \mathrm{Tr}(\rho O)$$

Spectral decomposition of a density operator

$$\rho = \sum_i p_i \,|\phi_i\rangle\langle\phi_i|$$

Note

$$\langle O\rangle_\rho = \mathrm{Tr}(\rho O) = \sum_i p_i \,\langle\phi_i|O|\phi_i\rangle$$

Ignorance interpretation: the system is in one of the states $|\phi_i\rangle$, we just don't know which.

Convex combination of states is a state:

$$\rho = \sum_i c_i \rho_i \quad 0 \le c_i \quad \sum_i c_i = 1$$

Pure state: ρ is pure if

$$\rho = \sum_i c_i \rho_i \quad 0 \le c_i \quad \sum_i c_i = 1 \;\to c_i = 0 \text{ except for a single one.}$$

Classical case

State: probability distribution $\Pi(q, p)$ on phase space.

Pure state: δ-distribution

$$\delta_{q_0,p_0} = \delta(q - q_0)\delta(p - p_0)$$

For a continuous distribution $\Pi(q,p)$

$$\Pi = \int dp_0 dq_0 \Pi(q_0, p_0)\, \delta_{q_0,p_0}$$

This the continuous version of a convex combination since

$$\Pi(q_0, p_0) \geq 0 \quad \text{and} \int dp_0 dq_0 \Pi(q_0, p_0) = 1$$

If the distribution of initial conditions is a δ-distribution, then the phase space trajectory is uniquely specified → all observables have a definite value without uncertainty.

Classically: pure states are dispersion-free states and vice versa!

Quantum case

Pure states: rank one density matrices corresponding to rays of the Hilbert space

$$\rho = |\Psi\rangle\langle\Psi|$$

However, such states are not dispersion-free:

$$\langle O^2\rangle - \langle O\rangle^2 = \langle\Psi|O^2|\Psi\rangle - \langle\Psi|O|\Psi\rangle^2 = |O|\Psi\rangle|^2 \left||\Psi\rangle\right|^2 - |\langle\Psi|O|\Psi\rangle|^2$$

By the Cauchy-Schwartz inequality this can only be zero if

$$\exists\lambda \in \mathbf{C}:\ O|\Psi\rangle = \lambda|\Psi\rangle$$

So, it is only zero if the state is an eigenstate of the observable. Since there exist non-commuting observables, this cannot be true for all of them.

Lecture 3

Hidden variables

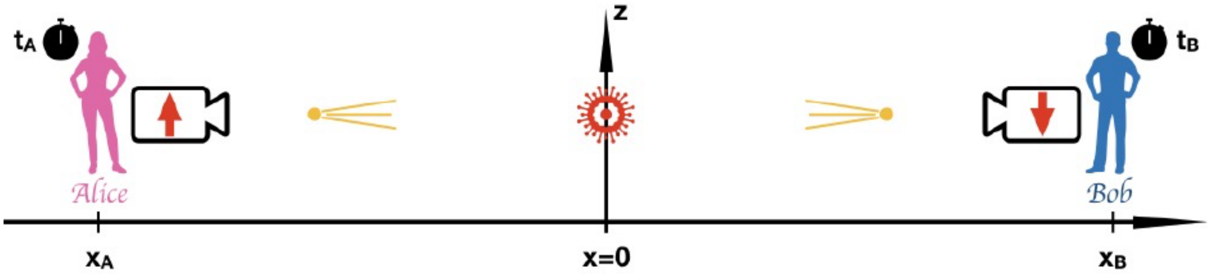
Hidden variable theories are hypothetical deterministic (classical) descriptions of quantum dynamics.

Hidden variables: unobservable hypothetical entities giving a complete description of physical reality.

Quantum mechanics is assumed to be incomplete, and indeterminacy follows from ignorance of the full physical configuration which can only be expressed in terms of the hidden variables.

Argument for incompleteness of quantum mechanics: Einstein-Podolsky-Rosen paradox.

EPR paradox



Assume that a system decays into a pair of spin-1/2 particles with total angular momentum zero, i.e. total spin state is a singlet:

$$\frac{1}{\sqrt{2}}(|\uparrow_z\rangle_1|\downarrow_z\rangle_2 - |\downarrow_z\rangle_1|\uparrow_z\rangle_2)$$

Assume that Alice measures first

$$t_A < t_B$$

and gets spin up/down: then Bob always measures spin down/up. However, if the two measurements are space-like separated i.e.

$$|\vec{x}_B - \vec{x}_A| > c|t_B - t_A|$$

then Bob only has the information provided by quantum mechanics but cannot know about Alice's result because of relativistic causality.

EPR argument: since Bob's result is already determined by Alice's measurement, there must exist an element of reality corresponding to the already determined outcome of measuring the second spin. But Bob cannot predict it on the basis of the quantum mechanical description, therefore quantum mechanics there is an *element of reality* not described by quantum mechanics.

EPR conclusion: quantum mechanics is not a complete description of physical reality.

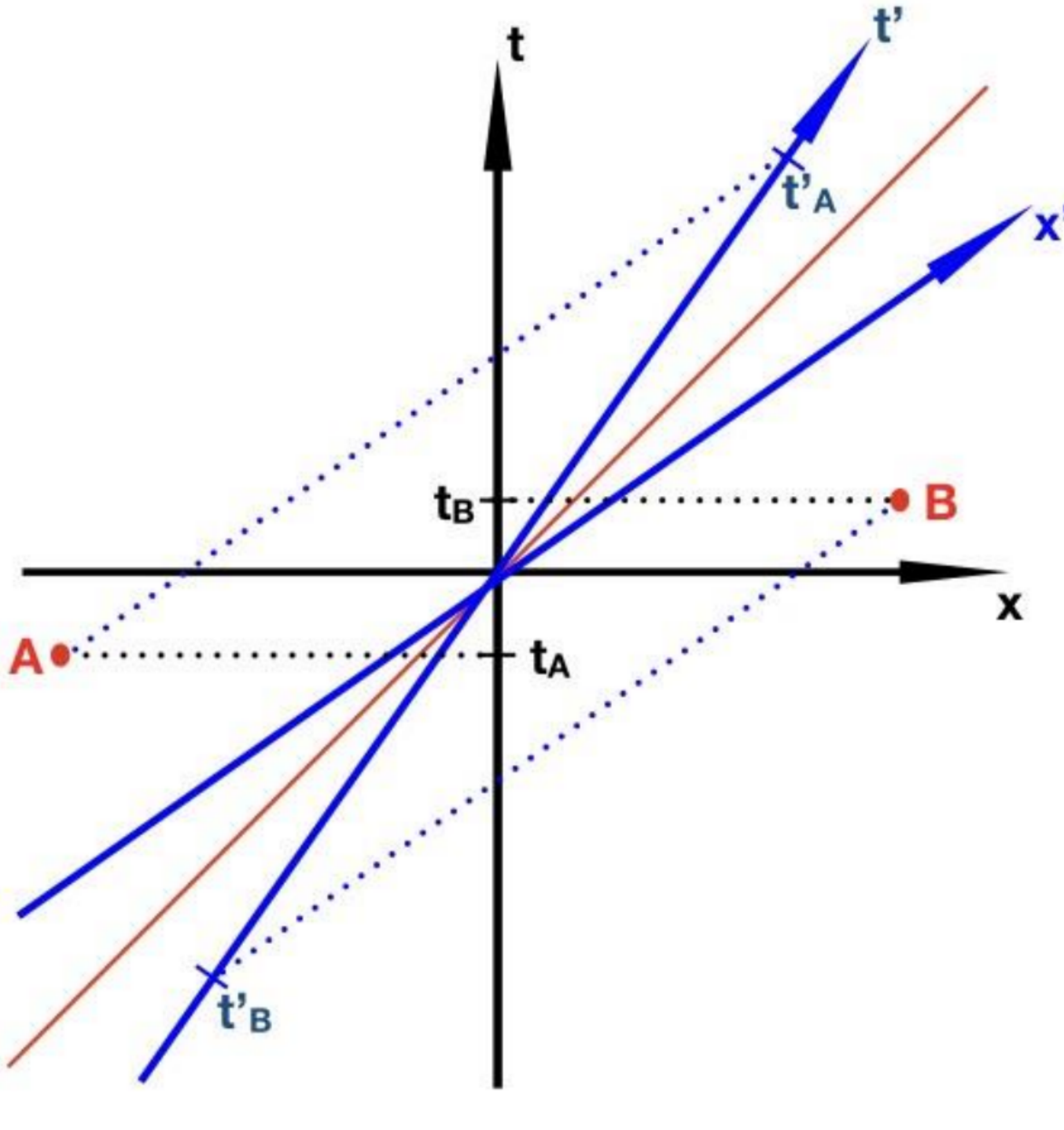
Problem with relativity: if we assume the first electron communicated the result with the second one to establish the outcome, then we have a *spooky action at a distance*.

<u>Even deeper problem</u>: for

$|\vec{x}_B - \vec{x}_A| > c|t_B - t_A|$

there always exists a Lorentz frame in which Bob's measurement comes first, i.e.

$t'_A > t'_B$



Lorentz transformation of distances and intervals (1+1D case)

$$\Delta x' = \frac{\Delta x - v\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \Delta t' = \frac{\Delta t - \frac{v\Delta x}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Note: if

$$x_B - x_A = \Delta x < c\Delta t = c(t_B - t_A)$$

then choosing

$$v > \frac{c^2}{\Delta x/\Delta t}$$

we get $\Delta t' < 0$. This is possible since by our assumption

$$\frac{c^2}{\Delta x/\Delta t} < c$$

So, we can find a system moving with an appropriate velocity v in which the order of events is reversed.

We cannot decide which measurement happened earlier: problem with wave-function collapse!

von Neumann hidden variable theorem and its critique

Theorem (von Neumann)

Take a quantum system with a Hilbert space of dimension larger than 2. Let's assume we can assign an expectation value $\langle O \rangle$ to any Hermitian operator O satisfying

i. $\langle \boldsymbol{I} \rangle = 1$
ii. $O \geq 0 \Rightarrow \langle O \rangle \geq 0$
iii. $\langle rO \rangle = r\langle O \rangle$
iv. $\langle O_1 + O_2 \rangle = \langle O_1 \rangle + \langle O_2 \rangle$

Then there is a positive semidefinite operator ρ such that
$\langle O \rangle = \mathrm{Tr}\, \rho O$

However: such a state is never dispersion free if there are non-commuting operators!
Linearity of the expectation value in the operator implies that it cannot be dispersion-free!

Neumann's argument against hidden parameters

Let's assume there are hidden variables with which we can extend the Hilbert space

$$\mathcal{H}_{\mathrm{HV}} = \mathcal{H} \otimes \mathcal{L}$$

so that the full state is

$$\Psi_{HV} = (|\Psi\rangle, \lambda)$$

We assume that the state including the hidden parameters is dispersion free, i.e.

$$E(\Psi_{HV}, O^2) = \big(E(\Psi_{HV}, O)\big)^2$$

for all observables O.

Let us take $P_\eta = |\eta\rangle\langle\eta|$ as our observable. We then have

$$\big(E(\Psi_{HV}, P_\eta)\big)^2 = E(\Psi_{HV}, {P_\eta}^2) = E(\Psi_{HV}, P_\eta)$$

As a result: $E(\Psi_{HV}, P_\eta) = 0{,}1$

But we also require that this must reproduce the QM expectation value $\langle\Psi|P_\eta|\Psi\rangle$ which is a continuous function of η. So that means it is identically either 0 or 1, which implies that

$$\langle\Psi|P_\eta|\Psi\rangle = |\langle\Psi|\eta\rangle|^2$$

is identically 0 or 1 for all η. This is impossible for a unit vector $|\Psi\rangle$ in a Hilbert space.

Taking the quantum state to be a density matrix ρ does not help either: repeating the same argument leads to

$$\langle\eta|\rho|\eta\rangle = 0 \text{ or } 1 \;\forall\, \eta$$

which implies either $\rho = 0$ or $\rho = \boldsymbol{I}$, none of which are valid density operators.

Criticism of the von Neumann argument

Bohm: constructed pilot wave theory which is a hidden variable description of ordinary one-particle quantum mechanics - counterexample!

Where does the argument go wrong? The quantum mechanical expectation value satisfies

$$\langle\alpha A + \beta B\rangle_\Psi = \alpha\langle A\rangle_\Psi + \beta\langle B\rangle_\Psi \quad \alpha, \beta \in \boldsymbol{R}$$

I.e. for any given state there are always operators which are not dispersion free.

If an operator is dispersion-free in Ψ, then its expectation value is one of its eigenvalues.

But if two operators $O_1,\ O_2$ do not commute, then the eigenvalues of $O_1 + O_2$ are not the sums of the eigenvalues of $O_1,\ O_2$, so $O_1,\ O_2$ and $O_1 + O_2$ cannot be dispersionless at the same time!

However: take a hidden variable theory the quantum average is reproduced via

$$E_{HV}(\Psi, O) = \sum_{\lambda} \omega(\lambda) v_{\Psi}(O, \lambda)$$

$v_{\Psi}(O, \lambda)$: expectation value of O with λ held fixed.

and linearity only holds after averaging over λ, but not for the $v_{\Psi}(O, \lambda)$ for fixed λ, i.e. it is perfectly possible that

$$v_{\Psi}(\alpha A + \beta B, \lambda) \neq \alpha v_{\Psi}(A, \lambda) + \beta v_{\Psi}(B, \lambda)$$

so that $v_{\Psi}(O, \lambda)$ can be dispersion less for any fixed λ.

Even if $v_{\Psi}(O, \lambda)$ is dispersion-free for any fixed λ, this does not imply that the quantum mechanical results are dispersion-free, since

$$E_{HV}(\Psi, O)^2 = \sum_{\lambda} \sum_{\lambda'} \omega(\lambda) \omega(\lambda') v_{\Psi}(O, \lambda) v_{\Psi}(O, \lambda')$$

$$E_{HV}(\Psi, O^2) = \sum_{\lambda} \omega(\lambda) v_{\Psi}(O^2, \lambda)$$

These are clearly not the same even if $v_{\Psi}(O, \lambda)^2 = v_{\Psi}(O^2, \lambda)$!

De Broglie-Bohm pilot wave theory

Postulates for a single particle:

(i) There's an objective configuration of the system described by coordinates $\vec{q}$ in the configuration space $Q = \mathbf{R}^3$, which evolve according to the guiding equation

$$\frac{d\vec{q}}{dt} = \frac{\vec{j}}{\rho} \qquad \vec{j} = \frac{\hbar}{2mi}\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right) \qquad \rho = \Psi^*\Psi$$

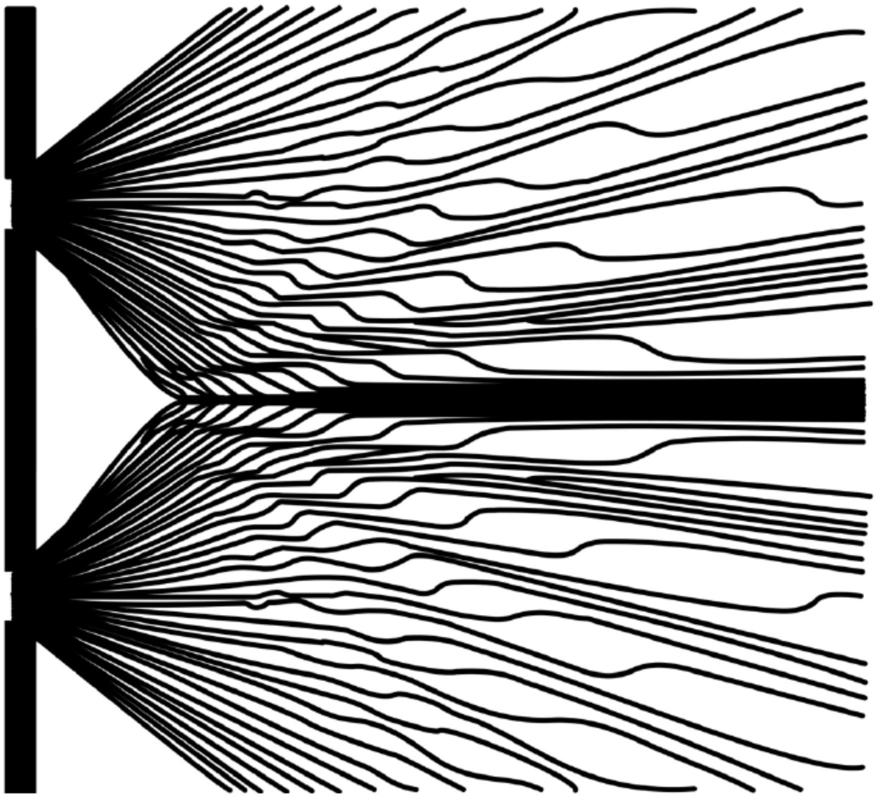
(ii) The wave function evolves according to the Schrödinger equation

$$i\hbar\partial_t\Psi(q,t) = -\frac{\hbar^2}{2m}\Delta\Psi(q,t) + V(q)\Psi(q,t)$$

(iii) At initial time t_0 the initial position has a distribution $|\Psi(q,t_0)|^2$ (called "quantum equilibrium")

It can then be proven that at any later time the system evolving according to the above dynamics leads to a distribution $|\Psi(q,t)|^2$ of the position in the configuration space Q.

Bohmian trajectories for the two-slit experiment reproduce interference pattern



My own criticism against Bohmian approaches is that they are parasitic upon, and reverse-engineered from, ordinary quantum theory. They just add excess baggage without yielding any deep insight or resolution of the puzzles we face.

Kochen-Specker theorem

Could there be a consistent value assignment to quantum mechanical observables at all?

Theorem (Kochen-Specker)

Take a Hilbert space of dimension ≥ 3. Then there is a set M of observables containing n elements, such that the following assumptions cannot be simultaneously satisfied:

KS1: All members of M simultaneously have vsalues i.e. there is a value assignment $v: M \to \mathbf{C}$
KS2: The map v satisfies
(a) If $[A, B] = 0$ then $v(aA + bB) = av(A) + bv(B)$ for all $a, b \in \mathbf{C}$
(b) If $[A, B] = 0$ then $v(AB) = v(A)v(B)$
(c) There is at least one observable $X \in M$ with $v(X) \neq 0$

Note:

- (a) is much weaker than for Neumann's where it is required to hold for non-commuting observables.
- (b) implies that for any analytic function f, $v\big(f(A)\big) = f\big(v(A)\big)$ holds.
- The value assignment v must depend on the quantum state of the system.

Common interpretation of the KS theorem: quantum theory fails to allow a non-contextual hidden variable model. More precisely, it states that it is impossible for the predictions of quantum mechanics to be in line with measurement outcomes which are pre-determined in a non-contextual manner i.e. without reference to a measurement arrangement.

Bohm-de Broglie theory escapes KS by stipulating that only position and functions of position have definite value. This is known as **partial value definiteness**: there is a subset of observables called "beables" that have simultaneous value assignments independent of observations (J. S. Bell).

Critique of the Kochen-Specker argument against hidden variables

Still, one can question whether (b) is a reasonable assumption. Without it, the value assignment does not reflect correlations between observables; however, it may be still too strong given that it implies $v\big(f(A)\big) = f\big(v(A)\big)$.
Assuming that a Hermitian operator can be expressed as $O = f(A) = g(B)$ where A and B are two non-commuting operators, there is no reason why $f(A)$ and $g(B)$ should represent the same physical quantity, so one can escape the problem by declaring that the mapping between Hermitian operators and physical observables is not one-to-one.
E.g. one can question that if the same projector P appears as a spectral projector in two non-commuting Hermitian operators A and B, why should we think that it corresponds to the same physical event, since measuring A and B can only happen under different physical arrangements?
Final note: assuming Gleason's theorem i.e. that the valuation v is inherited from a density matrix ρ, the KS theorem follows trivially. The novel thing it shows that it is impossible to find a valuation even if we do not require it to derive from a density matrix.

Some more details on the Kochen-Specker theorem

There is a simple proof if we assume that the Hilbert space is at least 4 dimensional. We consider 9 observables or "contexts". Each of them has four outcomes corresponding to four (unnormalized) orthogonal vectors $\left|u_i^{(\alpha)}\right\rangle, i = 1 \dots 4$ with $\alpha = 1 \dots 9$ indexing the contexts. We consider the eigenstate projectors

$$P_i^{(\alpha)} = \frac{\left|u_i^{(\alpha)}\right\rangle\left\langle u_i^{(\alpha)}\right|}{\left\langle u_i^{(\alpha)}\middle|u_i^{(\alpha)}\right\rangle}$$

which satisfy

$$P_i^{(\alpha)2} = P_i^{(\alpha)} \; ; \; P_i^{(\alpha)} = 0 \; for \; i \neq j \; ; \; \sum_{i=1}^{4} P_i^{(\alpha)} = \mathbf{1}$$

These imply that for any fixed context α, exactly one of these can be assigned the value 1, while the other three must be assigned zero. The nine sets are chosen so that each projector occurs in exactly two contexts; the pairs are coloured with the same colour in the table below. Each context is a different measurement arrangement, since they correspond to measuring some observable of the form

$$O^{(\alpha)} = \sum_{i=1}^{4} \lambda_i^{(\alpha)} P_i^{(\alpha)}$$

where $\lambda_i^{(\alpha)}$ are some real numbers giving the possible results of measuring $O^{(\alpha)}$ which we assume be distinct for different values of i. The nine operators $O^{(\alpha)}$ are all different, so the nine contexts really correspond to physically different measurements.

	Context 1	Context 2	Context 3	Context 4	Context 5	Context 6	Context 7	Context 8	Context 9
u_1	(0, 0, 0, 1)	(0, 0, 0, 1)	(1, −1, 1, −1)	(1, −1, 1, −1)	(0, 0, 1, 0)	(1, −1, −1, 1)	(1, 1, −1, 1)	(1, 1, −1, 1)	(1, 1, 1, −1)
u_2	(0, 0, 1, 0)	(0, 1, 0, 0)	(1, −1, −1, 1)	(1, 1, 1, 1)	(0, 1, 0, 0)	(1, 1, 1, 1)	(1, 1, 1, −1)	(−1, 1, 1, 1)	(−1, 1, 1, 1)
u_3	(1, 1, 0, 0)	(1, 0, 1, 0)	(1, 1, 0, 0)	(1, 0, −1, 0)	(1, 0, 0, 1)	(1, 0, 0, −1)	(1, −1, 0, 0)	(1, 0, 1, 0)	(1, 0, 0, 1)
u_4	(1, −1, 0, 0)	(1, 0, −1, 0)	(0, 0, 1, 1)	(0, 1, 0, −1)	(1, 0, 0, −1)	(0, 1, −1, 0)	(0, 0, 1, 1)	(0, 1, 0, −1)	(0, 1, −1, 0)

Finding a valuation now means that we must assign 0 or 1 to each cell of the table with the following rules:

- cells with the same colour have equal value, and
- the value 1 occurs exactly once in each column.

This is impossible since the first condition implies that 1 occurs an even number of times, while the second one implies that there can be exactly 9 occurrences of 1.

The criticism above can be formulated as follows: the first condition means that the valuation should only depend on the projector. However, one may argue that this is too restrictive since the different occurrences of the projectors correspond to measuring different observables, so there is no reason why this should be considered the same events. The event – projector correspondence is 1:1 in quantum theory, but since we are looking for hidden variables beyond quantum theory, they may well distinguish between the same projector when it occurs in the spectral decomposition of different observables. This is known as **contextuality**.

For any observable O if a projector P occurs in its spectral decomposition then there exists a function f such that $P=f(O)$ (for finite dimensions, this is a simple theorem of matrices, where f is eventually a polynomial function that can be constructed from the characteristic polynomial). If the projector P occurs in two observables A and B then there exist functions f and g such that $P=f(A)=g(B)$. Relaxing the event – projector correspondence in the way above exactly means that if we consider two observables A and B than even if for two functions f and g we have $f(A)=g(B)$ we still treat $f(A)$ and $g(B)$ as different physical quantities. This means that we must give up the 1:1 correspondence between Hermitian operators and observers.

Therefore, the Kochen-Specker theorem only proves that any hidden variable is necessary contextual. Compared to the von Neumann theorem, it relaxes linearity for non-commuting operators, which is important since this assumption can never be directly checked as non-commuting quantities cannot be measured simultaneously. However, it introduces a new condition for the product of commuting operators, and still relies on the conceptual framework of quantum theory which relates observables to Hermitian operators.

In the next lecture we discuss Bell's inequalities, which only requires to consider the probabilities of actual physical outcomes, without presupposing an underlying physical formalism. Therefore, it can directly rule out (certain classes of) hidden variable theories as descriptions of the real physical world directly based on experimental results.

Note that the existence of value assignments is only a prerequisite of a classical hidden variable description. The eventual description of experimental observations must also include a probabilistic aspect, which corresponds to the existence of multiple value assignments v_κ corresponding to different internal states κ of the system described by the hidden variables, which occur with probabilities p_κ.

Lecture 4

Deeper into the EPR paradox: Bell's inequality

Note that the singlet state looks the same with respect to a general axis $\vec{w}$:

$$|\Psi_{\mathrm{EPR}}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle_1|\downarrow_z\rangle_2 - |\downarrow_z\rangle_1|\uparrow_z\rangle_2) \equiv \frac{1}{\sqrt{2}}(|\uparrow_w\rangle_1|\downarrow_w\rangle_2 - |\downarrow_w\rangle_1|\uparrow_w\rangle_2)$$

Therefore, it contains infinitely many correlations. However, one measurement can only reveal one direction. However, one measurement can only reveal one direction and then the correlation could have been encoded in the system prior to the measurements.

J.S. Bell: we can get around this problem by repeating the experiment multiple times and measuring the relative frequencies of different outcomes.

Note: it is also necessary to select measurement directions after the particles are emitted so that the system cannot decide which correlation to encode into its state.

First experiment with measurement choices decided while particles already under way:

A. Aspect, P. Grangier and G. Roger: Experimental Tests of Realistic Local Theories via Bell’s Theorem, Phys. Rev. Lett. 47 (1983) 460–463.

Measuring the two spins along different directions

Let us assume that the measurements made by Alice and Bob happen with directions differing by angle θ:

Alice: $e_z = (0,0,1)$ Bob: $e_{z'} = (\sin\theta, 0, \cos\theta)$

Then we can use the results

$$|\uparrow_w\rangle = e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}|\downarrow_z\rangle \qquad |\downarrow_w\rangle = -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}|\uparrow_z\rangle + e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}|\downarrow_z\rangle$$

to write

$$|\uparrow_z\rangle_2 = \cos\frac{\theta}{2}|\uparrow_{z'}\rangle_2 - \sin\frac{\theta}{2}|\downarrow_{z'}\rangle_2 \qquad |\downarrow_z\rangle_2 = \sin\frac{\theta}{2}|\uparrow_{z'}\rangle_2 + \cos\frac{\theta}{2}|\downarrow_{z'}\rangle_2$$

$$|\Psi_{\mathrm{EPR}}\rangle = \frac{1}{\sqrt{2}}\left(\sin\frac{\theta}{2}|\uparrow_z\rangle_1|\uparrow_{z'}\rangle_2 + \cos\frac{\theta}{2}|\uparrow_z\rangle_1|\downarrow_{z'}\rangle_2 - \cos\frac{\theta}{2}|\downarrow_z\rangle_1|\uparrow_{z'}\rangle_2 + \sin\frac{\theta}{2}|\downarrow_z\rangle_1|\downarrow_{z'}\rangle_2\right)$$

The probabilities of the four different outcomes are

$$P\left(s_{1z} = \pm\frac{\hbar}{2}, s_{2z'} = \pm\frac{\hbar}{2}\right) = \frac{1}{2}\sin^2\theta \qquad P\left(s_{1z} = \pm\frac{\hbar}{2}, s_{2z'} = \mp\frac{\hbar}{2}\right) = \frac{1}{2}\cos^2\theta$$

Bell's "1st" inequality

Let's assume both Alice and Bob can choose between three directions $\vec{a}$, $\vec{b}$ and $\vec{c}$ to measure the spin and also that the outcome of all these measurements is determined by some property of the electron pair that exists before any measurement is made. Alice and Bob then each have eight combinations of possible outcomes with probabilities

Alice *abc*	Bob *abc*	Probability
↑↑↑	↓↓↓	p_1
↑↑↓	↓↓↑	p_2
↑↓↑	↓↑↓	p_3
↑↓↓	↓↑↑	p_4
↓↑↑	↑↓↓	p_5
↓↑↓	↑↓↑	p_6
↓↓↑	↑↑↓	p_7
↓↓↓	↑↑↑	p_8

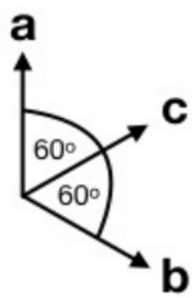
We then have

$P(\uparrow_a, \uparrow_b) = p_3 + p_4 \quad P(\uparrow_a, \uparrow_c) = p_2 + p_4 \qquad P(\uparrow_c, \uparrow_b) = p_3 + p_7$

(Simple version of) Bell's inequality

$$P(\uparrow_a, \uparrow_b) \leq P(\uparrow_a, \uparrow_c) + P(\uparrow_c, \uparrow_b)$$

This is violated by quantum probabilities! Take e.g.



$$P(\uparrow_a, \uparrow_b) = \frac{1}{2}\sin^2 60^o = \frac{3}{8} \qquad P(\uparrow_a, \uparrow_c) = P(\uparrow_c, \uparrow_b) = \frac{1}{2}\sin^2 30^o = \frac{1}{8}$$

Conclusion: the quantum probabilities cannot be interpreted as relative frequencies of properties of the measured system (i.e. of elements of reality)! Therefore there can be no hidden variables depending only on the state of the measured system that determine the outcome of the measurements.

Existence of Kolmogorovian representation

Question: given some probabilities for events, when can they be represented as probabilities of classical events i.e. on a Kolmogorovian event space?

We only get nontrivial restrictions if we include also correlations i.e. probabilities for conjunctions.

Pitowsky theorem

Assume we have some events $A_1, \dots, A_n$ and a subset of pairs

$S \subseteq \{(i,j): i < j; i,j = 1,2, \dots n\}$

and we are given some probability assignments

$$p_i = \mathrm{Prob}(A_i) \qquad i = 1,2, \dots n$$
$$p_{ij} = \mathrm{Prob}(A_i \wedge A_j) \qquad (i,j) \in S$$

assembled as a vector $\vec{p} = (p_1, p_2, \dots p_n, \dots, p_{ij}, \dots)$

Define the following 2^n elementary vectors:
$\epsilon \in \{0,1\}^n \to \vec{u}^\epsilon \in \mathbf{R}^{n+|S|}$
$u_i^\epsilon = \epsilon_i \quad i = 1,2, \dots, n \qquad u_{ij}^\epsilon = \epsilon_i \epsilon_j$

Classical correlation polytope

$$c(n,S) = \left\{ \vec{f} \in \mathbf{R}^{n+|S|} \mid \vec{f} = \sum_{\epsilon \in \{0,1\}^n} \lambda_\epsilon \vec{u}^\epsilon \; ; \; \lambda_\epsilon \geq 0 \; ; \; \sum_{\epsilon \in \{0,1\}^n} \lambda_\epsilon = 1 \right\}$$

Theorem (Pitowsky 1989)
$\vec{p}$ has a Kolmogorov representation if and only if $\vec{p} \in c(n,S)$.

Proof

(i) Assume there is a Kolmogorovian probability field (Σ, μ) such that there exists $X_1, \dots, X_n$ with $p_i = \mu(X_i)$ and $p_{ij} = \mu(X_i \wedge X_j)$.
Denote
$X(\epsilon) = {X_1}^{\epsilon_1} \wedge {X_2}^{\epsilon_2} \wedge \dots \wedge {X_n}^{\epsilon_n}$ where $X^0 = \neg X$ and $X^1 = X$
Note that these are mutually exclusive events that make up the certain event i.e.

$$X(\epsilon) \wedge X(\epsilon') \text{ for } \epsilon \neq \epsilon' \text{ and } \bigvee_{\epsilon \in \{0,1\}^n} X(\epsilon) = 1$$

and additionally satisfy

$$X_i = \bigvee_{\epsilon \in \{0,1\}^n : \epsilon_i = 1} X(\epsilon) = 1$$

Defining $\lambda_\epsilon = \mu(X(\epsilon)) \geq 0$ it automatically holds that

$$\sum_{\epsilon\in\{0,1\}^n} \lambda_\epsilon = 1$$

$$p_i = \mu(X_i) = \sum_{\epsilon\in\{0,1\}^n \,:\, \epsilon_i=1} \mu(X(\epsilon)) = \sum_{\epsilon\in\{0,1\}^n} \lambda_\epsilon \epsilon_i$$

$$p_{ij} = \mu(X_i \wedge X_j) = \sum_{\epsilon\in\{0,1\}^n \,:\, \epsilon_i=\epsilon_j=1} \mu(X(\epsilon)) = \sum_{\epsilon\in\{0,1\}^n} \lambda_\epsilon \epsilon_i\, \epsilon_j$$

so

$$\vec{p} = \sum_{\epsilon\in\{0,1\}^n} \lambda_\epsilon \vec{u}^\epsilon \in c(n,S)$$

(ii) Assume now $\vec{p} \in c(n,S)$ so it can be written as

$$\vec{p} = \sum_{\epsilon\in\{0,1\}^n} \lambda_\epsilon \vec{u}^\epsilon \quad \text{with} \quad \lambda_\epsilon \geq 0 \quad \text{and} \quad \sum_{\epsilon\in\{0,1\}^n} \lambda_\epsilon = 1$$

Let Σ be the Boolean lattice generated by the subsets of $\{0,1\}^n$ and

$$X_i = \{\epsilon \in \{0,1\}^n |\, \epsilon_i = 1\}$$

We can define a probability measure as

$$\mu: X \in \Sigma \mapsto \mu(X) = \sum_{\epsilon\in X} \lambda_\epsilon$$

Then we have

$$\mu(X_i) = \sum_{\epsilon\in\{0,1\}^n \,:\, \epsilon_i=1} \lambda_\epsilon = \sum_{\epsilon\in\{0,1\}^n} \lambda_\epsilon \epsilon_i = \sum_{\epsilon\in\{0,1\}^n} \lambda_\epsilon u_i^\epsilon = p_i$$

$$\mu(X_i \wedge X_j) = \sum_{\epsilon\in\{0,1\}^n \,:\, \epsilon_i=\epsilon_j=1} \lambda_\epsilon = \sum_{\epsilon\in\{0,1\}^n} \lambda_\epsilon \epsilon_i \epsilon_j = \sum_{\epsilon\in\{0,1\}^n} \lambda_\epsilon u_{ij}^\epsilon = p_{ij}$$

Q.e.d.

Pitowsky inequalities

$n = 2$ case: $S = \{(1,2)\}$ $\quad \{0,1\}^2 = \{\{0,0\},\{0,1\},\{1,0\},\{1,1\}\}$

The classical correlation polytope is three dimensional and has four vertices $\{0,0,0\},\{0,1,0\},\{1,0,0\},\{1,1,1\}$

Then we can write

$$\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_{12} \end{pmatrix} = (1 - p_1 - p_2 + p_{12}) \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + (p_1 - p_{12}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (p_2 - p_{12}) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + p_{12} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

therefore $\vec{p} \in c(n,S)$ is equivalent to the following system of inequalities

$0 \leq p_{12} < p_1 \leq 1$
$0 \leq p_{12} < p_2 \leq 1$
$p_1 + p_2 - p_{12} \leq 1$

These are just the obvious relations!

$n = 3$ case: $S = \{(1,2), (1,3), (2,3)\}$

The classical correlation polytope is six dimensional and has eight vertices.

The inequalities are
$0 \leq p_{ij} \leq p_i \leq 1 \quad 0 \leq p_{ij} \leq p_j \leq 1 \quad p_i + p_j - p_{ij} \leq 1$ are the obvious ones
and, in addition
$p_1 - p_{12} - p_{13} + p_{23} \geq 0 \quad p_2 - p_{12} - p_{23} + p_{13} \geq 0 \quad p_3 - p_{13} - p_{23} + p_{12} \geq 0$
$p_1 + p_2 + p_3 - p_{12} - p_{13} - p_{23} \leq 1$

$n = 4$ case with $S = \{(1,3), (1,4), (2,3), (2,4)\}$

$$0 \leq p_{ij} \leq p_i \leq 1 \quad 0 \leq p_{ij} \leq p_j \leq 1 \quad p_i + p_j - p_{ij} \leq 1$$
$$-1 \leq p_{13} + p_{14} + p_{24} - p_{23} - p_1 - p_4 \leq 0 \quad -1 \leq p_{23} + p_{24} + p_{14} - p_{13} - p_2 - p_4 \leq 0$$
$$-1 \leq p_{14} + p_{13} + p_{23} - p_{24} - p_1 - p_3 \leq 0 \quad -1 \leq p_{24} + p_{23} + p_{13} - p_{14} - p_2 - p_3 \leq 0$$

Deriving these inequalities gets exponentially harder with increasing n.

EPR probabilities violate Pitowsky inequalities

Assume Alice and Bob can both measure in two directions $\vec{a}_{1,2}$ for Alice, $\vec{b}_{1,2}$ for Bob, with angles
$\angle(\vec{a}_1, \vec{b}_1) = -\angle(\vec{a}_1, \vec{b}_2) = \angle(\vec{a}_2, \vec{b}_2) = 120^o$ and $\angle(\vec{a}_2, \vec{b}_1) = 0$

Let $A_{1,2}$ be the events that Alice gets that her spin points in direction $\vec{a}_{1,2}$ and $B_{1,2}$ be the events that Alice gets that his spin points in direction $\vec{b}_{1,2}$. Then we have

$$p_1 = p(A_1) = \frac{1}{2} \qquad p_2 = p(A_2) = \frac{1}{2} \qquad p_3 = p(B_1) = \frac{1}{2} \qquad p_4 = p(B_2) = \frac{1}{2}$$
$$p_{13} = p(A_1 \wedge B_1) = \frac{3}{8} \quad p_{14} = p(A_1 \wedge B_2) = \frac{3}{8} \quad p_{23} = p(A_2 \wedge B_1) = 0$$
$$p_{24} = p(A_2 \wedge B_2) = \frac{3}{8}$$

The resulting vector

$$\vec{p} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, 0, \frac{3}{8}\right)$$

violates the Pitowsky inequalities
$$p_{13} + p_{14} + p_{24} - p_{23} - p_1 - p_4 = \frac{3}{8} + \frac{3}{8} + \frac{3}{8} - 0 - \frac{1}{2} - \frac{1}{2} = \frac{1}{8} \nleq 0 \text{ !!}$$

Conclusion (again): the quantum probabilities cannot be interpreted as relative frequencies of properties of the measured system!

However: there is no problem in interpreting them as conditional probabilities, conditioned on the choice of measurement direction, i.e.

$p(A_1|a_1) = \frac{1}{2}$ $p(A_2|a_2) = \frac{1}{2}$ $p(B_1|b_1) = \frac{1}{2}$ $p(B_2|b_2) = \frac{1}{2}$

$p(A_1 \wedge B_1|a_1 \wedge b_1) = \frac{3}{8}$ $p(A_1 \wedge B_2|a_1 \wedge b_2) = \frac{3}{8}$

$p(A_2 \wedge B_1|a_2 \wedge b_1) = 0$ $p(A_2 \wedge B_2|a_2 \wedge b_2) = \frac{3}{8}$

"Quantum contextuality", again: they can be interpreted as probabilities of physical outcomes of measurements, i.e., always in a given measurement "context".

Note that this is not really a very deep (conceptual) contextuality, as it is very natural that the outcome of the measurement is determined not only by the system, but also the measurement setup!

The convex set of quantum correlations

Definition: the correlation vector $\vec{p}$ has a *quantum realisation* if there exists a Hilbert space with subspaces $E_1, \dots, E_n$ with projectors $P_1, \dots, P_n$ and a density operator ρ such that
$p_i = Tr\, \rho P_i \;\; and \;\; p_{ij} = Tr\, \rho P_{ij}$
where P_{ij} is the projector on the subspace $E_i \cap E_j$ (quantum AND operation).

Denote by $q(n, S)$ the set of correlation vectors $\vec{p}$ which have quantum realisations.

Definition: a vector $\vec{v}$ of the form
$v_i = 0$ or 1 $\quad v_{ij} = v_i v_j$ or 0 for $(i, j) \in S$
is called a *quantum vertex* and denote their set by $V_{n,S}$.

Note that classical vertices $\vec{u}^{\epsilon}$ are also quantum vertices.

Define the polytope of convex combination of quantum vertices as

$$l(n, S) = \left\{ \vec{f} \in R^{n+|S|} \,|\, \vec{f} = \sum_{\vec{v} \in V_{n,S}} \lambda_{\vec{v}} \vec{v} \;;\; \lambda_{\vec{v}} \geq 0 \;;\; \sum_{\vec{v} \in V_{n,S}} \lambda_{\vec{v}} = 1 \right\}$$

Theorem (Pitowsky)
(i) $c(n,S) \subset q(n,S) \subset l(n,S)$
(ii) $q(n,S)$ is convex, but not closed
(iii) $int(l(n,S)) \subset q(n,S)$

The last statement means that although not all vectors in $l(n,S)$ can be realised by quantum theory, any of them can be realised by arbitrary precision.

The proof is sketched in the book

L.E. Szabó: The Problem of Open Future - chance, causality, and determinism in physics
(in Hungarian, Section 7.2)

Where do the "quantum vertices" come from?

Let us consider a simple example of two events corresponding to projections $P_1 = |\psi_1\rangle\langle\psi_1|$ and $P_2 = |\psi_2\rangle\langle\psi_2|$ on normalised Hilbert space vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ which form an angle $\alpha > 0$ i.e. $\langle\psi_1|\psi_2\rangle = \cos\alpha$. Choose our quantum state as $\rho = |\psi_1\rangle\langle\psi_1|$. Then we have $P_{12} = 0$ and so

$$p_1 = \mathrm{Tr}\,\rho P_1 = 1 \qquad p_2 = \mathrm{Tr}\,\rho P_2 = \cos\alpha \qquad p_{12} = \mathrm{Tr}\,\rho P_{12} = 0$$

which can be arbitrarily close to the "quantum vertex" $p_1 = p_2 = 1, p_{12} = 0$ for small values of α. However, it can never exactly agree with these values as for $\alpha = 0$ all three projectors coincide ($P_1 = P_2 = P_{12}$) and $p_1 = p_2 = p_{12} = 0$, showing that the quantum vertices are not part of the convex set of quantum correlations $q(n,S)$.

Quantum correlations can be more general than what is classically allowed

The lesson is: **quantum theory allows correlations that cannot have a classical Kolmogorovian representation!**

However: note that we did not assume that the projectors to which we assigned a quantum AND should commute! In fact, to get quantum correlation that are not in the classical correlation polytope $c(n,S)$ it is necessary to include events with projectors that do not commute. So, **it is rather unclear whether the above theorem has any physical consequences, since only commuting observables can be measured simultaneously.**

The strength of the EPR-Bell setup is that it only involves outcomes of compatible measurements, and still shows a violation of the Pitowsky inequalities!

Note that if the Pitowsky inequalities are violated by the measured relative frequencies, then the outcomes cannot be attributed to pre-existent properties of the system ("elements of reality") irrespective of the validity of quantum theory.

The EPR-Bell argument is strong because the experiment directly tells us something about the nature of physical reality, without any assumption about whether it is described by quantum theory.

Lecture 5

EPR experiment loopholes

Communication loophole: if Alice and Bob's choice of measurement directions/measurements are not space-like separated.

Detection loophole: if we cannot detect all EPR pairs, our statistics may be biased and so does not reflect the actual frequencies of the physical properties - since Bell's inequality is violated by a finite amount, to avoid this we need to make sure that the detection efficiency is above a lower bound (approx. 85%).

Loophole-free experiment

W. Rosenfeld, D. Burchardt, R. Garthoff, K. Redeker, N. Ortegel, M. Rau and H. Weinfurter: Event-Ready Bell Test Using Entangled Atoms Simultaneously Closing Detection and Locality Loopholes, Phys. Rev. Lett. 119 (2017) 010402

The laboratory notebook argument

Cf. L.E. Szabó: The Problem of Open Future - chance, causality, and determinism in physics (in Hungarian)

Suppose a labor assistant is making notes of outcomes $A_1, \dots, A_n$ happening in a measurement

Run	A_1	A_2	...	A_n	$A_1 \wedge A_2$	...
1	1	0	...	0	0	...
2	1	1	...	0	1	...
$\vdots$						
N	...					

Each row is some classical vertex $\vec{u}^\epsilon$; assume a row corresponding to ϵ occurs N_ϵ times. Then the measured relative frequencies form the correlation vector

$$\vec{f} = \sum_\epsilon \frac{N_\epsilon}{N} \vec{u}^\epsilon \in \ c(n, S)$$

So, frequencies recorded in any such laboratory notebook always have a Kolmogorovian representation.

How is it possible that EPR measurements violate the Pitowsky inequalities?

Solution: the outcomes are conditioned on the choices of measurement settings!

If in any given run we can fill in say the outcome $\uparrow_a$ for Alice, then the outcome of the unperformed measurement in direction b is not known!

If the outcomes of measurements are determined by hidden variables (properties) inherent in the system (the EPR pair), then the outcome of unperformed measurements would be well-defined even the measurement is not performed, with relative frequencies that are measured in the runs when the appropriate settings are selected. But then these relative frequencies would satisfy the Pitowsky inequalities, which is not the case - neither using the predicted probabilities of quantum theory, nor for the experimentally measured relative frequencies (which happen to agree with the quantum predictions within experimental accuracy).

The real laboratory notebook can be formulated by recording the choices made by Alice and Bob, i.e., we should add to each run their measurement settings:

Run	a_1	a_2	...	a_n	b_1	b_2	...	b_m	A	B	$a_1 \wedge b_1$	$a_1 \wedge b_2$	...	$a_n \wedge b_m$	$A \wedge B$
1	1	0	...	0	0	1	...	0	0	1	0	1	...	0	0
2	1	0	...	0	1	0	...	0	1	0	1	0	...	0	0
⋮															

where $a_1, \ldots, a_n$ are the possible choices for Alice, $b_1, \ldots, b_m$ are the possible choices for Bob, A is that Alice measures her spin in her chosen direction, while B is that Bob measures his spin in his chosen direction.

The probabilities constructed from the above table obey Kolmogorovian statistics, but they cannot be interpreted as relative frequencies for different states of the system as they also include the choice of measurement settings.

What quantum mechanics predicts are the conditional probabilities

$p(A|a_i)$, $p(B|b_i)$, $p(\bar{A}|a_i)$, $p(\bar{B}|b_i)$

and

$p\left(A \wedge B \middle| a_i \wedge b_j\right)$, $p\left(\bar{A} \wedge B \middle| a_i \wedge b_j\right)$, $p\left(A \wedge \bar{B} \middle| a_i \wedge b_j\right)$, $p\left(\bar{A} \wedge \bar{B} \middle| a_i \wedge b_j\right)$

Even deeper into the EPR-Bell experiment

Even if there is no description in terms of system properties, a more pertinent question to ask is:

Can the outcome of EPR experiment be accommodated into a classical description?

Local Markovian determinism

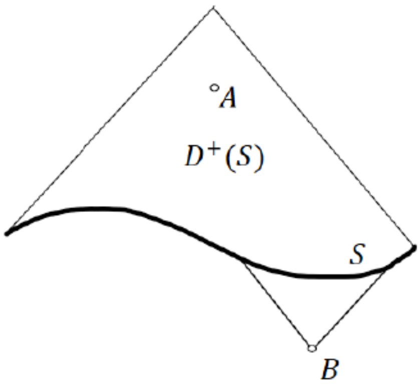
Classical physics satisfies local determinism: data on a Cauchy surface S determine all events A in its future dependence domain $D^+(S)$

It is also Markovian: earlier events B determine A only via the data in S that they influence.

Classical physics is LDM: local, deterministic and Markovian

Markovian: memories exist, but they are inscribed into the variables describing the temporal evolution of the system.

Sometimes we use non-Markovian descriptions, but they are only effective models obtained by eliminating some degrees of freedom.



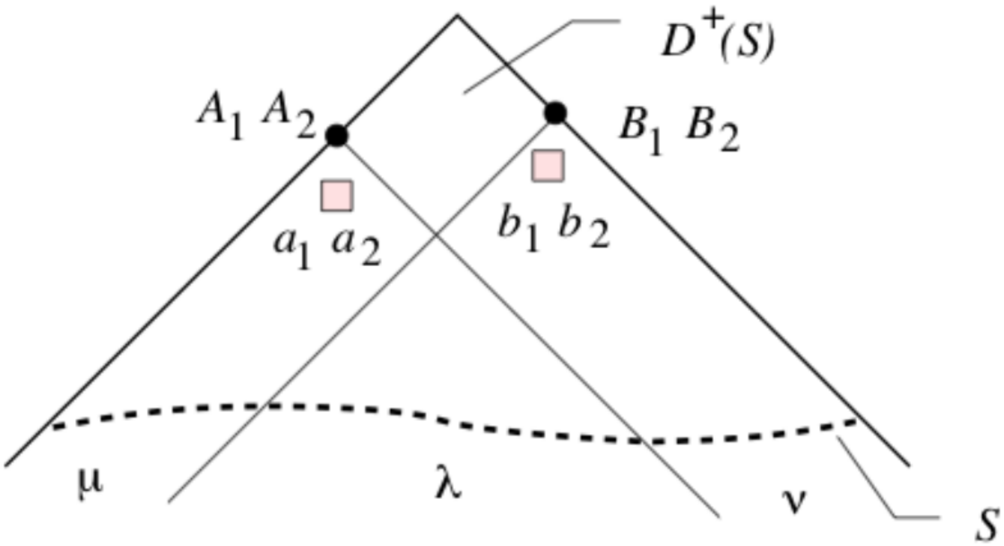
Let us assume that Alice and Bob can both make two measurement choices for the direction they can measure the spin:

a_1, a_2 for Alice and b_1, b_2 for Bob

Let us consider the events that their detectors measure the spins pointing in the selected directions

A_1, A_2 for Alice and B_1, B_2 for Bob

We assume that the selections of the directions and the measurements happen in a space-like separated way:



and that there is an LDM description from a Cauchy surface S with initial data $\mu \cup \lambda \cup \nu$. We can then reproduce the event probabilities in the following way:

$$p(A_i) = \sum_{\mu,\lambda} u^{A_i}(\mu,\lambda)p(\mu \wedge \lambda) \quad p(B_j) = \sum_{\nu,\lambda} u^{B_j}(\lambda,\nu)p(\lambda \wedge \nu)$$

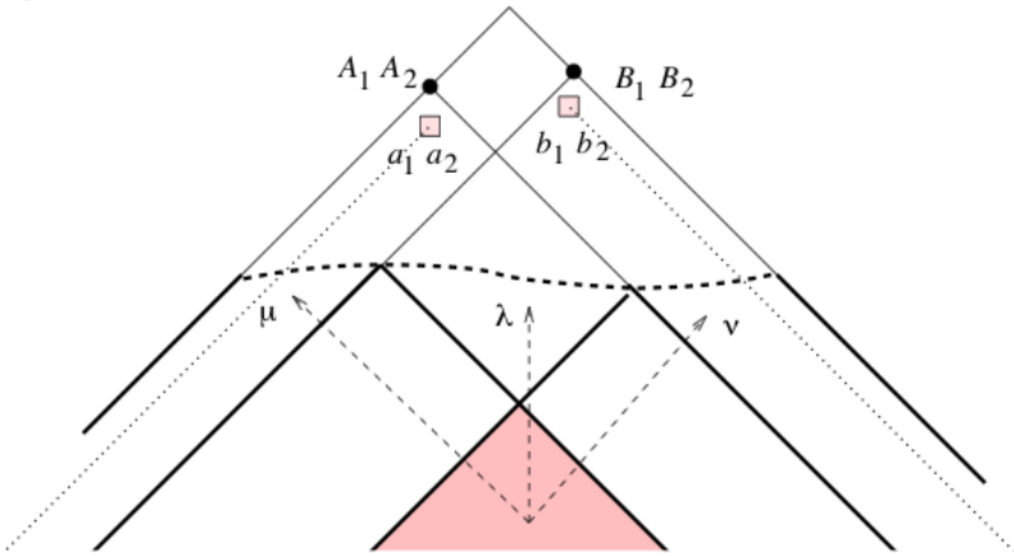
$$p(a_i) = \sum_{\mu,\lambda} u^{a_i}(\mu,\lambda)p(\mu \wedge \lambda) \quad p(b_j) = \sum_{\nu,\lambda} u^{b_j}(\lambda,\nu)p(\lambda \wedge \nu)$$
$$p(A_i \wedge B_j) = \sum_{\mu,\nu,\lambda} u^{A_i}(\mu,\lambda)u^{B_j}(\lambda,\nu)p(\mu \wedge \lambda \wedge \nu)$$
$$p(a_i \wedge b_j) = \sum_{\mu,\nu,\lambda} u^{a_i}(\mu,\lambda)u^{b_j}(\lambda,\nu)p(\mu \wedge \lambda \wedge \nu)$$

Here $p(\ldots)$ denotes the probability, while the function u is defined as $u^A(\alpha) = 1$ if outcome A occurs given the initial data α and 0 otherwise.

Assumption of separability:

$$p(\mu \wedge \lambda \wedge \nu) = p(\mu)p(\lambda)p(\nu)$$

Does not look natural since there is generally a common past that can induce correlations in the Cauchy data:



quasar 1 common past quasar 2

However, if we use this to avoid the conclusion, it leads to an infinite regression (in actual cosmology: regression to the Big Bang)!

Problem can be avoided by taking a random signal to determine the choices of directions, e.g. from distant quasars. Note: a recent experiment did just that!

D. Rauch, J. Handsteiner, A. Hochrainer, J. Gallicchio, A.S. Friedman, C Leung, B. Liu, L. Bulla, S. Ecker, F. Steinlechner, R. Ursin, B. Hu, D. Leon, C. Benn, A. Ghedina, M. Cecconi, A.H. Guth, D.I. Kaiser, T. Scheidl and A. Zeilinger:
Cosmic Bell Test using Random Measurement Settings from High-Redshift Quasars,
Phys. Rev. Lett. 121 (2018) 080403.

This experiment excluded that the common past responsible for the observed correlations was within 7.8 billion years in the past!

Note: Big Bang event is in the common past of the observable Universe - we cannot exclude that quantum correlations were encoded there ⇒ **superdeterminism**. It can always avoid the no-go theorems - but is it viable as an explanatory framework? Is it really a competitive alternative to quantum theory?

Screening property

$$\begin{aligned}
p(A_i \wedge B_j | a_i \wedge b_j \wedge \lambda) &= \frac{p(A_i \wedge B_j \wedge a_i \wedge b_j \wedge \lambda)}{p(a_i \wedge b_j \wedge \lambda)} \\
&= \frac{\sum_{\mu,\nu} u^{A_i}(\mu,\lambda) u^{a_i}(\mu,\lambda) u^{B_j}(\lambda,\nu) u^{b_j}(\lambda,\nu) p(\mu \wedge \lambda \wedge \nu)}{\sum_{\mu,\nu} u^{a_i}(\mu,\lambda) u^{b_j}(\lambda,\nu) p(\mu \wedge \lambda \wedge \nu)} \\
&= \frac{\sum_{\mu,\nu} u^{A_i}(\mu,\lambda) u^{a_i}(\mu,\lambda) u^{B_j}(\lambda,\nu) u^{b_j}(\lambda,\nu) p(\mu) p(\lambda) p(\nu)}{\sum_{\mu,\nu} u^{a_i}(\mu,\lambda) u^{b_j}(\lambda,\nu) p(\mu) p(\lambda) p(\nu)} \\
&= \frac{\sum_{\mu} u^{A_i}(\mu,\lambda) u^{a_i}(\mu,\lambda) p(\mu) p(\lambda)}{\sum_{\mu} u^{a_i}(\mu,\lambda) p(\mu) p(\lambda)} \frac{\sum_{\nu} u^{B_j}(\nu,\lambda) u^{b_j}(\nu,\lambda) p(\nu) p(\lambda)}{\sum_{\nu} u^{b_j}(\nu,\lambda) p(\nu) p(\lambda)} \\
&= p(A_i | a_i \wedge \lambda) p(B_j | b_j \wedge \lambda)
\end{aligned}$$

i.e.

$$p(A_i \wedge B_j | a_i \wedge b_j \wedge \lambda) = p(A_i | a_i \wedge \lambda) p(B_j | b_j \wedge \lambda)$$

Bell-Clauser-Horne theorem

Further assumption: choice of measurement direction is independent of the shared information λ:

$$u^{a_i}(\mu,\lambda) = u^{a_i}(\mu) \qquad u^{b_j}(\lambda,\nu) = u^{b_j}(\nu)$$

Motivation: if this is not true, then the “strange” EPR correlations originate from a “conspiracy” between the system state and measurement choices, encoded in λ (another form of superdeterminism).

Then we get

$$\begin{aligned}
p(A_i | a_i) &= \frac{p(A_i \wedge a_i)}{p(a_i)} = \frac{\sum_{\mu,\lambda} u^{A_i}(\mu,\lambda) u^{a_i}(\mu,\lambda) p(\mu \wedge \lambda)}{\sum_{\mu,\lambda} u^{a_i}(\mu,\lambda) p(\mu \wedge \lambda)} = \frac{\sum_{\mu,\lambda} u^{A_i}(\mu,\lambda) u^{a_i}(\mu) p(\mu) p(\lambda)}{\sum_{\mu,\lambda} u^{a_i}(\mu) p(\mu) p(\lambda)} \\
&= \frac{\sum_{\lambda} \sum_{\mu} u^{A_i}(\mu,\lambda) u^{a_i}(\mu) p(\mu) p(\lambda)}{\sum_{\mu} u^{a_i}(\mu) p(\mu)} = \sum_{\lambda} \frac{\sum_{\mu} u^{A_i}(\mu,\lambda) u^{a_i}(\mu) p(\mu)}{\sum_{\mu} u^{a_i}(\mu) p(\mu)} p(\lambda) \\
&= \sum_{\lambda} p(A_i | a_i \wedge \lambda)\, p(\lambda)
\end{aligned}$$

Similarly

$$p(A_i | a_i) = \sum_{\lambda} p(A_i | a_i \wedge \lambda)\, p(\lambda)$$

$$p(B_i | b_i) = \sum_{\lambda} p(B_i | b_i \wedge \lambda)\, p(\lambda)$$

$$p(A_i \wedge B_j | a_i \wedge b_j) = \sum_{\lambda} p(A_i \wedge B_j | a_i \wedge b_j \wedge \lambda)\, p(\lambda)$$

Theorem (Bell-Clauser-Horne)

Under the stated assumptions, the conditional probabilities $p\left(A_i \wedge B_j \middle| a_i \wedge b_j\right)$, $p(A_i|a_i)$ and $p(B_i|b_i)$ must satisfy the following inequalities

$$
\begin{aligned}
-1 \leq p(A_1 \wedge B_1|a_1 \wedge b_1) + p(A_1 \wedge B_2|a_1 \wedge b_2) + p(A_2 \wedge B_2|a_2 \wedge b_2) \\
- p(A_2 \wedge B_1|a_2 \wedge b_1) - p(A_1|a_1) - p(B_2|b_2) \leq 0 \\
-1 \leq p(A_2 \wedge B_1|a_2 \wedge b_1) + p(A_2 \wedge B_2|a_2 \wedge b_2) + p(A_1 \wedge B_2|a_1 \wedge b_2) \\
- p(A_1 \wedge B_1|a_1 \wedge b_1) - p(A_2|a_2) - p(B_2|b_2) \leq 0 \\
-1 \leq p(A_1 \wedge B_2|a_1 \wedge b_2) + p(A_1 \wedge B_1|a_1 \wedge b_1) + p(A_2 \wedge B_1|a_2 \wedge b_1) \\
- p(A_2 \wedge B_2|a_2 \wedge b_2) - p(A_1|a_1) - p(B_1|b_1) \leq 0 \\
-1 \leq p(A_2 \wedge B_2|a_2 \wedge b_2) + p(A_2 \wedge B_1|a_2 \wedge b_1) + p(A_1 \wedge B_1|a_1 \wedge b_1) \\
- p(A_1 \wedge B_2|a_1 \wedge b_2) - p(A_2|a_2) - p(B_1|b_1) \leq 0
\end{aligned}
$$

Note: these look formally the same as the corresponding Pitowsky inequalities, but their physical content is entirely different!

Proof

For any $0 \leq x_1, x_2, y_1, y_2 \leq 1$ it is automatically satisfied that

$$-1 \leq x_1 y_1 + x_1 y_2 + x_2 y_2 - x_2 y_1 - x_1 - y_2 \leq 0$$

Therefore

$$
\begin{aligned}
-1 \leq p(A_1|a_1 \wedge \lambda)p(B_1|b_1 \wedge \lambda) + p(A_1|a_1 \wedge \lambda)p(B_2|b_2 \wedge \lambda) + p(A_2|a_2 \wedge \lambda)p(B_2|b_2 \wedge \lambda) \\
- p(A_2|a_2 \wedge \lambda)p(B_1|b_1 \wedge \lambda) - p(A_1|a_1 \wedge \lambda) - p(B_2|b_2 \wedge \lambda) \leq 0
\end{aligned}
$$

Using the screening property

$$
\begin{aligned}
-1 \leq p(A_1 \wedge B_1|a_1 \wedge b_1 \wedge \lambda) + p(A_1 \wedge B_2|a_1 \wedge b_2 \wedge \lambda) + p(A_2 \wedge B_2|a_2 \wedge b_2 \wedge \lambda) \\
- p(A_2 \wedge B_1|a_2 \wedge b_1 \wedge \lambda) \\
- p(A_1|a_1 \wedge \lambda) - p(B_2|b_2 \wedge \lambda) \leq 0
\end{aligned}
$$

Multiplying by $p(\lambda)$ and summing over λ gives the first inequality. The three others are proven similarly.

Quantum probabilities violate the Clauser-Horne-Bell inequality

Assume Alice and Bob can both measure in two directions $\vec{a}_{1,2}$ for Alice, $\vec{b}_{1,2}$ for Bob, with angles

$$\angle\left(\vec{a}_1, \vec{b}_1\right) = \angle\left(\vec{a}_1, \vec{b}_2\right) = \angle\left(\vec{a}_2, \vec{b}_2\right) = 120^o \text{ and } \angle\left(\vec{a}_2, \vec{b}_1\right) = 0$$

Let $A_{1,2}$ be the events that Alice gets that her spin points in direction $\vec{a}_{1,2}$ and $B_{1,2}$ be the events that Bob gets that his spin points in direction $\vec{b}_{1,2}$. Then we have

$$p(A_1|a_1) = \frac{1}{2} \qquad p(A_2|a_2) = \frac{1}{2} \qquad p(B_1|b_1) = \frac{1}{2} \qquad p(B_2|b_2\,) = \frac{1}{2}$$

$$p(A_1 \wedge B_1|a_1 \wedge b_1) = \frac{3}{8} \;\; p(A_1 \wedge B_2|a_1 \wedge b_2) = \frac{3}{8}$$

$$p(A_2 \wedge B_1|a_2 \wedge b_1) = 0 \;\; p(A_2 \wedge B_2|a_2 \wedge b_2) = \frac{3}{8}$$

$$p(A_1 \wedge B_1|a_1 \wedge b_1) + p(A_1 \wedge B_2|a_1 \wedge b_2) + p(A_2 \wedge B_2|a_2 \wedge b_2) - p(A_2 \wedge B_1|a_2 \wedge b_1) - p(A_1|a_1) - p(B_2|b_2) = \frac{3}{8} + \frac{3}{8} + \frac{3}{8} - 0 - \frac{1}{2} - \frac{1}{2} = \frac{1}{8} \text{ !!!}$$

So, quantum correlations cannot be embedded in a local deterministic Markovian description which satisfies the screening property.

If we omit the screening property (or Bell's additional assumption): the LDM variables can "conspire" to determine both the state of the system and the measurement choices in a correlated way – but this would lead us down the path of superdeterminism.

A few common misconceptions

1. **Assumption of independence of measurements settings is not a "free will" assumption.**

It is just the statement that measurement choices are statistically independent of the state of the system. They can even be made by a random generator (even of cosmological origin) without reference to any conscious observer.

Example: in Brownian motion the Brownian particle moves under fluctuating random force of medium, whose probability distribution is independent of the particle's state of motion. At microscopic level this is typically valid even if classical dynamics holds and full motion of Brownian particle + molecules of medium is fully deterministic in terms of initial conditions + Newtonian dynamics!

2. **Superdeterminism is not just ordinary determinism.**

Superdeterminism is much more than ordinary determinism: it involves very specific correlations between measurement choices and system state variables to lead to the observed finite violation of Bell's inequality.

Cosmological Bell's experiment pushes the origin of this "conspiracy" to at least 7.8 billion years in the past - making any model using superdeterminism to explain EPR correlation very contrived!

Caveat: the cosmological Bell's experiment did not close other loopholes, but any attempt using these escape routes looks increasingly contrived.

3. **In principle, superdeterminism is not the only way out!**

It is possible to relax the property interpretation by a different way than giving up independence of measurement settings. It is also possible that the hidden parameter determines whether the property is measurable at all, and if yes, it also determines its value.

Proposal by A. Fine: the quantum probabilities are interpreted as

$$\text{Tr}\,\rho P_A = P(a_i|[A])$$

i.e., the conditional probability of measuring A gives the outcome a_i, provided the measurement gives a result (registers a hit) at all, which is denoted by $[A]$. An explicit realisation can be found in the book
L.E. Szabó: The Problem of Open Future - chance, causality, and determinism in physics
(in Hungarian)

However, reproducing the observed violation requires that the detection per emission rate is bounded:

$$\frac{\text{detection}}{\text{emission}} \leq \frac{1}{2} + \frac{\sqrt{2}}{4} \approx 0.85$$

This is now excluded by event-ready Bell tests simultaneously closing the detection and communication loopholes!!!

Lecture 6

Local and non-local realism

Local realism

Excluding LDM is interpreted as excluding **local realism** - where realism means existence of elements of reality, i.e. classical variables.

Only remaining way out for local realism is **superdeterminism** but it comes with a heavy price: the correlations inscribed into the state of the Universe billions of years ago must go through the dynamics of enormous number of degrees of freedom "buffeting" the corresponding variables until the eventual experiment - very hard to maintain in the face of 2nd law of thermodynamics!

This is the same mechanism that ensures that even if molecules of the medium can be described deterministically, the force exerted by them on the Brownian particle is an independent random noise. Preserving any correlation breaking statistical independence between the state of motion of the particle and the force exerted by the medium requires extreme fine-tuning!

A superdeterministic world could be classical, but at a very heavy price - it would not at all resemble usual classical physics! The requirement of **extreme fine-tuning** also prevents such a theory to be predictive. To reproduce quantum correlations, it would need to reverse engineer them, making any such theory parasitic on quantum theory.

Non-local realism

However: non-local hidden variables are possible as demonstrated by Bohmian mechanics
⇒ possibility of **non-local realism**.

Nevertheless, some forms of non-local realism can be excluded:

Leggett inequality ("crypto non-local theories")
A.J. Leggett, Foundations of Physics 33 (2003) 1469–1493.
Violated by QM, also violated experimentally (2007,2010).

Leggett-Garg inequality (realism violated in time evolution)
A.J. Leggett and A. Garg, Physical Review Letters **54** (1985) 857–860.

The inequality follows from two assumptions, both of which are parts of a classical world view:

1. Macrorealism: "A macroscopic object, which has available to it two or more macroscopically distinct states, is at any given time in a definite one of those states."
2. Noninvasive measurability: "It is possible in principle to determine which of these states the system is in without any effect on the state itself, or on the subsequent system dynamics."

Violated by QM, also violated experimentally (some loopholes may remain).

Tensor products, separable states and entanglement

Suppose we have a composite system of two parts:

Part 1: described by states $|\psi\rangle_1$ in a Hilbert space $\mathcal{H}_1$
Part 2: described by states $|\phi\rangle_2$ in a Hilbert space $\mathcal{H}_2$

Then the total state of the system can be given as a pair $(|\psi\rangle_1, |\phi\rangle_2)$

$(|\psi\rangle_1, |\phi\rangle_2) = |\psi\rangle_1 \otimes |\phi\rangle_2$ -- such a state is called a dyad.

However: superposition principle allows states of the general form

$$\sum_{i,j} C_{ij} |\psi_i\rangle_1 \otimes |\phi_j\rangle_2$$

The space of such linear combinations is called the **tensor product** of $\mathcal{H}_1$ and $\mathcal{H}_2$ and is denoted by $\mathcal{H}_1 \otimes \mathcal{H}_2$. The operation $\otimes$ is assumed to be linear in both arguments, i.e.

$$(\alpha|\psi\rangle_1 + \beta|\psi'\rangle_1) \otimes |\phi\rangle_2 = \alpha|\psi\rangle_1 \otimes |\phi\rangle_2 + \beta|\psi'\rangle_1 \otimes |\phi\rangle_2$$
$$|\psi\rangle_1 \otimes (\alpha|\phi\rangle_2 + \beta|\phi'\rangle_2) = \alpha|\psi\rangle_1 \otimes |\phi\rangle_2 + \beta|\psi\rangle_1 \otimes |\phi'\rangle_2$$

Tensor product basis and scalar product

Take a complete orthonormal basis $|e_i\rangle_1 \in \mathcal{H}_1$ and similarly $|f_j\rangle_2 \in \mathcal{H}_2$. Then any vector $|\Psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ can be (uniquely) written as

$$|\Psi\rangle = \sum_{i,j} \psi_{ij} \, |e_i\rangle_1 \otimes |f_j\rangle_2$$

Scalar product on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is defined for simple dyads $|\Psi\rangle = |\psi\rangle_1 \otimes |\phi\rangle_2$ and $|\Psi'\rangle = |\psi'\rangle_1 \otimes |\phi'\rangle_2$ as

$$\langle\Psi'|\Psi\rangle = \langle\psi'|\psi\rangle\langle\phi'|\phi\rangle$$

This has a uniquely defined linear extension to states which are linear combinations of multiple dyads. It is then easy to prove

$$\langle\Psi'|\Psi\rangle = \sum_{i,j} \left(\psi'_{ij}\right)^* \psi_{ij}$$

Operators on a tensor product

Given linear operators $O_1: \mathcal{H}_1 \to \mathcal{H}_1$ and $O_2: \mathcal{H}_2 \to \mathcal{H}_2$, their tensor product is

$$O_1 \otimes O_2: \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2 \qquad (O_1 \otimes O_2)(|\psi\rangle_1 \otimes |\phi\rangle_2) = O_1|\psi\rangle_1 \otimes O_2|\phi\rangle_2$$

Any linear operator $O: \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ can be written as linear combination of such product operators. Operators acting on $\mathcal{H}_{1,2}$ can be embedded into the space of operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$ by $O_1 \otimes \mathbf{1}$ and $\mathbf{1} \otimes O_2$.

E.g. if the subsystems have Hamiltonians H_1 and H_2 then the composite Hamiltonian in the absence of interactions is

$$H_1 \otimes \mathbf{1} + \mathbf{1} \otimes H_2$$

Interactions are describing by terms which act nontrivially in both factors.

Matrix elements of operators in a product basis $|v_{ij}\rangle = |e_i\rangle_1 \otimes |f_j\rangle_2$

$$O_{ij,i'j'} = \langle v_{i'j'}|O|v_{ij}\rangle$$

If $O = O_1 \otimes O_2$ this results in

$$O_{ij,i'j'} = (O_1)_{ii'}(O_2)_{jj'} \quad (O_1)_{ii'} = \langle e_i|O_1|e_{i'}\rangle \quad (O_2)_{jj'} = \langle f_j|O_2|f_{j'}\rangle$$

Partial trace

$$\mathrm{Tr}_1 O : \mathcal{H}_2 \to \mathcal{H}_2 \quad (\mathrm{Tr}_1 O)_{jj'} = \sum_i O_{ij,ij'}$$

$$\mathrm{Tr}_2 O : \mathcal{H}_1 \to \mathcal{H}_1 \quad (\mathrm{Tr}_2 O)_{ii'} = \sum_j O_{ij,i'j}$$

This is a linear operation, which can be proven to be independent of the choice of basis, and also

$$\mathrm{Tr}_1(O_1 \otimes O_2) = (\mathrm{Tr}\, O_1)\, O_2 \qquad \mathrm{Tr}_2(O_1 \otimes O_2) = (\mathrm{Tr}\, O_2)\, O_1$$

Separable states and entanglement

A pure state $|\Psi\rangle$ is **separable** if it can be written as

$$|\Psi\rangle = |\psi\rangle_1 \otimes |\phi\rangle_2$$

A mixed state ρ is called **separable** if it can be written as a convex combination

$$\rho = \sum_k \lambda_k \rho_1^k \otimes \rho_2^k \quad \lambda_k \geq 0 \quad \sum_k \lambda_k = 1$$

Why this definition? Because then ρ is just a probability distribution over uncorrelated product states.

Entanglement: a state is entangled if it is non-separable.

Note that classical probability distributions are always separable!

Classical systems: configuration (phase) spaces X and Y.

Composite system: configuration (phase) space $X \times Y$ (simple Cartesian product since no superposition principle applies).

Pure states: delta distribution $\delta_{x_0}(x)$ centred on a point $x_0 \in X$ which satisfies

$$\int_X dx\, \delta_{x_0}(x) f(x) = f(x_0) \quad \forall f: X \to \mathbf{R}$$

All classical states are trivially separable

Pure states:

$$\delta_{(x_0,y_0)}(x,y) = \delta_{x_0}(x)\delta_{y_0}(y)$$

The same goes for a classical mixed state on $X \times Y$ given by a probability distribution $\pi(x,y)$:

$$\pi(x,y) = \int_X dx_0 \int_Y dy_0\, \pi(x_0,y_0)\delta_{x_0}(x)\delta_{y_0}(y)$$

Note that this a (continuous) convex combination since

$$\pi(x_0,y_0) \geq 0 \qquad \text{and} \qquad \int_X dx_0 \int_Y dy_0\, \pi(x_0,y_0) = 1$$

Therefore, entanglement encodes quantum correlations that go beyond what is classically possible!

All the above can be extended to a multiple tensor product $H_1 \otimes H_2 \dots \otimes H_n$. Definition of separability for systems composed of n parts:

A pure state $|\Psi\rangle$ is **separable** if it can be written as

$$|\Psi\rangle = |\psi_1\rangle_1 \otimes |\psi_2\rangle_2 \dots \otimes |\psi_n\rangle_n$$

A mixed state ρ is called **separable** if it can be written as a convex combination

$$\rho = \sum_k \lambda_k\, \rho_1^k \otimes \rho_2^k \dots \otimes \rho_n^k \quad \lambda_k \geq 0 \quad \sum_k \lambda_k = 1$$

Again, it is simple to see that states in classical composite systems are always separable.

Is there superluminal communication in EPR-Bell?

Assuming wave function reduction is some physical event, let's consider *when* it happens!

Spin projection eigenstates in direction $\vec{w} = (\sin\phi, 0, \cos\phi)$:

$$|\uparrow_w\rangle = \cos\frac{\phi}{2}|\uparrow_z\rangle + \sin\frac{\phi}{2}|\downarrow_z\rangle = \begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2} \end{pmatrix}$$

$$|\downarrow_w\rangle = -\sin\frac{\phi}{2}|\uparrow_z\rangle + \cos\frac{\phi}{2}|\downarrow_z\rangle = \begin{pmatrix} -\sin\frac{\phi}{2} \\ \cos\frac{\phi}{2} \end{pmatrix}$$

Event projector to spin up in direction $\vec{w}$

$$P(\phi) = |\uparrow_w\rangle\langle\uparrow_w| = \begin{pmatrix} \cos^2\frac{\phi}{2} & \sin\frac{\phi}{2}\cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\phi}{2} & \sin^2\frac{\phi}{2} \end{pmatrix}$$

2-spin event projector for Alice measuring spin 1 in direction $\vec{w}$:

$$P_1(\phi) = P(\phi) \otimes 1 = \begin{pmatrix} \cos^2\frac{\phi}{2} & 0 & \sin\frac{\phi}{2}\cos\frac{\phi}{2} & 0 \\ 0 & \cos^2\frac{\phi}{2} & 0 & \sin\frac{\phi}{2}\cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\phi}{2} & 0 & \sin^2\frac{\phi}{2} & 0 \\ 0 & \sin\frac{\phi}{2}\cos\frac{\phi}{2} & 0 & \sin^2\frac{\phi}{2} \end{pmatrix}$$

2-spin event projector for Bob measuring spin 1 in direction $\vec{w}'$:

$$P_2(\phi') = 1 \otimes P(\phi') = \begin{pmatrix} \cos^2\frac{\phi'}{2} & \sin\frac{\phi'}{2}\cos\frac{\phi'}{2} & 0 & 0 \\ \sin\frac{\phi'}{2}\cos\frac{\phi'}{2} & \sin^2\frac{\phi'}{2} & 0 & 0 \\ 0 & 0 & \cos^2\frac{\phi'}{2} & \sin\frac{\phi'}{2}\cos\frac{\phi'}{2} \\ 0 & 0 & \sin\frac{\phi'}{2}\cos\frac{\phi'}{2} & \sin^2\frac{\phi'}{2} \end{pmatrix}$$

The joint probability of these events makes sense:

$$P_1(\phi)P_2(\phi') = P_2(\phi')P_1(\phi)$$

(Alice and Bob can choose their detector settings independent of each other)

The probability that both of their detectors clicks (i.e. they both measure spin up)

$$\langle \Psi_{EPR} | P_2(\phi') P_1(\phi) | \Psi_{EPR} \rangle = \frac{1}{2} \sin^2 \frac{\phi - \phi'}{2}$$

$$|\Psi_{EPR}\rangle = \frac{1}{\sqrt{2}} (|\uparrow_z\rangle_1 |\downarrow_z\rangle_2 - |\downarrow_z\rangle_1 |\uparrow_z\rangle_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Action of a projector is the same as reducing the wave function to the appropriate eigenstate (up to a normalisation) but since the two vectors commute, the order does not matter.

Since this holds even if the two measurements are space-like separated, the results are consistent with relativistic causality!

(Note: locality in Quantum Field Theory!)

No-signalling property

No-signalling property means that measurement choices made by Bob cannot influence the outcome statistics measured by Alice and vice versa:

$$\sum_B p(A \wedge B | a \wedge b) = p(A|a) \qquad \sum_A p(A \wedge B | a \wedge b) = p(B|b)$$

i.e. Bob cannot send a signal to Alice by manipulating the degrees of freedom locally available to him (and vice versa).

Correlation vectors $\vec{p}$ having this property form the **no-signalling polytope** $NS(n, S)$ (S. Popescu and D. Rohrlich, Found. Phys. 24 (1994) 379–385).

Local (Bell) polytope: correlations come from shared randomness between the two sides

$$p(A \wedge B | a \wedge b) = \sum_i \lambda_i \, P_i^{Alice}(A|a) P_i^{Bob}(B|b) \qquad \lambda_i \geq 0 \qquad \sum_i \lambda_i = 1$$

Here i runs over the possible configurations of the shared (hidden) variables, λ_i are the probabilities of the shared variables realising the configuration i. This gives exactly Pitowski's classical polytope $C(n, S)$, and also implies the no-signalling property

$$\sum_B p(A \wedge B | a \wedge b) = \sum_i \lambda_i \, P_i^{Alice}(A|a) \sum_B P_i^{Bob}(B|b) = \sum_i \lambda_i \, P_i^{Alice}(A|a)$$
$$= p(A|a) \qquad \text{since} \qquad \sum_B P_i^{Bob}(B|b) = 1$$

Quantum convex set: correlation vectors realisable in quantum theory, $q(n,S)$

<u>**Central property:**</u> $C(n,S) \subsetneq q(n,S) \subsetneq NS(n,S)$

Quantum correlations go beyond classical ones: **entanglement!**

However, quantum correlations still satisfy no-signalling. But they are not the most general no-signalling correlations possible!

The world could in principle be even more strongly correlated than in quantum theory without violating no-signalling!

Literature for those wanting to learn more about correlation polytopes:

- N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani and S. Wehner, Rev. Mod. Phys. 86 (2014) 419-478, arXiv:1303.2849 [quant-ph].
- J. Bub, Int. J. Theor. Phys. 53 (2014) 3346–3369, arXiv:1210.6371 [quant-ph].
- Jeffrey Bub: *Bananaworld: Quantum Mechanics for Primates*, Oxford University Press, 2016, 273 pages.

Quantum correlations satisfy the no-signalling property:

No-communication theorem

(P.H. Eberhard, Il Nuovo Cimento 46 (1978) 392-419).

Since all observables measured by Alice and Bob commute, one can write the system's Hilbert space as

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Observables measured by Alice: $O_A \otimes \mathbf{1}_B$

Observables measured by Bob: $\mathbf{1}_A \otimes O_B$

Assume Alice's measurements has (pairwise commuting) outcome projectors $P_{A_i} \otimes \mathbf{1}_B$, while Bobs has $\mathbf{1}_A \otimes P_{B_j}$ such that

$$\sum_i P_{A_i} = \mathbf{1}_A \qquad \sum_j P_{B_j} = \mathbf{1}_B$$

Note: their measurement settings are encoded in the choice of their system of outcome projectors.

Now suppose that the observed system is in a quantum state described by a density matrix ρ. Then Alice and Bob can describe the state from their point of view by separate density matrices ρ_A and ρ_B uniquely defined by specifying all the expectation values under them:

$$\mathrm{Tr}_{\mathcal{H}}\, \rho(O_A \otimes \mathbf{1}_B) = \mathrm{Tr}_{\mathcal{H}_A}\, \rho_A O_A \quad \forall O_A$$
$$\mathrm{Tr}_{\mathcal{H}}\, \rho(\mathbf{1}_A \otimes O_B) = \mathrm{Tr}_{\mathcal{H}_B}\, \rho_B O_B \quad \forall O_B$$

Note: existence of such ρ_A and ρ_B is guaranteed by von Neumann's theorem (cf. Lecture 3), and also by Gleason's theorem.

The states of subsystems are given by the partial trace operation:

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}\rho \qquad \rho_B = \mathrm{Tr}_{\mathcal{H}_A}\rho$$

How to compute the partial trace? The density matrix can always be written (not uniquely) in the form

$$\rho = \sum_k \alpha_k \, S_k \otimes T_k$$

Then it is easy to see that

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}\rho = \sum_k \alpha_k \left(\mathrm{Tr}_{\mathcal{H}_B} T_k\right) S_k \qquad \rho_B = \mathrm{Tr}_{\mathcal{H}_A}\rho = \sum_k \alpha_k \left(\mathrm{Tr}_{\mathcal{H}_A} S_k\right) T_k$$

Example: EPR state

$$\begin{aligned}\rho &= \frac{1}{2}(\,|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B\,)(\langle\uparrow|_A \otimes \langle\downarrow|_B - \langle\downarrow|_A \otimes \langle\uparrow|_B)\\ &= \frac{1}{2}|\uparrow\rangle_A\langle\uparrow|_A \otimes |\downarrow\rangle_B\langle\downarrow|_B - \frac{1}{2}|\downarrow\rangle_A\langle\uparrow|_A \otimes |\uparrow\rangle_B\langle\downarrow|_B - \frac{1}{2}|\uparrow\rangle_A\langle\downarrow|_A \otimes |\downarrow\rangle_B\langle\uparrow|_B\\ &\quad + \frac{1}{2}|\downarrow\rangle_A\langle\downarrow|_A \otimes |\uparrow\rangle_B\langle\uparrow|_B\end{aligned}$$

Using

$$\mathrm{Tr}_{\mathcal{H}_B}|\downarrow\rangle_B\langle\downarrow|_B = 1 = \mathrm{Tr}_{\mathcal{H}_B}|\uparrow\rangle_B\langle\uparrow|_B \qquad \mathrm{Tr}_{\mathcal{H}_B}|\downarrow\rangle_B\langle\uparrow|_B = 0 = \mathrm{Tr}_{\mathcal{H}_B}|\uparrow\rangle_B\langle\downarrow|_B$$

we get

$$\rho_A = \frac{1}{2}|\uparrow\rangle_A\langle\uparrow|_A + \frac{1}{2}|\downarrow\rangle_A\langle\downarrow|_A$$

and similarly

$$\rho_B = \frac{1}{2}|\uparrow\rangle_B\langle\uparrow|_B + \frac{1}{2}|\downarrow\rangle_B\langle\downarrow|_B$$

Note that the partial states are mixed, even though the full system state is pure!

Now

$$\sum_j p(A_i \wedge B_j | a \wedge b) = \sum_j \mathrm{Tr}_H\, \rho\left(P_{A_i} \otimes P_{B_j}\right) = \mathrm{Tr}_H\, \rho\left(P_{A_i} \otimes 1\right) = \mathrm{Tr}_{H_A}\, \rho_A P_{A_i} = p(A_i|a)$$

and similarly

$$\sum_i p(A_i \wedge B_j | a \wedge b) = p(B_j | b)$$

So, Alice cannot communicate to Bob by local manipulations on her side (choosing measurement settings and performing measurements) and vice versa for Bob. Even quantum entanglement does not allow superluminal signalling, i.e.

$$Q(n,S) \subset NS(n,S)$$

Communication is possible if Alice transfers her measurement outcomes to Bob over some direct communication channel - then Bob can deduce Alice's measurement settings from the correlations between his and Alice's measurement outcomes.

Indeed, this is how quantum communication and quantum cryptography work - they involve a direct (classical) channel between the participants.

No-communication theorem implies the **no-cloning theorem** - there are no quantum operations which allow the cloning of an arbitrary state on Alice's side to Bob's side.

Lecture 7

Quantum histories

Wave function reduction is problematic

- Incompatibility with relativistic causality
- Our EPR-Bell analysis: it is not possible to identify a time when it happens - hard to assign it to a real physical event!

Consistent histories: a reformulation of quantum theory

- Independent of interpretation: it works as an extension of orthodox Copenhagen QM, or can be accommodated into an Everett interpretation etc.

- It is also a research program to interpret QM solely in its own terms.
  - Description of closed quantum systems without outside observers (e.g. quantum cosmology)
  - Does not presuppose classical background in any sense
  - Manifestly compatible with relativity

Good book to learn the details from:

R.B. Griffiths: Consistent Quantum Theory. Cambridge University Press, 2003.

Consistent histories in the orthodox (Copenhagen) interpretation:

R. Omnes: The Interpretation of Quantum Mechanics. Princeton University Press, 1994. - with details

R. Omnes: Understanding Quantum Mechanics. Princeton University Press, 1999. - simpler presentation

Quantum sample space

Set of mutually orthogonal projectors P_i spanning all the Hilbert space

$$P_i^2 = P_i = P_i^\dagger \quad P_i P_j = 0 \quad \sum_i P_i = \mathbf{1}$$

(resolution of identity)

Select time instants $t_1 < \cdots < t_n$ and a quantum sample space $P_{i_k}(t_k)$ for each (they can be different for each instant)

History operator

$$\chi_{i_1 \dots i_n} = P_{i_n}(t_n) \dots P_{i_1}(t_1)$$

Compatibility with relativity: time ordering is only absolute if projectors correspond to non-spacelike separated events. However, whenever they are spacelike separated, their relative order does not matter due to locality in relativistic quantum field theory.

The full set of history operators constructed this way form a resolution of identity

$$\sum_{i_1,\ldots,i_n} \chi_{i_1\ldots i_n} = \mathbf{1}$$

So, we can interpret these histories as mutually exclusive and forming a complete set of possibilities.

We would like to assign probabilities to the system having the history $\chi_{i_1\ldots i_n}$:

$$p_{i_1\ldots i_n} = \mathrm{Tr}\, \chi_{i_1\ldots i_n} \rho \chi^{\dagger}_{i_1\ldots i_n}$$

Consistency with usual quantum rules for probability

This recovers the usual rules of wave function reduction. If we have an initial state $|\Psi(0)\rangle$ evolving as

$$|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle$$

and perform a measurement at time t_1 resulting in an outcome $P^{(1)}_{i_1}$ from a resolution of the identity, the reduced wave function after t_1 is

$$|\Psi(t_1)\rangle_+ = \frac{1}{\sqrt{p_{i_1}}} P^{(1)}_{i_1} e^{-iHt_1}|\Psi(0)\rangle$$

with probability $p_{i_1} = \langle\Psi(0)|e^{+iHt_1} P^{(1)}_{i_1} e^{-iHt_1}|\Psi(0)\rangle$

Now perform a measurement at time t_2 resulting in an outcome $P^{(2)}_{i_2}$ from another resolution of the identity

$$|\Psi(t_2)\rangle_+ = \frac{1}{\sqrt{p_{i_2|i_1}}} P^{(2)}_{i_2} e^{-iH(t_2-t_1)}|\Psi(t_1)\rangle_+$$

with probability $p_{i_2|i_1} = {}_+\langle\Psi(t_1)|e^{+iH(t_2-t_1)} P^{(2)}_{i_2} e^{-iH(t_2-t_1)}|\Psi(t_1)\rangle_+$

We then get a two-step history with the probability

$$\begin{aligned} p_{i_1 i_2} = p_{i_2|i_1} p_{i_1} &= \langle\Psi(0)|e^{+iHt_1} P^{(1)}_{i_1} e^{+iH(t_2-t_1)} P^{(2)}_{i_2} e^{-iH(t_2-t_1)} P^{(1)}_{i_1} e^{-iHt_1}|\Psi(0)\rangle \\ &= \langle\Psi(0)|P^{(1)}_{i_1}(t_1) P^{(2)}_{i_2}(t_2) P^{(1)}_{i_1}(t_1)|\Psi(0)\rangle \\ &= \langle\Psi(0)|\chi^{\dagger}_{i_1 i_2} \chi_{i_1 i_2}|\Psi(0)\rangle \qquad \chi_{i_1 i_2} = P^{(2)}_{i_2}(t_2) P^{(1)}_{i_1}(t_1) \end{aligned}$$

We can also compute the reduced state back at the initial time

$$|\Psi_{12}(0)\rangle_+ = e^{iHt_2}|\Psi(t_2)\rangle_+ = e^{iHt_2} \frac{1}{\sqrt{p_{i_2|i_1}}} P^{(2)}_{i_2} e^{-iH(t_2-t_1)}|\Psi(t_1)\rangle_+$$

$$= e^{iHt_2}\frac{1}{\sqrt{p_{i_2|i_1}}}P_{i_2}^{(2)}e^{-iH(t_2-t_1)}\frac{1}{\sqrt{p_{i_1}}}P_{i_1}^{(1)}e^{-iHt_1}|\Psi(0)\rangle = \frac{1}{\sqrt{p_{i_1 i_2}}}P_{i_2}^{(2)}(t_2)P_{i_1}^{(1)}(t_1)|\Psi(0)\rangle$$

$$= \frac{1}{\sqrt{p_{i_1 i_2}}}\chi_{i_1 i_2}|\Psi(0)\rangle$$

Repeating this procedure for n consecutive measurements, we find the probability for specified outcomes

$$p_{i_1\dots i_n} = \langle\Psi(0)|\chi_{i_1\dots i_n}^{\dagger}\chi_{i_1\dots i_n}|\Psi(0)\rangle \quad \text{with history operator} \quad \chi_{i_1\dots i_n} = P_{i_n}^{(n)}(t_n)\dots P_{i_1}^{(1)}(t_1)$$

and the reduced state

$$|\Psi_{1\dots n}(0)\rangle_+ = \frac{1}{\sqrt{p_{i_1\dots i_n}}}\chi_{i_1\dots i_n}|\Psi(0)\rangle$$

For a density matrix initial state, we can simply write it in eigen basis

$$\rho = \sum_k \lambda_k\, |\Psi_k(0)\rangle\langle\Psi_k(0)|$$

and get

$$p_{i_1\dots i_n} = \sum_k \lambda_k\, \langle\Psi_k(0)|\chi_{i_1\dots i_n}^{\dagger}\chi_{i_1\dots i_n}|\Psi_k(0)\rangle = \mathrm{Tr}\left(\chi_{i_1\dots i_n}\rho\chi_{i_1\dots i_n}^{\dagger}\right)$$

with the reduced state moved back to $t = 0$ given by

$$\rho_{i_1\dots i_n} = \frac{1}{p_{i_1\dots i_n}}\chi_{i_1\dots i_n}\rho\chi_{i_1\dots i_n}^{\dagger}$$

Composite histories can be formed as sums of a finite number of elementary histories $\chi_{i_1\dots i_n}$ and are interpreted meaning that one of the alternatives included in the sum happened.

Let us consider two different histories χ_1 and χ_2 and their combination $\chi_1 + \chi_2$ then the rules of probability dictate that

$$\begin{aligned}\mathrm{Tr}\,\chi_1\rho\chi_1{}^{\dagger} + \mathrm{Tr}\,\chi_2\rho\chi_2{}^{\dagger} &= \mathrm{Tr}\,(\chi_1+\chi_2)\rho(\chi_1+\chi_2)^{\dagger} \\ &= \mathrm{Tr}\,\chi_1\rho\chi_1{}^{\dagger} + \mathrm{Tr}\,\chi_2\rho\chi_2{}^{\dagger} + 2\,\mathrm{Re}\,\mathrm{Tr}\,\chi_2\rho\chi_1{}^{\dagger}\end{aligned}$$

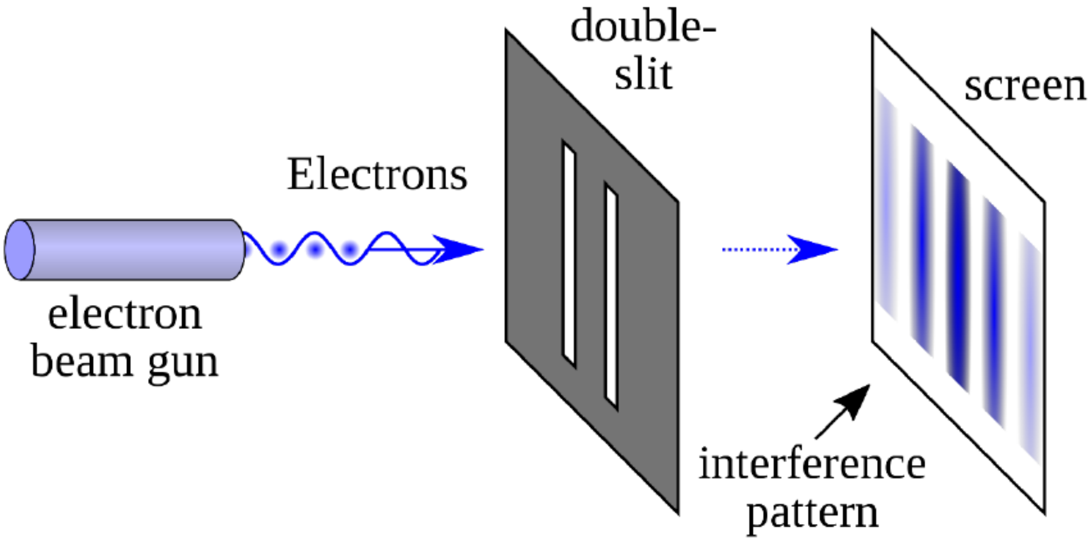
Definition: a set of histories $\{\chi_\alpha\}$ is called **consistent** if $\mathrm{Re}\,\mathrm{Tr}\,\chi_\alpha\rho\chi_\beta{}^{\dagger} = 0$ whenever $\alpha \neq \beta$.

For a consistent set of histories, we can consider each member of the set as mutually inequivalent alternatives for a temporal history of evolution of the quantum system, assigned with the probability

$$p_\alpha = \mathrm{Tr}\,\chi_\alpha\rho\chi_\alpha{}^{\dagger}$$

Probabilities of composite histories are simply obtained by summing the probabilities of the elementary histories involved.

Example: double-slit experiment



The wave function of the particle passing the slits (symmetric arrangement for simplicity's sake)

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\psi_a\rangle + |\psi_b\rangle)$$

where $|\psi_{a,b}\rangle$ are the wavefunctions obtained by passing slit *a/b*.

In coordinate representation, intensity in a patch X on the screen can be computed as

$$\begin{aligned}\int_X dx\Psi(x)^*\Psi(x) &= \frac{1}{2}\int_X dx\big(\psi_a(x)+\psi_b(x)\big)^*\big(\psi_a(x)+\psi_b(x)\big)\\ &= \underbrace{\frac{1}{2}\int_X dx|\psi_a(x)|^2}_{\text{slit } a} + \underbrace{\frac{1}{2}\int_X dx|\psi_b(x)|^2}_{\text{slit } b} + \underbrace{\mathrm{Re}\int_X dx\psi_a(x)^*\psi_b(x)}_{\text{interference term}}\end{aligned}$$

We can take the following sample spaces:

- At time t_1:
  - $P_a(t_1) = |\psi_a\rangle\langle\psi_a|$ (particle passed slit a)
  - $P_b(t_1) = |\psi_b\rangle\langle\psi_b|$ (particle passed slit b)

- At time t_2:
  $P_X(t_2) = P_X$: particle is in some patch X of the detector screen.

Compatibility condition for histories $P_X(t_2)P_a(t_1)$ and $P_X(t_2)P_b(t_1)$

i.e. that particle landed at patch X and earlier passed through slit a/b

$$\mathrm{Re}\ \mathrm{Tr}\left(P_X(t_2)P_a(t_1)|\Psi\rangle\langle\Psi|P_b(t_1)P_X(t_2)\right) = \mathrm{Re}\ \langle\Psi|P_bP_XP_XP_a|\Psi\rangle$$
$$= \frac{1}{2}\mathrm{Re}\ \langle\psi_a|P_X|\psi_b\rangle = \frac{1}{2}\mathrm{Re}\ \int_X dx\psi_a(x)^*\psi_b(x)$$
$$\neq 0 \text{ in general (this is the interference term!)}$$

So: if we observe interference, it is not consistent to include in the history the which-slit information!

Histories are consistent exactly when their interference vanishes.

Consistency of histories is a precise and general formulation of Niels Bohr's complementarity principle.

Refinement, coarsening and frameworks

A consistent set of histories $\{\chi_{i_1\dots i_n} = P_{i_n}(t_n)\dots P_{i_1}(t_1)\}$ can be **refined** in two ways:

1. Adding another time instant t' with a sample space $P_{i'}(t')$.

   Assume that the added instant fits in as $t_{k+1} > t' > t_k$.

   This gives a refinement of the original set if the extended histories

   $$\chi_{i_1\dots i_k i' i_{k+1}\dots i_n} = P_{i_n}(t_n)\dots P_{i_{k+1}}(t_{k+1})P_{i'}(t')P_{i_k}(t_k)\dots P_{i_1}(t_1)$$

   are still consistent. The original histories can be identified with the composites

   $$\sum_{i'}\chi_{i_1\dots i_k i' i_{k+1}\dots i_n}$$

2. Refining any of the sample spaces such that the new set of histories is still consistent.

   A sample space $S' = \{P'_{i'}\}$ is a refinement of the sample space $S = \{P_i\}$ if for any projector $P'_{i'} \in S'$ there is a projector $P_i \in S$ such that $P'_{i'} \leq P_i$.

   This implies that any element $P_i \in S$ can be written as sum of elements of S', so the original histories can again be embedded into the refined set.

The original set of histories is then called a **coarsening** of the refined one.

Single framework rule: any physical situation must be described by using histories which are contained in a single consistent family.

A framework can be viewed as a class of compatible families which have a common refinement.

Two consistent sets of histories are called **incompatible** if there is no consistent set of histories which is a refinement of both.

When two histories are incompatible (i.e. inconsistent) they cannot be members of a consistent set and so cannot be part of a single framework.

Most (all?) paradoxes of quantum theory arise from comparing histories which are incompatible, i.e. from violating the single framework rule.

Note: consistent histories do not presuppose wave function reduction - the history projectors are not considered as actual measurements.

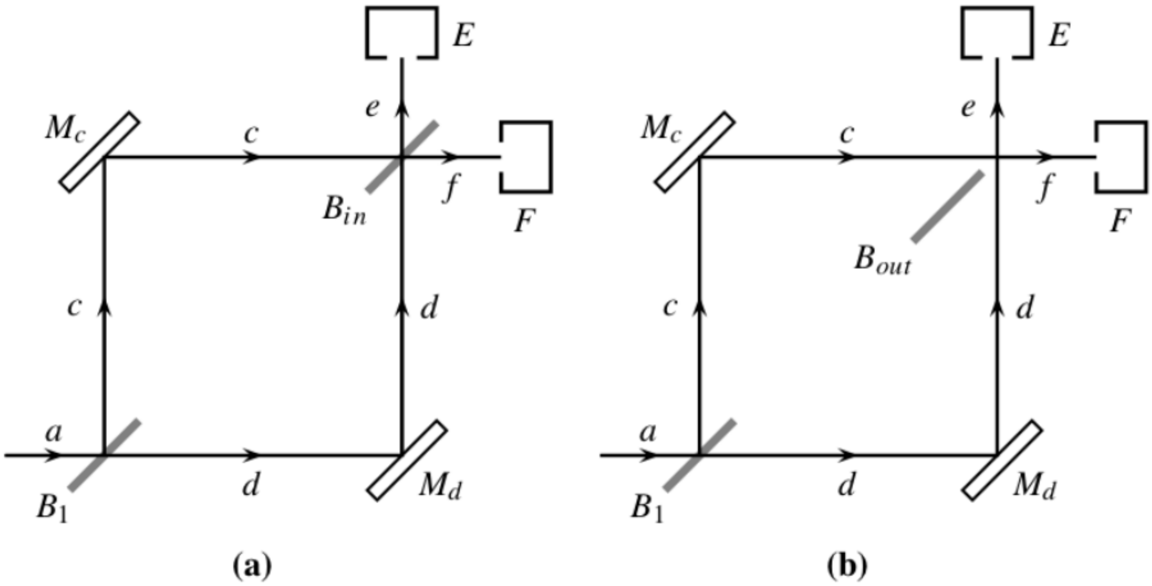
Therefore, using quantum histories we can talk about intermediate states (i.e. the history) of the system and the probabilities of alternative histories without performing any actual measurement during the evolution!

The price to pay is that this description depends on the choice of the consistent family (single framework rule!) - which is nothing else but the already encountered contextuality.

Lecture 8

Wheeler's delayed choice paradox

Mach-Zehnder interferometer with second beam splitter in (a) and out (b) of the way



Time evolution for the two situations

The beam splitters are unitary time-evolution operators

$$B_1\colon |0a\rangle \to |1\bar{a}\rangle = \frac{1}{\sqrt{2}}(|1c\rangle + |1d\rangle) \qquad |0b\rangle \to |1\bar{b}\rangle = \frac{1}{\sqrt{2}}(-|1c\rangle + |1d\rangle)$$

$$B_2\colon\ |1c\rangle \to |2\bar{c}\rangle = \frac{1}{\sqrt{2}}(|2e\rangle + |2f\rangle) \qquad |1d\rangle \to |2\bar{d}\rangle = \frac{1}{\sqrt{2}}(-|2e\rangle + |2f\rangle)$$

(a) Second beam splitter in

$$|0a\rangle \to |1\bar{a}\rangle = \frac{1}{\sqrt{2}}(|1c\rangle + |1d\rangle) \to |2f\rangle \to |E^o\rangle|F^*\rangle$$

In this case only detector F registers hits. What B_2 does is to **erase which-path information** by combining the two paths again, so we observe an interference pattern, with detector E as the place of destructive and detector F as the place of constructive interference.

(b) Second beam splitter out

$$|0a\rangle \to |1\bar{a}\rangle = \frac{1}{\sqrt{2}}(|1c\rangle + |1d\rangle) \to \frac{1}{\sqrt{2}}(|2f\rangle + |2e\rangle) \to \frac{1}{\sqrt{2}}(|E^o\rangle|F^*\rangle + |E^*\rangle|F^o\rangle)$$

In this case either E and F registers hits (but never both), with equal probabilities.

The paradox seems to arise if we only decide whether to remove B_2 when the photon already passed B_1 and almost arrived at B_2. Then QM says we do not see interference (case (b)), but the photon is already in the device so that we think we can say it had already "chosen" the arm $|f\rangle$ which should lead to outcome according to case (a).

Consistent families of histories

Let us consider four times for the evolution

1. t_0: photon in channel a, state is $|0a\rangle|E^o\rangle|F^o\rangle$
2. t_1: photon in channels c or d, state is $|1\bar{a}\rangle|E^o\rangle|F^o\rangle = \frac{1}{\sqrt{2}}(|1c\rangle + |1d\rangle)|E^o\rangle|F^o\rangle$
3. t_2: photon in channels e or f
   B_{out}: state is $\frac{1}{\sqrt{2}}(|2\bar{c}\rangle + |2\bar{d}\rangle)|E^o\rangle|F^o\rangle = \frac{1}{\sqrt{2}}(|2f\rangle + |2e\rangle)|E^o\rangle|F^o\rangle$
   B_{in}: state is $|2f\rangle|E^o\rangle|F^o\rangle$
4. t_3: photon detected
   B_{out}: state is $\frac{1}{\sqrt{2}}(|E^*\rangle|F^o\rangle + |E^o\rangle|F^*\rangle)$
   B_{in}: state is $|E^o\rangle|F^*\rangle$

Notation: $[\psi] = |\psi\rangle\langle\psi|$

Family 1: convenient for case with second beam splitter out, made of two histories

$\chi_E = [E^*][2e][1d][0a] \qquad \chi_F = [F^*][2f][1c][0a]$

These are consistent:

$\text{Tr}\ \chi_E\rho\chi_F^\dagger = \text{Tr}\ [E^*][2e][1d][0a][0a][0a][1c][2f][F^*] = 0$

since $[2e]$ and $[2f]$ are orthogonal projectors (they commute with $[E^*]$ and $[F^*]$)

and both have probability $1/2$:

$$\begin{aligned}\text{Tr}\ \chi_E\rho\chi_E^\dagger &= \text{Tr}\ [E^*][2e][1d][0a][0a][0a][1d][2e][E^*] = \text{Tr}\ [E^*][2e][1d][0a]\\ &= (\text{Tr}\ [E^*])(\text{Tr}\ [2e][1d][0a]) = \text{Tr}\ |d\rangle\langle d|\left(\frac{1}{2}(|c\rangle + |d\rangle)(\langle c| + \langle d|)\right)\\ &= \frac{1}{2}\langle d|\big((|c\rangle + |d\rangle)(\langle c| + \langle d|)\big)|d\rangle = \frac{1}{2}\end{aligned}$$

and similarly, for the other case.

Note that in this case χ_E contains the projection $[1d]$, so we can infer that whenever detector E is triggered, the photon went along the arm d, while similarly we can infer that whenever detector F is triggered, the photon went along arm c.

Family 2: convenient when 2nd beam splitter is in. We only need one history in this case

$\chi = [F^*][2f][1\bar{a}][0a]$

This has probability 1, so F is always triggered. However, we can only infer now the intermediate state

$$|1\bar{a}\rangle = \frac{1}{\sqrt{2}}(|1c\rangle + |1d\rangle)$$

Therefore, we cannot say which way the photon went!

Can we try to use a family which allows us to infer the which way information? We can attempt to do that with

Family 3:

$\chi_1 = S_+[2\bar{c}][1c][0a] \qquad \chi_2 = S_-[2\bar{e}][1d][0a]$

S_+: projector on state $\frac{1}{\sqrt{2}}(|E^*\rangle|F^o\rangle + |E^o\rangle|F^*\rangle)$
S_-: projector on state $\frac{1}{\sqrt{2}}(-|E^*\rangle|F^o\rangle + |E^o\rangle|F^*\rangle)$

This family is consistent, but it does not allow us to draw the conclusion since in this framework the events that one or the other detector clicked make no sense (their state is a macroscopic quantum superposition).

Conclusion: if we detect interference, we cannot infer the path! So the rules of quantum mechanics (as reflected by quantum histories) simply do not allow for any paradox to arise here.

Delayed choice with "quantum coin"

What if the position of the second beam splitter is decided by a "quantum coin toss"? We assume that some time while the photon is between the two beam splitters, a quantum coin toss occurs and the second beam splitter's state changes into a "Schrödinger's cat" state:

$$|B_0\rangle = \frac{1}{\sqrt{2}}(|B_{in}\rangle + |B_{out}\rangle)$$

Then the time evolution is

1. t_0: photon in channel a, state is $|\Psi_0\rangle = |0a\rangle|E^o\rangle|F^o\rangle|B_0\rangle$
2. t_1: photon in channel c or d,
   state is $|1\bar{a}\rangle|E^o\rangle|F^o\rangle = \frac{1}{\sqrt{2}}(|1c\rangle + |1d\rangle)|E^o\rangle|F^o\rangle|B_0\rangle$
3. t_2: photon in channel c or d,

state is $|1\bar{a}\rangle|E^o\rangle|F^o\rangle\frac{1}{\sqrt{2}}(|B_{in}\rangle + |B_{out}\rangle) = \frac{1}{2}(|1c\rangle + |1d\rangle)|E^o\rangle|F^o\rangle(|B_{in}\rangle + |B_{out}\rangle)$

4. t_3: photon in channel e or f

   state is $\frac{1}{\sqrt{2}}|2f\rangle|E^o\rangle|F^o\rangle|B_{in}\rangle + \frac{1}{2}(|2f\rangle + |2e\rangle)|E^o\rangle|F^o\rangle|B_{out}\rangle$

5. t_4: photon detected

   state is $\frac{1}{\sqrt{2}}(|B_{in}\rangle|E^o\rangle|F^*\rangle + |B_{out}\rangle|S^+\rangle)$ where $|S^+\rangle = \frac{1}{\sqrt{2}}(|E^*\rangle|F^o\rangle + |E^o\rangle|F^*\rangle)$

Statement of paradox

Suppose we measure the coin later than t_4. Then whether we see interference or not seems to be decided later than the eventual detection of the photon. Does the future influence the past?

Crucial observation

Until we do not measure the coin, we can only say that detector E fired in 25% of the cases and detector F fired in 75% of the cases since the quantum state is

$$\frac{1}{2}|B_{out}\rangle(|E^*\rangle|F^o\rangle + |E^o\rangle|F^*\rangle) + \frac{1}{\sqrt{2}}|B_{in}\rangle|E^o\rangle|F^*\rangle$$

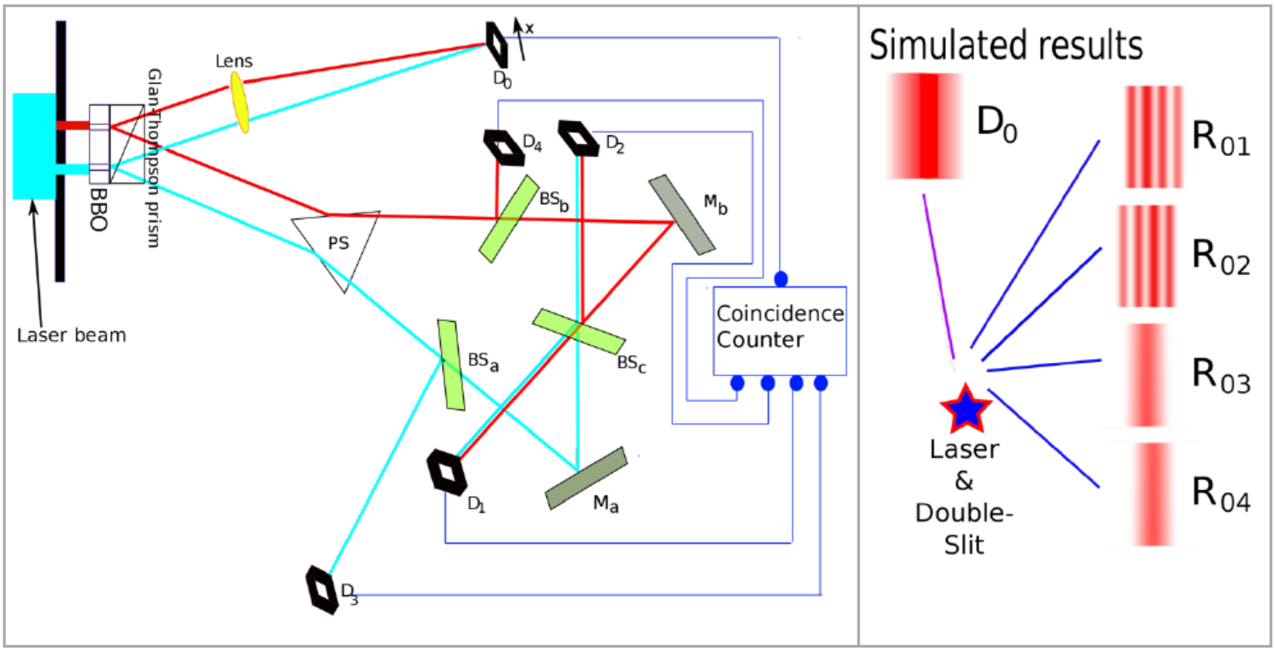
Outcome table and interpretation

Outcome	B_{in}	B_{out}	$E^*\&B_{out}$	$F^*\&B_{out}$	$E^*\&B_{in}$	$F^*\&B_{in}$
Probability	1/2	1/2	1/4	1/4	0	1/2

We can be sure that firing of E corresponds with a certainty to "no interference" i.e. to the result B_{out} for the coin toss. However, for the cases when F fired, we cannot say which one corresponded to interference or not before we measured the coin toss. The outcome of the coin toss then **post-selects** the F^* events: 1/3 of them will be attributed to cases of "no interference", and 2/3 of them are selected to be cases with interference, but we cannot tell which is which before we measured the quantum coin. It does not matter how far in the future the measurement of the coin takes place, the correlation can be interpreted both ways: either as the coin toss influencing the probabilities of firings, or vice versa, as the result of the firings influencing the probability of the outcome of coin toss. The choice depends on the temporal order of the detection of the photon vs. measurement of the coin toss.

This resolves the apparent paradox posed by the famous delayed choice quantum eraser experiment and removes the need for any influence to propagate backwards in time.

Delayed choice quantum eraser



BBO: Barium-borate crystal $Ba(BO_2)_2$ – spontaneous parametric down conversion $\gamma \rightarrow \gamma\gamma$

Upper photon: "signal photon" Red path: upper slit
Lower photon: "idler photon" Blue path: lower slit

Before detecting idler photons, we only have full statistics $\mathrm{D}_0 = \mathrm{R}_{01} + \mathrm{R}_{02} + \mathrm{R}_{03} + \mathrm{R}_{04}$

Detecting idler photon in
D_1 or D_2: no which-path information
- selected subset of signal photons show interference ($\mathrm{R}_{01}/\mathrm{R}_{02}$)

D_3 or D_4: which-path information provided
- selected subset of signal photons show no interference ($\mathrm{R}_{03}/\mathrm{R}_{04}$)

Temporal sequence: what happens is that position of the signal photon in D_0 determines the probabilities of the idler photon to show up in the detectors $\mathrm{D}_{1,2,3,4}$.

No "retro-causal" influence of future on the past occurs at all!

Consistent families for delayed choice

Family 1

$[F^*]_4[3f]_3[B_{in}]_2[1\bar{a}]_1$
$[F^*]_4[3\bar{c}]_3[B_{out}]_2[1\bar{a}]_1$
$[E^*]_4[3\bar{c}]_3[B_{out}]_2[1\bar{a}]_1$

This only allows to draw conclusions about which-path info conditionally. The only unconditional conclusion is that $E^* \rightarrow$ interference and B_{out}.

Family 2

$[F^*]_4[3f]_3[B_{in}]_2[1\bar{a}]_1$
$[F^*]_4[3f]_3[B_{out}]_2[1c]_1$
$[E^*]_4[3e]_3[B_{out}]_2[1d]_1$

We can now infer from E^* that the photon came through arm d, but that can apply only for the cases when we had B_{out} since E^* implies B_{out}. For B_{in} we cannot identify the arms at all, as expected for a case with interference.

Family 3

$[S^+]_4[3\bar{c}]_3[B_{in}]_2[1c]_1$
$[F^*]_4[3f]_3[B_{out}]_2[1c]_1$
$[S^-]_4[3\bar{d}]_3[B_{in}]_2[1d]_1$
$[E^*]_4[3e]_3[B_{out}]_2[1d]_1$

This family cannot be used to construct a paradox, since for the cases with B_{in} the outgoing state of the photon does not correspond to either E or F firing, which is the assumption we start from (that we eventually detected the photon in either E or F).

Conclusion: the paradox cannot even be stated consistently!

Lecture 9

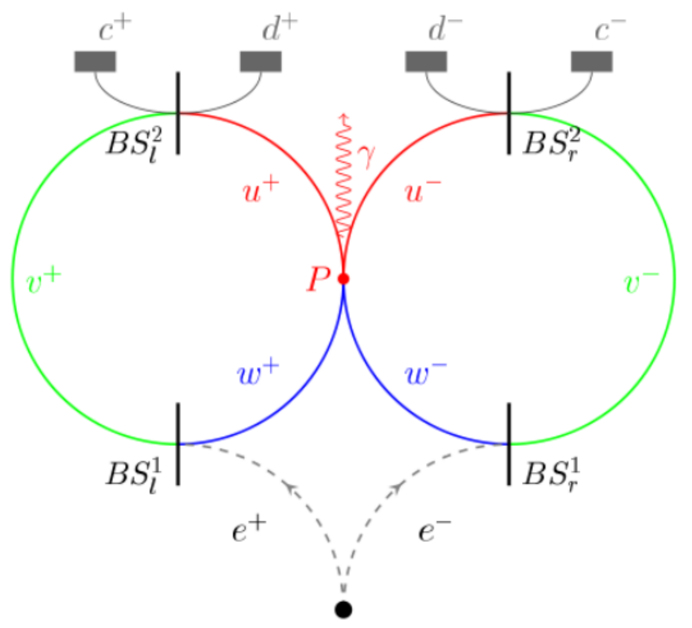
Hardy's paradox: original formulation

Left/right MZ alone:

$$|e^\pm\rangle \to \frac{1}{\sqrt{2}}(|v^\pm\rangle + |w^\pm\rangle)$$
$$|w^\pm\rangle \to |u^\pm\rangle$$
$$|v^\pm\rangle \to \frac{1}{\sqrt{2}}(|c^\pm\rangle + |d^\pm\rangle)$$
$$|u^\pm\rangle \to \frac{1}{\sqrt{2}}(|c^\pm\rangle - |d^\pm\rangle)$$

resulting in

$$|e^\pm\rangle \to \frac{1}{\sqrt{2}}(|v^\pm\rangle + |w^\pm\rangle) \to \frac{1}{\sqrt{2}}(|u^\pm\rangle + |w^\pm\rangle) \to |c^\pm\rangle$$



So, if the interferometers are separated, then $d^\pm$ never fires.

However, when they are connected, we have annihilation in P:

$$|w^+\rangle|w^-\rangle \to |\gamma\rangle|\gamma\rangle$$

The full evolution is then

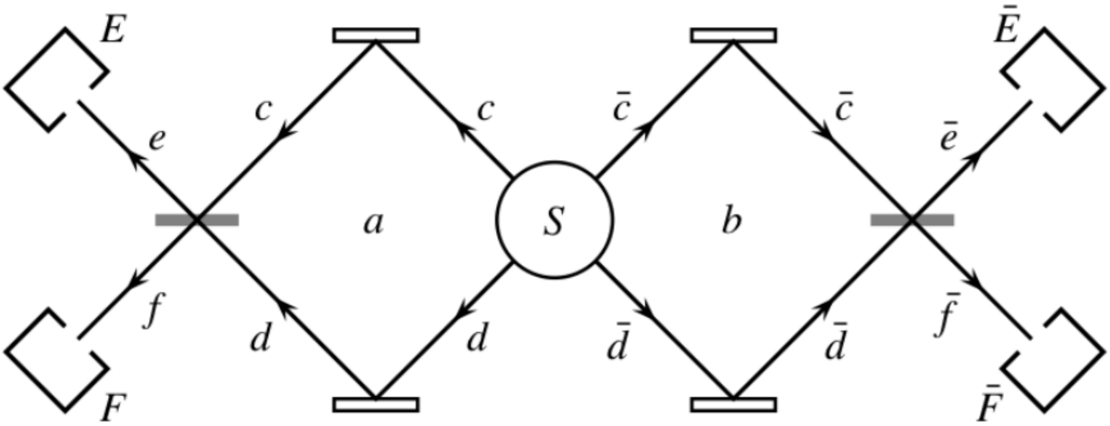
$$\begin{aligned}
&|e^+\rangle|e^-\rangle \\
&\to \frac{1}{2}(|v^+\rangle|v^-\rangle + |v^+\rangle|w^-\rangle + |w^+\rangle|v^-\rangle + |w^+\rangle|w^-\rangle) \\
&\to \frac{1}{2}(|v^+\rangle|v^-\rangle + |v^+\rangle|u^-\rangle + |u^+\rangle|v^-\rangle + |\gamma\rangle|\gamma\rangle) \\
&\to \frac{1}{4}\big((|c^+\rangle + |d^+\rangle)(|c^-\rangle + |d^-\rangle) + (|c^+\rangle + |d^+\rangle)(|c^-\rangle - |d^-\rangle) \\
&\qquad + (|c^+\rangle - |d^+\rangle)(|c^-\rangle + |d^-\rangle) + 2|\gamma\rangle|\gamma\rangle\big) \\
&= \frac{3}{4}|c^+\rangle|c^-\rangle + \frac{1}{4}|c^+\rangle|d^-\rangle + \frac{1}{4}|d^+\rangle|c^-\rangle - \frac{1}{4}|d^+\rangle|d^-\rangle + \frac{1}{2}|\gamma\rangle|\gamma\rangle
\end{aligned}$$

Paradox:

- Naively speaking, $d^\pm$ can only fire if access to $u^\pm$ was blocked by antiparticle arriving via $w^\mp$.
- But it is possible that $d^\pm$ fires simultaneously, with probability 1/16.

- How can this happen when this requires that both the electron and positron take the w-arm of their side, so they should have annihilated, leaving no particle to be detected?
- Indeed, note that the probability of annihilation is 1/4, as expected.

Hardy's paradox: photonic realisation



Assume that the source emits the GHZ (Greenberger-Horne-Zeilinger) state

$$|\Psi_0\rangle = \frac{1}{\sqrt{3}}\left(|c\bar{c}\rangle + |c\bar{d}\rangle + |d\bar{c}\rangle\right)$$

The two beam splitters act as

$$B: |c\rangle \to \frac{1}{\sqrt{2}}(|e\rangle + |f\rangle) \quad |d\rangle \to \frac{1}{\sqrt{2}}(-|e\rangle + |f\rangle)$$
$$\bar{B}: |\bar{c}\rangle \to \frac{1}{\sqrt{2}}(|\bar{e}\rangle + |\bar{f}\rangle) \quad |\bar{d}\rangle \to \frac{1}{\sqrt{2}}(-|\bar{e}\rangle + |\bar{f}\rangle)$$

We can compute the state $|\Psi\rangle$ evolved until before detection

$$B\bar{B}: \frac{1}{\sqrt{12}}\Big(\left(|e\bar{e}\rangle + |f\bar{e}\rangle + |e\bar{f}\rangle + |f\bar{f}\rangle\right) + \left(-|e\bar{e}\rangle - |f\bar{e}\rangle + |e\bar{f}\rangle + |f\bar{f}\rangle\right) + \left(-|e\bar{e}\rangle + |f\bar{e}\rangle - |e\bar{f}\rangle + |f\bar{f}\rangle\right)\Big)$$

$$|\Psi\rangle = \frac{1}{\sqrt{12}}\left(-|e\bar{e}\rangle + |f\bar{e}\rangle + |e\bar{f}\rangle + 3|f\bar{f}\rangle\right)$$

So, we have the following probabilities for detectors firing:

$$E\bar{E}: \frac{1}{12} \quad F\bar{E}: \frac{1}{12} \quad E\bar{F}: \frac{1}{12} \quad F\bar{F}: \frac{3}{4}$$

Paradox

1. If we assume E fired (which has probability 1/6) then we can be sure particle *b* was in arm $\bar{d}$ since

$$|\Psi\rangle = \frac{1}{\sqrt{6}}|e\bar{d}\rangle + \frac{1}{\sqrt{12}}\left(|f\bar{e}\rangle + 3|f\bar{f}\rangle\right)$$

   This is often called a **weak or interaction-free measurement** since by detecting *a* in *E*, we obtain which-path information about *b* without ever interacting with it. Note that it only works in 1/6 of cases - weak measurements are never 100% efficient! An interesting application of weak measurement is the **Elitzur–Vaidman bomb-tester.**

2. If we assume $\bar{E}$ fired (which has probability 1/6) then we can be sure particle *a* was in arm d since

$$|\Psi\rangle = \frac{1}{\sqrt{6}}|d\bar{e}\rangle + \frac{1}{\sqrt{12}}\left(|e\bar{f}\rangle + 3|f\bar{f}\rangle\right)$$

   This is another example of a weak measurement.

3. But then if both E and $\bar{E}$ fired, *a* must have been in arm d AND *b* must have been in arm $\bar{d}$.

4. However, this contradicts the fact that the initial state

$$|\Psi_0\rangle = \frac{1}{\sqrt{3}}\left(|c\bar{c}\rangle + |c\bar{d}\rangle + |d\bar{c}\rangle\right)$$

   has no $|d\bar{d}\rangle$ component!

Analysis of the paradox using consistent histories

We need to use a number of consistent families to describe the paradox.

F_1: $\{[e][\bar{c}], [f][\bar{c}], [e][\bar{d}], [f][\bar{d}]\}$

This family is consistent since the unitary dynamics of the two particles are totally independent (tensor product), so

$\mathrm{Tr}\ \chi_\alpha \rho \chi_\beta^\dagger = \langle\Psi_0|\ \chi_\beta^\dagger \chi_\alpha\ |\Psi_0\rangle = 0$
simply due to $[e][f] = [f][e] = 0$ and $[\bar{c}][\bar{d}] = [\bar{d}][\bar{c}] = 0$

We can then compute that history $[e][\bar{c}]$ has zero probability since

$$[e][\bar{c}]|\Psi_0\rangle = \frac{1}{\sqrt{3}}[e](|c\bar{c}\rangle + |d\bar{c}\rangle) = \sqrt{\frac{2}{3}}[e]|f\bar{c}\rangle = 0$$

giving 1.

Similarly, the family

F_2: $\{[\bar{e}][c], [\bar{f}][c], [\bar{e}][d], [\bar{f}][d]\}$

implies that history $[\bar{e}][c]$ has zero probability, giving 2.

The consistent family

F_3: $\{[e\bar{e}]\boldsymbol{I}, [f\bar{e}]\boldsymbol{I}, [e\bar{f}]\boldsymbol{I}, [f\bar{f}]\boldsymbol{I}\}$

implies the probabilities 1/12, 1/12, 1/12, 3/4 for the four outcomes.

Finally, the family

F_4: $\{[c\bar{c}]\boldsymbol{I}, [d\bar{c}]\boldsymbol{I}, [c\bar{d}]\boldsymbol{I}, [d\bar{d}]\boldsymbol{I}\}$

implies that $[d\bar{d}]$ has zero probability simply due to $[d\bar{d}]|\Psi_0\rangle = 0$.

Now the problem is that the following pairs are incompatible as they have no common refinement:

5. F_1 and F_2

   Any common refinement would include the histories

   $[e][c]$, $[f][c]$, $[e][d]$, $[f][d]$

   and additionally

   $[\bar{e}][\bar{c}]$, $[\bar{f}][\bar{c}]$, $[\bar{e}][\bar{d}]$, $[\bar{f}][\bar{d}]$

   But all these histories are internally inconsistent even individually - they can never be part of any consistent family, since they combine interference and which-path information for the same particle!

Similarly, the following pairs are also incompatible:

6. F_1 and F_3
7. F_1 and F_4
8. F_2 and F_3
9. F_2 and F_4
10. F_3 and F_4

Remark: F_3 and F_4 can be substitued by the consistent family

$$F_5 = \{[e\bar{e}][d\bar{d}], [e\bar{e}](\boldsymbol{I} - [d\bar{d}]), [f\bar{e}][d\bar{d}], [f\bar{e}](\boldsymbol{I} - [d\bar{d}]), [e\bar{f}][d\bar{d}], [e\bar{f}](\boldsymbol{I} - [d\bar{d}]), [f\bar{f}][d\bar{d}], [f\bar{f}](\boldsymbol{I} - [d\bar{d}])\}$$

which implies that the probability of $[e\bar{e}]$ is 1/12, while that of $[d\bar{d}]$ is 0.

This would be enough for the paradox, but: any combination of F_1, F_2 and F_5 is incompatible!

Experimental realisation agrees with quantum mechanics, and confirms incompatibility:

J. S. Lundeen and A. M. Steinberg, Phys. Rev. Lett. **102**: 020404, 2009.

Closing remarks

All these paradoxes play with putting together reasoning in incompatible frameworks. Why does this count?

A step-by-step analysis

We know that experimental outcomes obey Kolmogorov axioms (laboratory notebook argument) -- which is why we can call them "outcomes" (i.e. events) at all!

1. Therefore, in any experimental setup the histories containing only the actual measured alternatives are always consistent.

2. Note that every measurement procedure consists of interactions – the measurement acts are themselves part of the time evolution of the system, and therefore enter the consistency conditions for the histories.

3. In the paradoxes we use the actual measured values to derive conclusions for unmeasured alternatives. This is possible to combine into logical reasonings as far as they can be incorporated into a single framework.

4. But we do not measure these properties! In fact, if we want to measure the properties that are incompatible in the original measurement setup, we must alter the apparatus in a way that changes the outcomes. This is often stated as the maxim "unmeasured properties have no values". In fact, this is not 100% true: they can be assigned values, but one must choose a consistent framework for that, which is again the notorious "contextuality" of quantum theory.

Basic example: the two-slit interference experiment

- When we observe interference, we cannot incorporate the which-path information in the logical framework.
- We can change the setup to get which-path information, but that corresponds to another experimental setup, and the interference disappears.
- So, there could be no logical contradictions, because the apparently contradictory states of affairs constituting the paradox cannot be realised simultaneously, in a fixed experimental setting.

Note that all the above paradoxes essentially were just more elaborate version of interference/which-path complementarity – demonstrating Feynman's statement that the two-slit experiment already contains all the weirdness of quantum theory!!!

1. The interference picture in the two-slit experiment

For two narrow slits, the wave function is the sum of two components from the slits acting as point sources

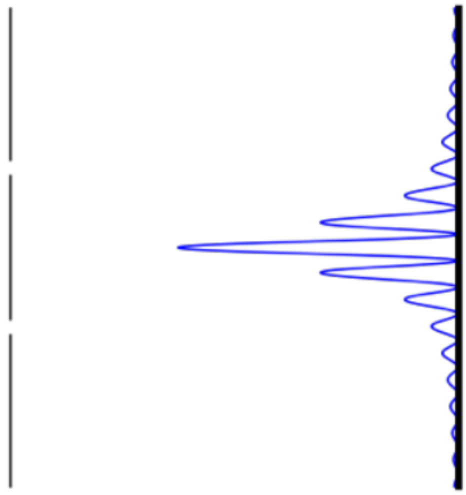
$$\Psi(x) = \frac{1}{\sqrt{2}}\big(\Psi_1(x) + \Psi_2(x)\big)$$

so, the intensity on the screen takes the form

$$|\Psi(x)|^2 = \frac{1}{2}\big(|\Psi_1(x)|^2 + |\Psi_2(x)|^2 + 2\,\mathrm{Re}\,\Psi^*{}_1(x)\Psi_2(x)\big)$$

where the last term is the interference contribution.

In this case there is no consistent history that allows one to speak about which-way information.



2. Detecting which-way information

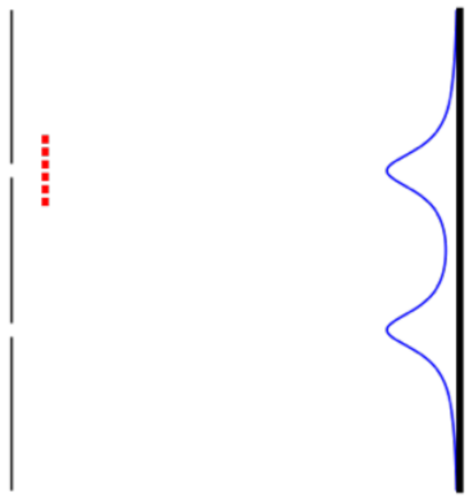
We can, however, decide to detect which-way information by placing a detector behind one of the slits. Now the wave function is

$$\Psi(x) = \frac{1}{\sqrt{2}}(\Psi_1(x)|1\rangle + \Psi_2(x)|0\rangle)$$

where $|0\rangle$ and $|1\rangle$ are the states corresponding to the detector fired or intact. If we assume that it has no further effect on the particle, and that it allows perfect discrimination ($\langle 1|0\rangle = 0$), then the detector states make the two components orthogonal. If the detector state is not read out (corresponding to tracing out the detector degrees of freedom), the intensity becomes

$$\frac{1}{2}(|\Psi_1(x)|^2 + |\Psi_2(x)|^2)$$

We can also decide to read out the detector state, and conditionalize the outcome on it, which results in the intensity distributions $|\Psi_1(x)|^2$ or $|\Psi_2(x)|^2$, depending on whether we select outcomes where the detector did or did not fire, respectively.

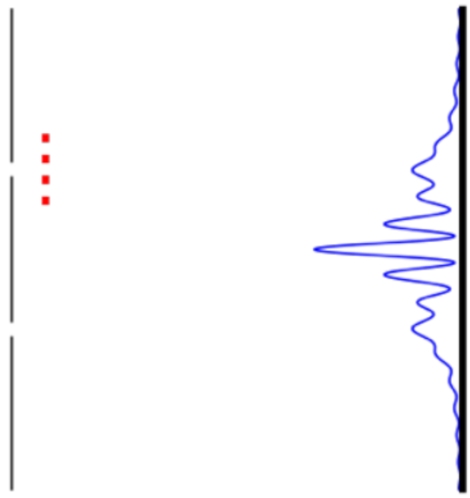


3. Partial which-way information

If the detector is not perfect: $\langle 1|0\rangle = \rho e^{i\phi}$ with $0 < \rho < 1$, then the intensity is given by

$$|\Psi(x)|^2 = \frac{1}{2}\Big(|\Psi_1(x)|^2 + |\Psi_2(x)|^2 + 2\rho\,\mathrm{Re}\,e^{i\phi}\Psi^*{}_1(x)\Psi_2(x)\Big)$$

and the parameter ρ determines the visibility of the interference fringes, while ϕ corresponds to an additional phase gained when the particle passes through slit 1 that results in the shift of the interference pattern. The case shown in the figure corresponds to $\rho = 1/2$ and $\phi = 0$.



The issue of macroscopic superpositions a.k.a. Schrödinger cats

However, one cannot deny that something strange is going on. In the Frauchiger-Renner paradox (see Supplementary Material 1) in order to build a consistent picture, we must admit so called "macroscopic superposition states" such as

$$\frac{1}{\sqrt{2}}(|H\rangle \pm |T\rangle)$$

corresponding to the whole of laboratory 1 being in a superposition of two outcomes for the result of the measurement of the coin, and

$$\frac{1}{\sqrt{2}}(|\Uparrow\rangle \pm |\Downarrow\rangle)$$

corresponding to the whole of laboratory 2 being in a superposition of two outcomes for the result of the measurement of the q-bit. Similarly, for the delayed choice paradox our reasoning involved the macroscopic superposition states

$$|S^{\pm}\rangle$$

These are "Schrödinger cat states", in which whole macroscopic measurement devices are in a superposition of two outcomes. Then the question is:

Why is it that we never see macroscopic bodies in such states, or equivalently, why do measurements (apparently) have definite outcomes?

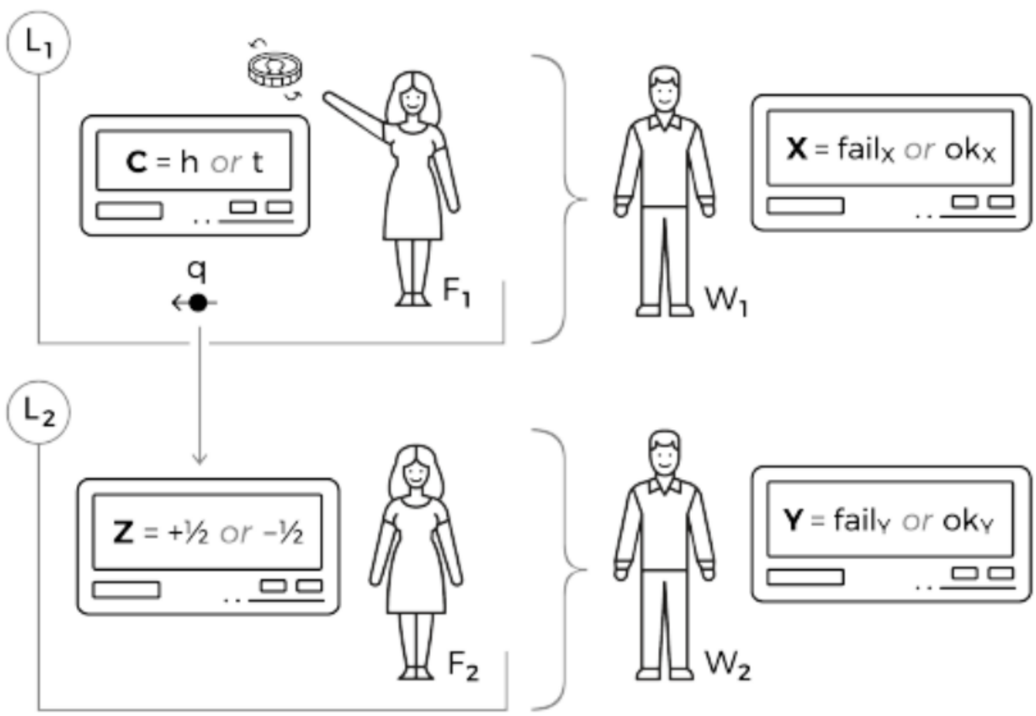
Supplementary Material 1. Frauchiger-Renner paradox

D. Frauchiger and R. Renner, arXiv:1604.07422 [quant-ph]
D. Frauchiger and R. Renner, Nat. Commun. 9, 3711 (2018)

Quantum theory cannot consistently describe the use of itself

Abstract: Quantum theory provides an extremely accurate description of fundamental processes in physics. It thus seems likely that the theory is applicable beyond the, mostly microscopic, domain in which it has been tested experimentally. Here, we propose a Gedankenexperiment to investigate the question whether quantum theory can, in principle, have universal validity. The idea is that, if the answer was yes, it must be possible to employ quantum theory to model complex systems that include agents who are themselves using quantum theory. Analysing the experiment under this presumption, we find that one agent, upon observing a particular measurement outcome, must conclude that another agent has predicted the opposite outcome with certainty. The agents' conclusions, although all derived within quantum theory, are thus inconsistent. This indicates that quantum theory cannot be extrapolated to complex systems, at least not in a straightforward manner.

Setting: Hardy's paradox combined with Wigner's friend



Sequence of operations

1. Toss of a biased quantum coin: head with probability 1/3 and tail with probability 2/3

$$|\phi\rangle = \sqrt{\frac{1}{3}}|\mathrm{h}\rangle + \sqrt{\frac{2}{3}}|t\rangle$$

2. F_1 prepares a qubit in the state $|\downarrow\rangle$ if the coin is measured in the state $|\mathrm{h}\rangle$ and $|\rightarrow\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ if $|t\rangle$: this means that the outside observer W1 describes the state of the laboratory L_1 as

$$\sqrt{\frac{1}{3}}|\mathrm{h}\rangle|\downarrow\rangle + \sqrt{\frac{1}{3}}|t\rangle(|\uparrow\rangle + |\downarrow\rangle)$$

3. F_1 sends the qubit to F_2, who measures its observable S^z. As a result, the system of laboratories L_1 and L_2 get into the state

$$|\Psi\rangle = \sqrt{\frac{1}{3}}|H\rangle|\Downarrow\rangle + \sqrt{\frac{1}{3}}|T\rangle(|\Uparrow\rangle + |\Downarrow\rangle) = \sqrt{\frac{1}{3}}|H\rangle|\Downarrow\rangle + \sqrt{\frac{2}{3}}|T\rangle|\Rightarrow\rangle$$

where

- $|H\rangle$ and $|T\rangle$ are eigenstates of an observable *A*, which W_1 can measure on laboratory L_1 e.g., by simply asking F_1 about the outcome and the result is *H/T* if the reply is *h/t*;
- $|\Uparrow\rangle$ and $|\Downarrow\rangle$ are eigenstates of an observable *B*, which W_2 can measure on laboratory L_2 e.g., by simply asking F_2 about the outcome and the result is $\Uparrow/\Downarrow$ if the reply is $\uparrow/\downarrow$;
- $|\Rightarrow\rangle = (|\Uparrow\rangle + |\Downarrow\rangle)/\sqrt{2}$

However, these are not the observables W_1 and W_2 choose to measure!

- W_1 measures an observable *X* which has the eigenvectors
  $|fail_X\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |T\rangle) \quad |ok_X\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |T\rangle)$
- W_2 measures an observable *Y* which has the eigenvectors
  $|fail_Y\rangle = \frac{1}{\sqrt{2}}(|\Uparrow\rangle + |\Downarrow\rangle) \quad |ok_Y\rangle = \frac{1}{\sqrt{2}}(|\Uparrow\rangle - |\Downarrow\rangle)$

In terms of these states, it is easy to rewrite

$$|\Psi\rangle = \frac{1}{\sqrt{12}}|ok_X\rangle|ok_Y\rangle - \frac{1}{\sqrt{12}}|ok_X\rangle|fail_Y\rangle + \frac{1}{\sqrt{12}}|fail_X\rangle|ok_Y\rangle + \sqrt{\frac{3}{4}}|fail_X\rangle|fail_Y\rangle$$

This is the Hardy's paradox state! So, note that all this setup just prepared a Hardy's paradox.

Now they consider three assumptions:

(Q) Quantum theory applies universally to systems of any complexity, including labs and observers. Moreover, if an agent knows that a given proposition is true whenever Born's rule assigns probability 1 to it.

(C) Self-consistency i.e., different agent's predictions are not contradictory.

(S) Single world: from the point of view of an agent who carries out a particular measurement, this measurement has one single outcome.

Statement of the paradox

(Q), (C) and (S) cannot be simultaneously valid so quantum theory cannot be self-consistent when applied to describe the use of itself.

Argument

Consider the probability that F_2 obtains ↓ in her S^z measurement while W_1 gets $|ok_X\rangle$ in her X measurement. Rewriting

$$|\Psi\rangle = -\frac{1}{\sqrt{6}}|ok_X\rangle|\Uparrow\rangle + \frac{1}{\sqrt{6}}|fail_X\rangle|\Uparrow\rangle + \sqrt{\frac{2}{3}}|fail_X\rangle|\Downarrow\rangle$$

this has zero possibility. So we conclude that if W_1 gets $|ok_X\rangle$ she can infer with certainty that F_2 obtained ↑.

Similarly, using

$$|\Psi\rangle = \sqrt{\frac{1}{3}}|H\rangle|\Downarrow\rangle + \sqrt{\frac{1}{3}}|T\rangle(|\Uparrow\rangle + |\Downarrow\rangle)$$

So, if F_2 gets $|\uparrow\rangle$ she can infer with certaintly that F_1 obtained t in her measurement of the coin. Now if F_1 obtained t she can infer that W_2 gets $|fail_Y\rangle$.
Putting together these steps we see that if W_1 obtains $|ok_X\rangle$ she can be certain that W_2 gets $|fail_Y\rangle$, but this is a contradiction since it is possible to get the outcome $|ok_X\rangle|ok_Y\rangle$ with probability 1/12.

Where is the catch? The problem is again that to construct the paradox it is necessary to put together inconsistent frameworks. But then the logical deduction cannot be completed and so the conclusion does not apply.

Interpretation using quantum histories

M. Losada, R. Laura and O. Lombardi, Phys. Rev. A100: 052114, 2019. arXiv:1907.10095

The relevant Hilbert space is very complicated: it is tensor product of the space of the coin, the qubit, and the Hilbert spaces for L_1, L_2, W_1 and W_2. However, at each time the manipulations only happen in a single factor, so we omit putting in the identity operators for the rest.

Temporal sequence

1. Initial state is prepared by time t_0:

   $$|\Psi_0\rangle = |\phi\rangle|q_0\rangle\,|l_{10}\rangle|l_{20}\rangle|w_{10}\rangle|w_{20}\rangle$$

The last four factors are some starting ('ready') states for the labs and for the outside observers.

2. Time interval (t_0, t_1): F_1 measures the coin which corresponds to a time evolution

   $U_{10}(|h\rangle|l_{10}\rangle) = |h\rangle|l_{1h}\rangle = |H\rangle \quad U_{10}(|t\rangle|l_{10}\rangle) = |t\rangle|l_{1t}\rangle = |T\rangle$

3. Time interval (t_1, t_2): F_1 prepares the q-bit

   $U_{21}(|l_{1\mathrm{h}}\rangle|q_0\rangle) = |l_{1\mathrm{h}}\rangle|\downarrow\rangle \quad U_{21}(|l_{1t}\rangle|q_0\rangle) = |l_{2t}\rangle|\rightarrow\rangle$

4. Time interval (t_2, t_3): F_2 observes the q-bit

   $U_{32}(|l_{20}\rangle|\downarrow\rangle) = |l_{2\downarrow}\rangle|\downarrow\rangle \quad U_{32}(|l_{20}\rangle|\uparrow\rangle) = |l_{2\uparrow}\rangle|\uparrow\rangle$

5. Time interval (t_3, t_4): W_1 measures laboratory L_1

   $U_{43}(|w_{10}\rangle|fail_X\rangle) = |w_{1fail}\rangle|fail_X\rangle \quad U_{43}(|w_{10}\rangle|ok_X\rangle) = |w_{1ok}\rangle|ok_X\rangle$

6. Time interval (t_4, t_5): W_2 measures laboratory L_2

   $U_{54}(|w_{20}\rangle|fail_Y\rangle) = |w_{2fail}\rangle|fail_Y\rangle \quad U_{54}(|w_{20}\rangle|ok_Y\rangle) = |w_{2ok}\rangle|ok_Y\rangle$

Note: 5 can happen any time after t_2 and 6 can happen any time after t_3.

Description with histories

Family 1: used to deduce that $|w_{1ok}\rangle$ implies $|l_{2\uparrow}\rangle$. This needs to contain the following history:

$\boldsymbol{I}[w_{1ok}][l_{2\downarrow}]\boldsymbol{I}[\Psi_0]$

which has zero probability as computed above.

Family 2: used to deduce that $|l_{2\uparrow}\rangle$ implies $|l_{1t}\rangle$. This needs to contain the following history:

$\boldsymbol{II}[l_{2\uparrow}][l_{1\mathrm{h}}][\Psi_0]$

which has zero probability as computed above.

Family 3: used to deduce that $|l_{1t}\rangle$ implies $|w_{2fail}\rangle$. This needs to contain the following history:

$[w_{2ok}]\boldsymbol{II}[l_{1t}][\Psi_0]$

which has zero probability as computed above.

However, these histories are all mutually incompatible!

Families 1&2: the projector $[w_{1ok}]$ when written out actually contains a projection on

$$\frac{1}{\sqrt{2}}(|H\rangle - |T\rangle)$$

which is incompatible with $[l_{1h}]$ which contains a projection on $|H\rangle$. This is the same incompatibility as in the basic two-slit experiment: interference is incompatible with which-path information.

Families 2&3: the projector $[l_{2\uparrow}]$ when written out actually contains a projecton on $|\Uparrow\rangle$, but this is incompatible with $[w_{2ok}]$ which contains a projection on

$$\frac{1}{\sqrt{2}}(|\Uparrow\rangle - |\Downarrow\rangle)$$

Families 3&1: the projector $[w_{2ok}]$ when written out actually contains a projection on

$$\frac{1}{\sqrt{2}}(|\Uparrow\rangle - |\Downarrow\rangle)$$

which is incompatible with $[l_{2\downarrow}]$ which contains a projection on $|\Downarrow\rangle$.

So, the rules of quantum mechanics forbid all but one step of the argument. One can find a framework for one of the steps, but then it is impossible to deduce the other two implications.

Some prevalent misunderstandings

- Originally (1^{st} version of the manuscript) the authors claimed that the paradox can only be resolved by using the Many-World Interpretation i.e. dropping (S). Since we showed that consistent histories resolve the paradox, it is effectively resolved by any consistent interpretation of quantum theory.

- By the same token, the argument shows no inconsistency whatsoever with quantum theory. It just shows like all similar arguments do that quantum theory is incompatible with naively applied (classical) realism.

(My) conclusion

Question: is it true that "quantum theory cannot describe the use of itself"?

Answer: no, quantum theory is consistent. Instead, it is the argument leading to the paradox which fails in applying quantum theory consistently!

Quantum theory simply does not permit the reasoning described by the authors. The problem is that the conjunction of the three assumptions (Q, C, S) as construed by the authors contradicts quantum theory, but that does not mean that quantum theory is not consistent or that it "cannot describe the use of itself".

However, just as the other paradoxes, this one is also useful in elucidating the counter-intuitive features of quantum theory.

Lecture 10

Three related problems

The problem of outcomes a.k.a. the measurement problem

Assume we have a system S for which an observable O is measured by a device D

Eigenstates of O: $O|\phi_i\rangle = \lambda_i|\phi_i\rangle \; i = 1,2,\ldots$

State of system: $|\Psi\rangle$

States of device: $|\psi_0\rangle$ starting state, $|\psi_i\rangle$ outcome states

Measurement: time evolution

$|\phi_i\rangle|\psi_0\rangle \to |\phi_i\rangle|\psi_i\rangle$

Assuming

$$|\Psi\rangle = \sum_i C_i\,|\phi_i\rangle \to |\phi_i\rangle|\Psi\rangle \to \sum_i C_i\,|\phi_i\rangle|\psi_i\rangle$$

The end state is a superposition - no definite outcome.

Observer (e.g. Wigner) comes and looks at device.

States of observer: $|\chi_0\rangle$ starting state, $|\chi_i\rangle$ state of having observed outcome i

$$\left(\sum_i C_i\,|\phi_i\rangle|\psi_i\rangle\right)|\chi_0\rangle \to \sum_i C_i\,|\phi_i\rangle|\psi_i\rangle|\chi_i\rangle$$

Friend comes and asks Wigner about outcome

States of friend: $|\xi_0\rangle$ starting state, $|\xi_i\rangle$ state of having been told outcome i

$$\left(\sum_i C_i\,|\phi_i\rangle|\psi_i\rangle|\chi_i\rangle\right)|\xi_0\rangle \to \sum_i C_i\,|\phi_i\rangle|\psi_i\rangle|\chi_i\rangle|\xi_i\rangle$$

Note that there is no point at which a definite outcome is selected. The system gets entangled with more and more degrees of freedom, but there is no point where a selection of outcomes happens. This is a feature of unitary time evolution (U), which is also reversible.

One can postulate that there is some point where another sort of process happens: (R) (reduction). It is usually thought to happen right after the entanglement with the device i.e.

$$\sum_i C_i\,|\phi_i\rangle|\psi_i\rangle \to \sum_i |C_i|^2\,|\phi_i\rangle|\psi_i\rangle\langle\phi_i|\langle\psi_i|$$

This is a transition from a pure state (the superposition) to a mixed state which can be interpreted as definite outcomes with probabilities $|C_i|^2$. Written fully in terms of density matrices

$$\text{(R)}: \sum_i C_i\, C_j^* |\phi_i\rangle|\psi_i\rangle\langle\phi_j|\langle\psi_j| \to \sum_i |C_i|^2\, |\phi_i\rangle\langle\phi_i||\psi_i\rangle\langle\psi_i|$$

(R) is not unitary and it is also irreversible (phase information is lost).

Possibilities:

1. (R) is a physical process. Quantum theory is not universally valid: there are some processes that collapse the wave functions of macroscopic objects.

   ⇒ collapse theories (spontaneous/gravitational etc.)

2. (R) is "in the eye of the beholder": the wave function is not reality, just a tool to predict probabilities. The wave function represents the observer's knowledge about the state of the system. (R) is the prescription for updating the wave function when a given measurement was performed and the observer gained knowledge about the system.

   ⇒ e.g., quantum Bayesianism (qubism).

3. (R) is not a real physical process. The wave function is ontological - it describes reality. There is some process during the measurement that leads to apparent collapse.

   ⇒ Everett type interpretations (e.g., "many-world").

(Note that I ignored lots of subtle distinctions above and only mentioned few interpretations. Many interpretations do not fit neatly in the above scheme. Nevertheless, the above options illustrate the sort of choice one faces here.

Preferred basis/framework problem

It is easy to state something like "there are no macroscopic superpositions". However, any quantum state is a superposition in most of the possible basis sets.

Which is the preferred set of basis vectors in which the "no macroscopic superposition" principle should be applied?

Closely related problems:

- Given a device D, which are the states $|\psi_i\rangle$ corresponding to the outcomes, in which the reduction (R) happens? Clearly these must be states whose superposition is not stable.

- In the consistent histories approach, which is the framework in which we can interpret the history of the Universe, i.e. in which we can draw the conclusions about the past that we all agree upon? (E.g., that given our current historical records, we can infer that Napoleon lost the battle of Waterloo).

This is a real problem since it can be shown that choosing different frameworks one can arrive at contradictory inferences about the past.
(c.f. A. Kent, Phys. Rev. Lett. 78 (1997) 2874–2877.)

This can be remedied by considering stronger consistency conditions, e.g. by replacing

$\mathrm{Re}\,\mathrm{Tr}\,\chi_\alpha\rho\chi_\beta{}^\dagger = 0$ whenever $\alpha \neq \beta$ **with** $\mathrm{Tr}\,\chi_\alpha\rho\chi_\beta{}^\dagger = 0\ \forall\,\rho$ and whenever $\alpha \neq \beta$

(global consistency, c.f. M. Losada, Physica A 503 (2018) 379–389).

However: what motivates such a replacement apart from the mere wish that the Universe had a unique past?

What we need is that at least the "macroscopic past" of the Universe can be uniquely "retrodicted". Is it possible that this can be obtained some way from quantum theory?

- **Emergence of classical behaviour**

  Why do macroscopic objects behave classically and are never seen in superposition - especially in superpositions of different locations in space?

  Leaving aside the option that quantum theory is not valid for macroscopic objects (objective collapse theories), is there a way this can be understood from quantum theory?

Environmental decoherence

Measurement outcomes

Assume we have a system S for which an observable O is measured by a device D.

System interacting with the apparatus

$$H_{int} = \sum_n |n\rangle\langle n| \otimes A_n$$

$|n\rangle$: measured states of S discriminated by D A_n: operators acting on states of the apparatus D

$$|n\rangle|\Phi_0\rangle \to e^{-itH_{int}}|n\rangle|\Phi_0\rangle = |n\rangle e^{-itA_n}|\Phi_0\rangle = |n\rangle|\Phi_n(t)\rangle$$

$$\left(\sum_n C_n\,|n\rangle\right)|\Phi_0\rangle \to \sum_n C_n\,|n\rangle|\Phi_n(t)\rangle$$

$$\rho = \left(\sum_{n,m} C_n C_m^*\ |n\rangle\langle m|\right)|\Phi_0\rangle\langle\Phi_0| \to \sum_{n,m} C_n C_m^*\ (|n\rangle\langle m|)(|\Phi_n(t)\rangle\langle\Phi_m(t)|)$$

The state of the system is

$$\rho_S = \mathrm{Tr}_D \rho = \sum_{n,m} C_n C_m^* \ (|n\rangle\langle m|)\langle \Phi_m(t)|\Phi_n(t)\rangle$$

The apparatus discriminates the "target" states if after some time t_0

$$\langle \Phi_m(t)|\Phi_n(t)\rangle \approx \delta_{mn} \ \ t > t_0 \qquad |\Phi_n(t)\rangle\text{: pointer states}$$

Then we have

$$\rho_S \approx \sum_n |C_n|^2 \ |n\rangle\langle n| \ \ !!!$$

Remarks:

- Improper mixture: system + detector still in superposition.
- Leads to apparent collapse.
- Nevertheless, if the detector state is not observed in detail, then we cannot distinguish between apparent and real collapse - this is sufficient FAPP (for all practical purposes).
- We shall see that t_0 is an extremely short time scale whenever D is macroscopic.

Mechanism:

- Provided by entanglement: system is entangled with detector, but this entanglement quickly delocalises between the many degrees of freedom of D.
- Irreversibility - related to 2nd law of thermodynamics.

Localisation by scattering

Consider some macroscopic object, with position eigenstates $|\boldsymbol{x}\rangle$, subject to a random environment of particles (air molecules, photons). Let us consider the scattering of a single particle with initial state and assume that the recoil of the macroscopic object can be neglected:

$$|\boldsymbol{x}\rangle|\chi\rangle \rightarrow |\boldsymbol{x}\rangle S_{\boldsymbol{x}}|\chi\rangle \quad S_{\boldsymbol{x}}\text{: scattering operator} \quad S_{\boldsymbol{x}} S_{\boldsymbol{x}}^\dagger = S_{\boldsymbol{x}}^\dagger S_{\boldsymbol{x}} = \mathbf{1}$$

For an object in a general delocalised state

$$\int d^3\boldsymbol{x}\, \psi(\boldsymbol{x})|\boldsymbol{x}\rangle|\chi\rangle \rightarrow \int d^3\boldsymbol{x}\, \psi(\boldsymbol{x})|\boldsymbol{x}\rangle S_{\boldsymbol{x}}|\chi\rangle$$

Density matrix of the object after the scattering in position basis:

$$\rho = \int d^3\boldsymbol{x} \int d^3\boldsymbol{x}'\, \rho(\boldsymbol{x},\boldsymbol{x}')|\boldsymbol{x}\rangle\langle \boldsymbol{x}'|$$

$$\rho(\boldsymbol{x},\boldsymbol{x}') = \psi(\boldsymbol{x})\psi^*(\boldsymbol{x}')\langle\chi|S_{\boldsymbol{x}'}^\dagger S_{\boldsymbol{x}}|\chi\rangle$$

Translationally invariant interaction: position dependence of scattering matrix in momentum space can be written as

$$\langle \boldsymbol{k}'|S_{\boldsymbol{x}}|\boldsymbol{k}\rangle = S_{\boldsymbol{x}}(\boldsymbol{k},\boldsymbol{k}') = S(\boldsymbol{k},\boldsymbol{k}')e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}}$$

Writing

$$|\chi\rangle = \int d^3\boldsymbol{k}\, c(\boldsymbol{k})|\boldsymbol{k}\rangle$$

we can compute

$$\begin{aligned}\langle\chi|S^\dagger_{\boldsymbol{x}'}S_{\boldsymbol{x}}|\chi\rangle &= \int d^3\boldsymbol{k}\,\langle\chi|S^\dagger_{\boldsymbol{x}'}|\boldsymbol{k}\rangle\langle\boldsymbol{k}|S_{\boldsymbol{x}}|\chi\rangle \\ &= \int d^3\boldsymbol{k}''d^3\boldsymbol{k}'d^3\boldsymbol{k}\, c^*(\boldsymbol{k}'')c(\boldsymbol{k}')\langle\boldsymbol{k}''|S^\dagger_{\boldsymbol{x}'}|\boldsymbol{k}\rangle\langle\boldsymbol{k}|S_{\boldsymbol{x}}|\boldsymbol{k}'\rangle \\ &= \int d^3\boldsymbol{k}''d^3\boldsymbol{k}'d^3\boldsymbol{k}\, c^*(\boldsymbol{k}'')c(\boldsymbol{k}')S^*(\boldsymbol{k}'',\boldsymbol{k})e^{+i(\boldsymbol{k}''-\boldsymbol{k})\cdot\boldsymbol{x}'}S(\boldsymbol{k}',\boldsymbol{k})e^{-i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{x}}\end{aligned}$$

From QM scattering theory we know that

$$S(\boldsymbol{k}',\boldsymbol{k}) = \delta^{(3)}(\boldsymbol{k}'-\boldsymbol{k}) + \frac{i}{2\pi k}f(\boldsymbol{k}',\boldsymbol{k})\delta(k'-k) \quad f(\boldsymbol{k}_0,\boldsymbol{k})\text{: scattering amplitude}$$

Ineffective single scattering: typical wavelength of particle is much longer than $|\boldsymbol{x}-\boldsymbol{x}'|$ i.e. $k_0|\boldsymbol{x}-\boldsymbol{x}'| \ll 1$. In a cube of length L we can then replace the wave function of the particle by a plane wave

$$\frac{1}{L^{3/2}}e^{i\boldsymbol{k}_0\cdot\boldsymbol{x}} \Rightarrow c(\boldsymbol{k}) \simeq \left(\frac{1}{L}\right)^{3/2}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}_0)$$

$$\begin{aligned}\langle\chi|\chi\rangle &= \int d^3\boldsymbol{k}\int d^3\boldsymbol{k}'\, c(\boldsymbol{k})^*c(\boldsymbol{k}')\langle\boldsymbol{k}|\boldsymbol{k}'\rangle = \int d^3\boldsymbol{k}\int d^3\boldsymbol{k}'\, c(\boldsymbol{k})^*c(\boldsymbol{k}')(2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}') \\ &= \int d^3\boldsymbol{k}\,(2\pi)^3|c(\boldsymbol{k})|^2 = 1\end{aligned}$$

since $\int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2 = \left(\frac{1}{L}\right)^3\int d^3\boldsymbol{k}\,\delta^{(3)}(\boldsymbol{0})\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}_0) = \frac{1}{(2\pi)^3}$

$$\begin{aligned}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}_0) = \int\frac{d^3\boldsymbol{x}}{(2\pi)^3}e^{i(\boldsymbol{k}-\boldsymbol{k}_0)\cdot\boldsymbol{x}} &\Rightarrow \left(\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}_0)\right)^2 = \delta^{(3)}(\boldsymbol{0})\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}_0) \\ &= \left(\frac{L}{2\pi}\right)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}_0)\end{aligned}$$

Also note that

$$d^3\boldsymbol{k} = k^2dkd\Omega \Rightarrow \delta^{(3)}(\boldsymbol{k}'-\boldsymbol{k}) = \frac{1}{k^2}\delta(k'-k)\delta(\Omega'-\Omega)$$

Let's recall how the cross-section is defined. First, the energy of a particle is

$$E = +\sqrt{k^2 + m^2} \;\Rightarrow \frac{dE}{dk} = \frac{k}{E} = v$$

$$\delta(k' - k) = v\delta(E' - E) \Rightarrow \; \delta(k' - k)^2 = v^2 \frac{T}{2\pi}\delta(E' - E) = v\frac{T}{2\pi}\delta(k' - k)$$

where T is the time-window of the single particle scattering process. We assume that the particles are dilute, so one scattering is completely finished by the time the next particle arrives.

Assume that (θ, ϕ) give the angular direction of $\boldsymbol{k}'$ relative to $\boldsymbol{k}$, and also for simplicity that the scattering interaction is isotropic. The probability of scattering is then

$$dP = \frac{1}{4\pi^2 k^2}|f(\theta)|^2 v \frac{T}{2\pi}\delta(k' - k)$$

The integration over the final state is

$$\int d^3\boldsymbol{k}'\,(2\pi)^3|c(\boldsymbol{k}')|^2 = L^{-3}(2\pi)^3\int k'^2 dk' d\Omega$$

We can do the integral over dk' to get the probability per solid angle

$$dp = |f(\theta)|^2 d\Omega\frac{vT}{L^3} \qquad \text{where } \frac{vT}{L^3} \text{ is the flux factor}$$

Dividing by the flux factor finally gives the well-known expression

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

Now we consider

$$\begin{aligned}
\langle\chi|S^\dagger_{\boldsymbol{x}'}S_{\boldsymbol{x}}|\chi\rangle = &\int d^3\boldsymbol{k}''d^3\boldsymbol{k}'d^3\boldsymbol{k}\,c^*(\boldsymbol{k}'')c(\boldsymbol{k}')e^{+i(\boldsymbol{k}''-\boldsymbol{k})\cdot\boldsymbol{x}'}e^{-i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{x}} \\
&\left(\delta^{(3)}(\boldsymbol{k}' - \boldsymbol{k}) + \frac{i}{2\pi k}f(\boldsymbol{k}',\boldsymbol{k})\delta(k' - k)\right)\left(\delta^{(3)}(\boldsymbol{k}'' - \boldsymbol{k})\right. \\
&\left.- \frac{i}{2\pi k}f^*(\boldsymbol{k}'',\boldsymbol{k})\delta(k'' - k)\right) \\
= &\int d^3\boldsymbol{k}\,|c(\boldsymbol{k})|^2 + \int d^3\boldsymbol{k}'d^3\boldsymbol{k}\,c^*(\boldsymbol{k})c(\boldsymbol{k}')e^{-i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{x}}\frac{i}{2\pi k}f(\boldsymbol{k}',\boldsymbol{k})\delta(k' - k) \\
&- \int d^3\boldsymbol{k}''d^3\boldsymbol{k}\,c^*(\boldsymbol{k}'')c(\boldsymbol{k})e^{+i(\boldsymbol{k}''-\boldsymbol{k})\cdot\boldsymbol{x}'}\frac{i}{2\pi k}f^*(\boldsymbol{k}'',\boldsymbol{k})\delta(k'' - k) \\
&+ \int d^3\boldsymbol{k}''d^3\boldsymbol{k}'d^3\boldsymbol{k}\,c^*(\boldsymbol{k}'')c(\boldsymbol{k}')e^{+i(\boldsymbol{k}''-\boldsymbol{k})\cdot\boldsymbol{x}'}e^{-i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{x}} \\
&\qquad\times\frac{1}{4\pi^2k^2}f(\boldsymbol{k}',\boldsymbol{k})\delta(k' - k)f^*(\boldsymbol{k}'',\boldsymbol{k})\delta(k'' - k)
\end{aligned}$$

Note that due to the unitarity of scattering we have

$$\langle\chi|S_{\boldsymbol{x}}^{\dagger}S_{\boldsymbol{x}}|\chi\rangle = \langle\chi|S_{\boldsymbol{x}'}^{\dagger}S_{\boldsymbol{x}'}|\chi\rangle = 1$$

so

$$\langle\chi|S_{\boldsymbol{x}'}^{\dagger}S_{\boldsymbol{x}}|\chi\rangle = 1 + \int d^3\boldsymbol{k}''d^3\boldsymbol{k}'d^3\boldsymbol{k}\, c^*(\boldsymbol{k}'')c(\boldsymbol{k}')\left(e^{+i(\boldsymbol{k}''-\boldsymbol{k})\cdot\boldsymbol{x}'}e^{-i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{x}} - (\boldsymbol{x}=\boldsymbol{x}' \text{ term})\right)$$
$$\times\frac{1}{4\pi^2k^2}f(\boldsymbol{k}',\boldsymbol{k})\delta(k'-k)f^*(\boldsymbol{k}'',\boldsymbol{k})\delta(k''-k)$$
$$= 1 + \int d^3\boldsymbol{k}\frac{1}{4\pi^2k^2L^3}\left(e^{+i(\boldsymbol{k_0}-\boldsymbol{k})\cdot(\boldsymbol{x}'-\boldsymbol{x})} - 1\right)|f(\boldsymbol{k_0},\boldsymbol{k})|^2\left(\delta(k_0-k)\right)^2$$
$$= 1 + \int d^3\boldsymbol{k}\frac{1}{4\pi^2k^2L^3}\left(e^{+i(\boldsymbol{k_0}-\boldsymbol{k})\cdot(\boldsymbol{x}'-\boldsymbol{x})} - 1\right)|f(\boldsymbol{k_0},\boldsymbol{k})|^2 v_0\frac{T}{2\pi}\delta(k_0-k)$$

where v_0 is the velocity of the incoming particle. We can now expand the exponential as

$$\left(e^{+i(\boldsymbol{k_0}-\boldsymbol{k})\cdot(\boldsymbol{x}'-\boldsymbol{x})} - 1\right) = i(\boldsymbol{k_0}-\boldsymbol{k})\cdot(\boldsymbol{x}'-\boldsymbol{x}) - \frac{1}{2}\left((\boldsymbol{k_0}-\boldsymbol{k})\cdot(\boldsymbol{x}'-\boldsymbol{x})\right)^2$$

Assuming that the particles have random direction $\widehat{\boldsymbol{k}}_{\boldsymbol{0}}$, and taking into account that $f(\boldsymbol{k_0},\boldsymbol{k})$ depends only on the relative angle θ between $\boldsymbol{k_0}$ and $\boldsymbol{k}$, averaging over $\widehat{\boldsymbol{k}}_{\boldsymbol{0}}$ in the first term gives zero.

Therefore, we are left with

$$\overline{\langle\chi|S_{\boldsymbol{x}'}^{\dagger}S_{\boldsymbol{x}}|\chi\rangle} = 1 - \frac{1}{2}\int d\Omega_{\boldsymbol{k}}\frac{k_0^2}{4\pi^2L^3}\overline{\left((\widehat{\boldsymbol{k}}_{\boldsymbol{0}}-\widehat{\boldsymbol{k}})\cdot(\boldsymbol{x}'-\boldsymbol{x})\right)^2|f(\boldsymbol{k_0},\boldsymbol{k})|^2}v_0\frac{T}{2\pi}$$

Now note that the integral

$$\int d\Omega_{\boldsymbol{k}}\left((\widehat{\boldsymbol{k}}_{\boldsymbol{0}}-\widehat{\boldsymbol{k}})\cdot(\boldsymbol{x}'-\boldsymbol{x})\right)^2|f(\boldsymbol{k_0},\boldsymbol{k})|^2$$

when averaged over the random directions of the incoming particles, cannot depend on the direction of $\boldsymbol{x}-\boldsymbol{x}'$. So, finally we get

$$\langle\chi|S_{\boldsymbol{x}'}^{\dagger}S_{\boldsymbol{x}}|\chi\rangle = 1 - \frac{k_0^2|\boldsymbol{x}-\boldsymbol{x}'|^2}{8\pi^2}\frac{v_0T}{L^3}\sigma_{eff}$$

where

$$\sigma_{\text{eff}} = \frac{1}{2\pi}\int d\Omega_{\boldsymbol{k}}\overline{\left(\widehat{\boldsymbol{k}}_{\boldsymbol{0}}-\widehat{\boldsymbol{k}}\right)^2(\cos\alpha-\cos\alpha')^2|f(\boldsymbol{k_0},\boldsymbol{k})|^2}$$

α,α': angle of $\boldsymbol{x}$ resp. $\boldsymbol{x}'$ to $\widehat{\boldsymbol{k}}_{\boldsymbol{0}}-\widehat{\boldsymbol{k}}$. Note that while σ_{eff} looks complicated, in practice its magnitude can be estimated by the total cross-section of the scattering.

For a full evaluation cf. E. Joos and H.D. Zeh, Zeitschrift für Physik B59, 223-243 (1985)

Therefore, the scattering of a single particle changes the density matrix as

$$\rho(\boldsymbol{x},\boldsymbol{x}') \to \rho(\boldsymbol{x},\boldsymbol{x}')\langle\chi|S_{x'}^\dagger S_x|\chi\rangle \simeq \rho(\boldsymbol{x},\boldsymbol{x}')\exp\left(-\frac{k_0^2|\boldsymbol{x}-\boldsymbol{x}'|^2}{8\pi^2}\frac{v_0 T}{L^3}\sigma_{eff}\right)$$

If the particles have a density $n = N/V$, then after the passage of a time t we have

$$\rho(\boldsymbol{x},\boldsymbol{x}') \to \rho(\boldsymbol{x},\boldsymbol{x}')\exp(-\Lambda t|\boldsymbol{x}-\boldsymbol{x}'|^2) \qquad \Lambda = \frac{k_0^2\sigma_{eff} n v_0}{8\pi^2}$$

More precisely, it is necessary to take the statistical average $\langle k_0^2\sigma_{eff}(k_0) n v_0\rangle$.

Typical values:

Table 1. Localisation rate Λ in $\text{cm}^{-2}\text{s}^{-1}$ for three sizes of "dust particles" and various types of scattering processes (from Joos and Zeh 1985). This quantity measures how fast interference between different positions disappears as a function of distance in the course of time, see (13).

	$a = 10^{-3}$cm dust particle	$a = 10^{-5}$cm dust particle	$a = 10^{-6}$cm large molecule
Cosmic background radiation	10^6	10^{-6}	10^{-12}
300 K photons	10^{19}	10^{12}	10^6
Sunlight (on earth)	10^{21}	10^{17}	10^{13}
Air molecules	10^{36}	10^{32}	10^{30}
Laboratory vacuum (10^3 particles/cm^3)	10^{23}	10^{19}	10^{17}

Note: peak wavelength of

- CMB is about 1.9 mm = 0.2 cm
- 300 K photons is about 17 μm = $1.7 \cdot 10^{-4}$ cm
- sunlight is about 635 nm = $6.35 \cdot 10^{-5}$ cm
- air molecule at 300 K is 0.03 nm = $3 \cdot 10^{-9}$ cm

Cross section $\propto a^2$ for $k_0 a \gg 1$, i.e. when size of object is much larger than wave-length of incoming particle.

For CMB we have dipole scattering on a dielectric sphere with cross section $\propto a^6$. For sunlight and thermal photons dependence is more complex.

Decoherence of position of macroscopic objects is an extremely fast process. Take for example a dust particle of size 10^{-5} cm. Then a superposition of spatial distance of 10^{-4} cm $=$ 1 μm decoheres in 10^{-9} seconds by sunlight, 10^{-11} seconds in a laboratory vacuum and 10^{-24} seconds by air molecules!

Decoherence time scale:

$$t_d = \frac{1}{\Lambda \Delta x^2}$$

Larger objects, such as tennis balls, cats, people's brains etc. decohere even faster. In particular, neural processes are entirely classical - decoherence is many orders of magnitudes faster than typical neural signal time scales of milliseconds! Proposed links between quantum theory and consciousness are unfounded.

The unfortunate cat in Schrödinger's Gedanken experiment either dies or survives before the box is opened - in fact this happens almost in an instant!

Note that the above process is described by a differential equation

$$\partial_t \rho(\boldsymbol{x}, \boldsymbol{x}', t) = -\Lambda |\boldsymbol{x} - \boldsymbol{x}'|^2 \rho(\boldsymbol{x}, \boldsymbol{x}', t)$$

This equation is a particular example of the dynamical description of open quantum systems, see later.

Experimental evidence for position localisation by decoherence

K. Hornberger, S. Uttenthaler, B. Brezger, L. Hackermüller, M. Arndt, and A. Zeilinger, Phys. Rev. Lett. 90: 160401, 2003.

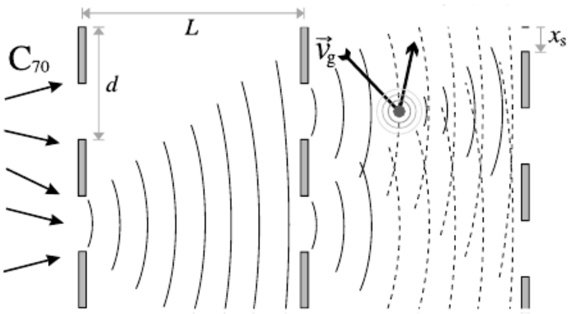


FIG. 1. Schematic setup of the near-field interferometer for C_{70} fullerenes. The third grating uncovers the interference pattern by yielding an oscillatory transmission with lateral shift x_s. Collisions with gas molecules localize the molecular center-of-mass wave function leading to a reduced visibility of the interference pattern.

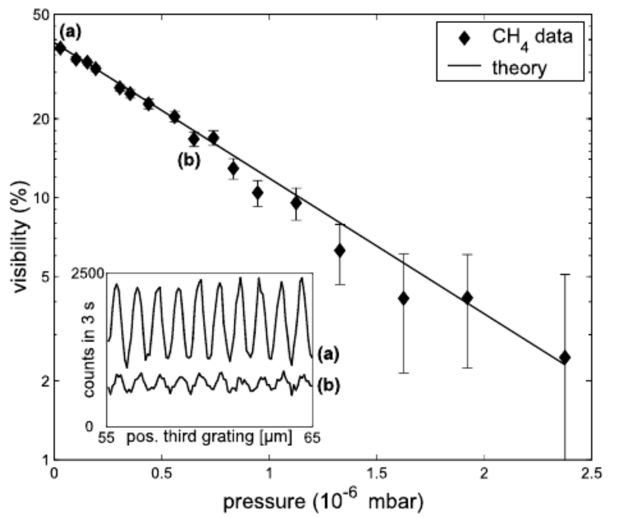


FIG. 2. Fullerene fringe visibility vs methane gas pressure on a semilogarithmic scale. The exponential decay indicates that each collision leads to a complete loss of coherence. The solid line gives the prediction of decoherence theory; see text. The inset shows the observed interference pattern at (*a*) $p = 0.05 \times 10^{-6}$ mbar and (*b*) $p = 0.6 \times 10^{-6}$ mbar.

Lecture 11

Qubit toy model for decoherence

Take a two-state apparatus interacting with an environment of N qubits

$$\mathcal{H}_{AE} = \mathcal{H}_A \otimes \mathcal{H}_E \qquad \mathcal{H}_A = \text{span}\,\{|\Uparrow\rangle, |\Downarrow\rangle\} \quad \mathcal{H}_E = \otimes_k \mathcal{H}_k \quad \mathcal{H}_k = \text{span}\,\{|\uparrow\rangle_k, |\downarrow\rangle_k\}$$

Interaction between apparatus and environment

$$H_{AE} = (|\Uparrow\rangle\langle\Uparrow| - |\Downarrow\rangle\langle\Downarrow|) \otimes \sum_{k=1}^{N} g_k\, (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)_k$$

Initial state:

$$|\Phi(0)\rangle = (a|\Uparrow\rangle + b|\Downarrow\rangle) \bigotimes_{k=1}^{N} (\alpha_k|\uparrow\rangle_k + \beta_k|\downarrow\rangle_k) \qquad |a|^2 + |b|^2 = 1 = |\alpha_k|^2 + |\beta_k|^2$$

Time evolution:

$$|\Phi(t)\rangle = a|\Uparrow\rangle|\mathcal{E}_{\Uparrow}(t)\rangle + b|\Downarrow\rangle|\mathcal{E}_{\Downarrow}(t)\rangle$$
$$|\mathcal{E}_{\Uparrow}(t)\rangle = \bigotimes_{k=1}^{N}\left(\alpha_k e^{ig_k t}|\uparrow\rangle_k + \beta_k e^{-ig_k t}|\downarrow\rangle_k\right)$$
$$|\mathcal{E}_{\Downarrow}(t)\rangle = \bigotimes_{k=1}^{N}\left(\alpha_k e^{-ig_k t}|\uparrow\rangle_k + \beta_k e^{+ig_k t}|\downarrow\rangle_k\right)$$

Reduced density matrix of the apparatus:

$$\begin{aligned}\rho_A(t) = \text{Tr}_E\, |\Phi(t)\rangle\langle\Phi(t)| = {} & |a|^2\langle\mathcal{E}_{\Uparrow}(t)|\mathcal{E}_{\Uparrow}(t)\rangle|\Uparrow\rangle\langle\Uparrow| + ab^*\langle\mathcal{E}_{\Uparrow}(t)|\mathcal{E}_{\Downarrow}(t)\rangle|\Uparrow\rangle\langle\Downarrow| \\ & + |b|^2\langle\mathcal{E}_{\Downarrow}(t)|\mathcal{E}_{\Downarrow}(t)\rangle|\Downarrow\rangle\langle\Downarrow| + a^*b\langle\mathcal{E}_{\Downarrow}(t)|\mathcal{E}_{\Uparrow}(t)\rangle|\Downarrow\rangle\langle\Uparrow|\end{aligned}$$

i.e.

$$\rho_A(t) = |a|^2|\Uparrow\rangle\langle\Uparrow| + ab^* r(t)|\Uparrow\rangle\langle\Downarrow| + a^*b\, r(t)^*|\Downarrow\rangle\langle\Uparrow| + |b|^2|\Downarrow\rangle\langle\Downarrow|$$

$$r(t) = \langle\mathcal{E}_{\Uparrow}(t)|\mathcal{E}_{\Downarrow}(t)\rangle = \prod_{k=1}^{N}[\cos 2g_k t - i(|\alpha_k|^2 - |\beta_k|^2)\sin 2g_k t]$$

For an environment with many spins at long times

$$|r(t)|^2 \simeq 2^{-N}\prod_{k=1}^{N}[1 + (|\alpha_k|^2 - |\beta_k|^2)^2]$$

which is typically exponentially small.

$$\rho_A(t) \simeq |a|^2|\Uparrow\rangle\langle\Uparrow| + |b|^2|\Downarrow\rangle\langle\Downarrow|$$

so superpositions of the states $|\Uparrow\rangle, |\Downarrow\rangle$ are unstable under interaction with environment.

However: interaction with the environment preserves purity of the states $|\Uparrow\rangle\langle\Uparrow|$ and $|\Downarrow\rangle\langle\Downarrow|$!

Pointer states and einselection

$|\Uparrow\rangle, |\Downarrow\rangle$: **pointer states**

Environment induced superselection: **einselection** (Zurek)

Pointer states are selected by the interaction with the environment. If the apparatus has its own Hamiltonian, them it is typically going to rotate the pointer states into superpositions; however, if the decoherence is fast, the superpositions are short-lived, and the effective dynamics becomes classical in terms of the pointer variables.

Fundamental example of pointer states: **position of macroscopic objects**

We know that the wave-function spreads for an isolated object. However, for macroscopic objects localisation by the environment is very fast, and the position is decohered.

Measurement outcomes

$$H_{int} = \sum_n |n\rangle\langle n| \otimes A_n$$

$|n\rangle$: measured states discriminated by D $\quad A_n$: operators acting on states of the apparatus D

$$|n\rangle|\Phi_0\rangle \to e^{-itH_{int}}|n\rangle|\Phi_0\rangle = |n\rangle e^{-itA_n}|\Phi_0\rangle = |n\rangle|\Phi_n(t)\rangle$$

$$\left(\sum_n C_n\,|n\rangle\right)|\Phi_0\rangle \to \sum_n C_n\,|n\rangle|\Phi_n(t)\rangle$$

$$\rho = \left(\sum_{n,m} C_n C_m^*\ |n\rangle\langle m|\right)|\Phi_0\rangle\langle\Phi_0| \to \sum_{n,m} C_n C_m^*\ (|n\rangle\langle m|)(|\Phi_n(t)\rangle\langle\Phi_m(t)|)$$

The state of the system is

$$\rho_S = \mathrm{Tr}_D\rho = \sum_{n,m} C_n C_m^*\ (|n\rangle\langle m|)\langle\Phi_m(t)|\Phi_n(t)\rangle$$

The apparatus discriminates the "target" states if after some time t_0

$\langle\Phi_m(t)|\Phi_n(t)\rangle \approx \delta_{mn}\ \ t > t_0 \qquad |\Phi_n(t)\rangle$: pointer states

Apparatus D = "pointer" $Q \otimes$ environment E

Typically: pointer states are locations of macroscopic objects (literally "pointers") which are very efficiently decohered by environment:

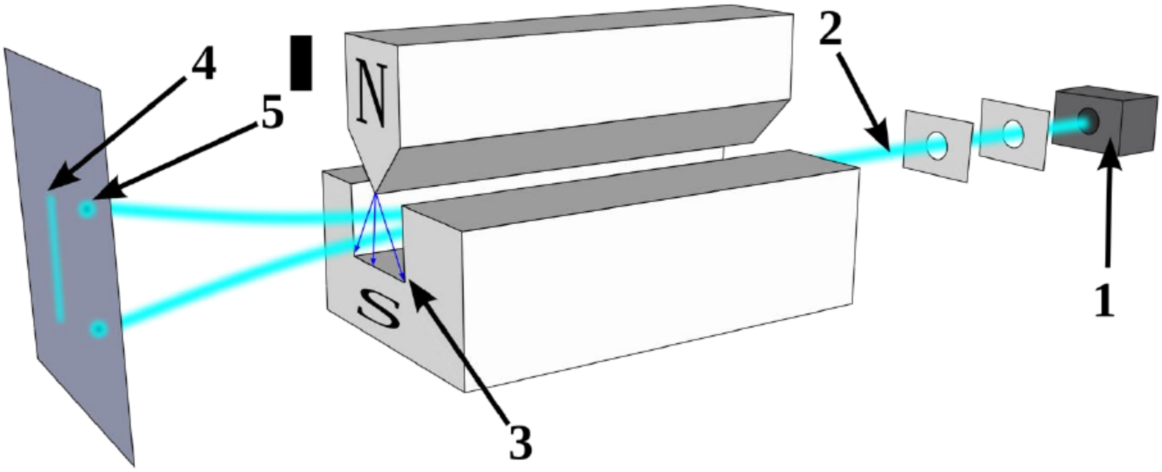
$$|\Phi_n(t)\rangle = |q_n\rangle \otimes |\mathcal{E}_n(t)\rangle \quad \langle\mathcal{E}_n(t)|\mathcal{E}_m(t)\rangle \approx \delta_{mn}$$

Then we have

$$\rho_{SQ} = \mathrm{Tr}_E\rho \approx \sum_n |C_n|^2 \ |n\rangle\langle n| \otimes |q_n\rangle\langle q_n| \text{ !!!}$$

So, if we do not keep track of the environment (which is usually impossible) - we get **apparent collapse** into definite outcomes from the perspective of the system + the pointer.

Example: Stern-Gerlach device



Silver atoms travelling through an inhomogeneous magnetic field, and being deflected up or down depending on their spin; (1) furnace, (2) beam of silver atoms, (3) inhomogeneous magnetic field, (4) classically expected result, (5) observed result (from Wikipedia)

The pointers are the pixels on the screen - these pixels are macroscopic objects whose position is subject to extremely fast decoherence on the scale of their spatial separation!

When an experiment is designed to measure an observable O, the interaction between the apparatus and the system is set up so that during the experiment the eigenstates of O corresponding to the outcomes to be distinguished are entangled with pointer states of the apparatus which evolve very fast into decohered alternatives.

Note that the original superposition is still there - if we could observe the environmental states $|\mathcal{E}_n(t)\rangle$ (which is entirely unfeasible), we would be able to see that the outcome of the measurement is a Schrödinger cat

$$\sum_n C_n |n\rangle|\Phi_n(t)\rangle = \sum_n C_n |n\rangle|q_n\rangle|\mathcal{E}_n(t)\rangle$$

But since we cannot monitor the environment, all we can confirm is that the system + the apparatus is in the state

$$\rho_{SQ} = \mathrm{Tr}_E \rho \approx \sum_n |C_n|^2 \ |n\rangle\langle n| \otimes |q_n\rangle\langle q_n|$$

which is interpreted as the outcome corresponding to $|n\rangle$ occurring with probability $|C_n|^2$, and the pointer of the apparatus correspondingly being in the state $|q_n\rangle$.

What is the most general possible time evolution for a quantum system?

Completely positive maps and quantum operations

General time evolution of quantum system: mapping density matrices to density matrices.

Space of density matrices: $\rho(\mathcal{H}_S) = \{A: \mathcal{H}_S \to \mathcal{H}_S \mid A \geq 0 \,,\ \mathrm{Tr}\, A = 1\}$

Time evolution over some interval is a map $\hat{V}: \rho(\mathcal{H}_S) \to \rho(\mathcal{H}_S)$

Definition: a map $\hat{V}$ is **positive** if for all $A \geq 0$ it follows that $\hat{V}(A) \geq 0$.

However, this is not enough since systems are usually subsystems of even larger systems. Performing an operation only affecting the system, we must arrive at a state of the full system. Therefore, we need

Definition: a map $\hat{V}$ is **completely positive** if for all $N \in \mathbf{N}$, $V \otimes \mathbf{1}_N$ is positive.

Example of a positive but not completely positive map is matrix transposition:

1. $A \geq 0 \Rightarrow A^T \geq 0$

2. Composite system

$$|\psi_B\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \Rightarrow \rho_B = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We can apply $\mathbf{1} \otimes T$ (transposition in second Hilbert space)

$$(\mathbf{1} \otimes T)(\rho_B) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

This matrix has negative eigenvalues!

Let us denote the bounded linear operators $\mathcal{H}_S \to \mathcal{H}_S$ by $\mathcal{B}(\mathcal{H}_S)$. For simplicity we assume $\dim \mathrm{H}_S$ is finite (we would need theory of C* algebras for infinite-dimensional cases, but there is nothing essentially new to learn from that as far as physics is concerned).

Lemma: for any $V \in \mathcal{B}(\mathcal{H}_S)$, the map $A \to \hat{V}(A) = V^\dagger A V$ is positive.

Proof: $A \geq 0 \Rightarrow \exists B\colon A = BB^\dagger \Rightarrow V^\dagger A V^\dagger = V^\dagger B B^\dagger V = (V^\dagger B)(V^\dagger B)^\dagger \geq 0$

Theorem (Choi): a linear map $\hat{V}\colon \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$ is completely positive iff it can be expressed as

$$\hat{V}(A) = \sum_i V_l^\dagger A V_l \quad \text{for some } V_l \in \mathcal{B}(\mathcal{H}_S)$$

Proof:

(1) One direction is trivial: if

$$\hat{V}(A) = \sum_l V_l^\dagger A V_l$$

then extending it to the space $\mathcal{H}_S \otimes \mathbf{C}^N$ is trivially positive since we can extend the operators V_l to the full space as $V_l \otimes \mathbf{1}_N$.

(2) Now assume $\hat{V}$ is completely positive and let $d = \dim \mathcal{H}_S$. Extend the system with a Hilbert space of the same dimension d and define a maximally entangled pure state on $\mathrm{H}_S \otimes \mathbf{C}^d$ by

$$|\Gamma\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d} |i\rangle_S |i\rangle \quad |\Gamma\rangle\langle\Gamma| = \frac{1}{d}\sum_{i,j=1}^{d} |i\rangle_S\langle j|_S \otimes |i\rangle\langle j|$$

where $|i\rangle_S$ and $|i\rangle$ are arbitrary orthonormal bases of the two factors. We can extend the action of $\hat{V}$ to the full space as

$$\hat{V}_2 = \hat{V} \otimes \mathbf{1}$$

$$\hat{V}_2(|\Gamma\rangle\langle\Gamma|) = \frac{1}{d}\sum_{i,j=1}^{d} \hat{V}(|i\rangle_S\langle j|_S) \otimes |i\rangle\langle j| \quad \Rightarrow \quad \hat{V}(|i\rangle_S\langle j|_S) = d\langle i|\hat{V}_2(|\Gamma\rangle\langle\Gamma|)|j\rangle$$

But $\hat{V}_2$ is positive so we can write

$$\hat{V}_2(|\Gamma\rangle\langle\Gamma|) = \sum_{l=1}^{d^2} |v_l\rangle\langle v_l| \quad \text{for some } |v_l\rangle \in \mathcal{H}_S \otimes \mathbf{C}^d$$

Now for any vector $|\psi\rangle \in \mathcal{H}_S \otimes \mathbf{C}^d$ we have a linear operator $V_\psi \in \mathcal{B}(\mathcal{H}_S)$ such that $|\psi\rangle = (V_\psi \otimes \mathbf{1})|\Gamma\rangle$. Explicitely writing a general state as

$$|\psi\rangle = \sum_{i,j=1}^{d} \alpha_{ij} |i\rangle_S |j\rangle$$

we define the operator by giving its action on the basis of H_S as

$$V_\psi |j\rangle_S = \sqrt{d} \sum_{i=1}^{d} \alpha_{ij} |i\rangle_S$$

Let us define now the operators V_l as the ones corresponding to the states $|v_l\rangle$:
$|v_l\rangle = (V_l \otimes \mathbf{1})|\Gamma\rangle$

Now we can express the action of $\hat{V}$ on an arbitrary basis element $|i\rangle_S\langle j|_S$ of $\mathcal{B}(\mathcal{H}_S)$ as

$$\hat{V}(|i\rangle_S\langle j|_S) = d\langle i|\hat{V}_2(|\Gamma\rangle\langle\Gamma|)|j\rangle = d\sum_{l=1}^{d^2} \langle i|(|v_l\rangle\langle v_l|)|j\rangle$$

$$= d\sum_{l=1}^{d^2} \langle i|\big(V_l \otimes \mathbf{1}|\Gamma\rangle\langle\Gamma|V_l^\dagger \otimes \mathbf{1}\big)|j\rangle$$

$$= \sum_{l=1}^{d^2} \langle i|\left(V_l \otimes \mathbf{1}\left(\sum_{k,l=1}^{d} |k\rangle_S\langle l|_S \otimes |k\rangle\langle l|\right)V_l^\dagger \otimes \mathbf{1}\right)|j\rangle$$

$$= \sum_{l=1}^{d^2} \langle i|\left(\left(\sum_{k,l=1}^{d} V_l|k\rangle_S\langle l|_S V_l^\dagger \otimes |k\rangle\langle l|\right)\right)|j\rangle$$

$$= \sum_{l=1}^{d^2} V_l(|i\rangle_S\langle j|_S)\, V_l^\dagger$$

which proves the theorem, since once we have it for any $|i\rangle_S\langle j|_S$ we have it for all linear operators because $|i\rangle_S\langle j|_S$ forms a basis in $\mathcal{B}(\mathcal{H}_S)$.

Theorem (Choi-Krauss): a linear map $\hat{V}: \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$ is completely positive and trace-preserving iff it can be expressed as

$$\hat{V}(A) = \sum_l V_l^\dagger A V_l \quad \text{for some } V_l \in \mathcal{B}(\mathcal{H}_S)$$

where

$$\sum_l V_l V_l^\dagger = \mathbf{1}$$

Proof: we only need to consider the trace preservation property.

One way is trivial:
$\hat{V}(A) = \sum_l V_l^\dagger A V_l \Rightarrow \mathrm{Tr}\,\hat{V}(A) = \sum_l \mathrm{Tr}\,V_l^\dagger A V_l = \sum_l \mathrm{Tr}\,V_l V_l^\dagger A = \mathrm{Tr}\,A$

The other way: let's consider the starting point
$\mathrm{Tr}\,\hat{V}(|i\rangle\langle j|) = \mathrm{Tr}\,|i\rangle\langle j| = \delta_{ij}$

But we can calculate
$\mathrm{Tr}\,\hat{V}(|i\rangle\langle j|) = \mathrm{Tr}\sum_l V_l^\dagger |i\rangle\langle j| V_l = \langle j|\sum_l V_l V_l^\dagger |i\rangle = \delta_{ij} \Rightarrow \sum_l V_l V_l^\dagger = \mathbf{1}$

Definition: completely positive trace preserving (CPTP) maps are called **quantum operations**.

The V_l are called **Krauss operators**.

Examples of quantum operations:

- Unitary time evolution

  $\rho \to U\rho U^\dagger \quad U$ is unitary $UU^\dagger = \mathbf{1}$

- Ideal (von Neumann) measurement

  $\rho \to \sum_i P_i \rho P_i$
  $P_i = P_i^\dagger = P_i^2$: projectors for measurement outcomes satisfying completeness
  $\sum_i P_i = \mathbf{1}$

- Non-ideal (positive operator valued) measurement

  $\rho \to \sum_i M_i \rho M_i^\dagger: \quad M_i \geq 0$ and $\sum_i M_i M_i^\dagger = \mathbf{1}$

If time evolution is non-unitary, the system is open: the Stinespring dilation theorem (below) states that it can be embedded into a larger system with unitary time evolution.

Purification and Stinespring dilation

Purification of mixed states

Assume we have a quantum system $\mathcal{H}_S$ in a mixed state ρ. Diagonalising the density matrix we can write

$$\rho = \sum_i p_i |i\rangle\langle i| \quad 0 \leq p_i \leq 1 \quad \sum_i p_i = 1$$

where $|i\rangle$ is an orthonormal basis for $\mathcal{H}_S$.

(It is a theorem that $\rho \geq 0$ implies that ρ is Hermitian so it has an orthonormal eigen basis. It also implies that all its eigenvalues p_i are non-negative.)

Now we can take another system (ancilla) $\mathcal{H}_A$ with $\dim \mathcal{H}_S = \dim \mathcal{H}_A$ and an orthonormal basis $|i\rangle_A$. We can then define the state

$$|\Psi_\rho\rangle = \sum_i \sqrt{p_i}\, |i\rangle \otimes |i\rangle_A \in \mathcal{H}_S \otimes \mathcal{H}_A$$

This is a pure state satisfying

$$\mathrm{Tr}_A\, |\Psi_\rho\rangle\langle\Psi_\rho| = \rho$$

So, every mixed state of a system can be extended to a pure state of a larger system - this is called *purification*.

Purification is impossible in classical systems

Let us assume we have a classical system with phase space $X \times Y$ which is in a pure state given by a Dirac delta distribution $\delta_{x_0,y_0}(x,y)$: every observable is a function $f(x,y)$ on the phase space with a definite value in the pure state given by

$$\langle f\rangle = \int_{X\times Y} dxdy\, \delta_{x_0,y_0}(x,y) f(x,y) = f(x_0,y_0)$$

The observables *g(x)* of subsystem *X* have values

$$\langle g\rangle = \int_{X\times Y} dxdy\, \delta_{x_0,y_0}(x,y) g(x) = g(x_0) \equiv \int_X dx\, \delta_{x_0}(x) g(x)$$

So, if a classical system is in a pure state, all of its subsystems are in a pure state.

Pure state corresponds to having the maximum possible (i.e., complete) information about the system.

Classically, if we have complete information about a system, then we have complete information about all of its subsystems.

Quantum theory: it is possible to have only partial information about a subsystem (mixed state) yet have complete information about the complete system (pure state)!

This is due to entanglement: the purified state

$$|\Psi_\rho\rangle = \sum_i \sqrt{p_i}\, |i\rangle \otimes |i\rangle_A \in \mathcal{H}_S \otimes \mathcal{H}_A$$

has a nontrivial entanglement between the subsystem and the ancilla!

Purification of time evolution: Stinepsring dilation

Let us now assume a system undergoes a general quantum operation:

$$\hat{V}(\rho) = \sum_{l=1}^{n} V_l^{\dagger} \rho V_l \quad \text{for some } V_l \in \mathcal{B}(\mathcal{H}_S) \text{ with } \sum_l V_l V_l^{\dagger} = \mathbf{1}$$

Let us now take an ancilla $\mathcal{H}_A$ of dimension n, with a basis $|i\rangle_A \;\; i = 1, \dots, n$

We can now define an isometry

$$I_V\colon \mathcal{H}_S \to \mathcal{H}_S \otimes \mathcal{H}_A \quad |\psi\rangle \to U_V|\psi\rangle = \sum_{l=1}^{n} V_l^{\dagger}|\psi\rangle \otimes |l\rangle$$

$$\begin{aligned}\langle\psi|I_V^{\dagger} I_V|\psi\rangle &= \sum_{k,l} \langle k| \otimes \langle\psi|V_k V_l^{\dagger}|\psi\rangle \otimes |l\rangle = \sum_{k,l} \langle k|l\rangle \langle\psi|V_k V_l^{\dagger}|\psi\rangle \\ &= \sum_k \langle\psi|V_k V_k^{\dagger}|\psi\rangle = \langle\psi|\psi\rangle\end{aligned}$$

Note that

$$\mathrm{Tr}_A\left(I_V|\psi_1\rangle\langle\psi_2|I_V^{\dagger}\right) = \mathrm{Tr}_A \sum_{k,l} V_l^{\dagger}|\psi_1\rangle \otimes |l\rangle\langle k| \otimes \langle\psi_2|V_k = \sum_l V_l^{\dagger}(|\psi_1\rangle\langle\psi_2|)V_l$$

$$\Rightarrow \mathrm{Tr}_A\left(I_V \rho I_V^{\dagger}\right) = \sum_{l=1}^{n} V_l^{\dagger} \rho V_l$$

Any isometry can be trivially extended to a unitary map

$U_V\colon \mathcal{H}_S \otimes \mathcal{H}_A \to \mathcal{H}_S \otimes \mathcal{H}_A$

Therefore: any quantum operation on a system can be obtained by a partial trace from a unitary time evolution on a larger system!

As a result, quantum systems with non-unitary time evolution can always be considered open systems interacting with an environment, whereby the full system is closed, since it evolves unitarily.

Lecture 12

Markovian time evolution: the Lindblad equation

We look for an equation of the form

$$\partial_t \rho_S(t) = \tilde{L} \rho_S(t)$$

so that the time evolution after a period t has the form

$$\hat{V}_t = e^{\tilde{L}t} \text{ with } \hat{V}_t(A) = \sum_l V_l(t)^\dagger A V_l(t)$$

Let us choose a basis for the operators on $\mathcal{B}(\mathcal{H}_S)$ on $\mathcal{H}_S$. Assuming that $\dim \mathcal{H}_S = d$ we can choose

$$F_i\colon\ i = 0, \dots d^2 - 1 \quad \mathrm{Tr}\, F_i^\dagger F_j = \delta_{ij} \quad F_0 = \frac{1}{d}\mathbf{1}$$

We note that the bilinear form

$$A, B \in \mathcal{B}(\mathcal{H}_S) \to \langle A, B \rangle = Tr(A^\dagger B)$$

defines a Hilbert-space structure on $\mathcal{B}(\mathcal{H}_S)$. In particular it is non-degenerate:

$$\langle A, B \rangle = 0\ \forall A \Rightarrow B = 0$$

Now we expand

$$V_k(t) = \sum_l C_{kl}(t) F_l$$

$$\partial_t \hat{V}_t(\rho) = \partial_t \sum_{l,m} c_{lm}(t) F_l^\dagger \rho F_m \qquad c_{lm}(t) = \sum_k C_{kl}(t)^* C_{km}(t)$$

Note that in a matrix form $c = C^\dagger C$ i.e. c is a positive Hermitian matrix. Define the matrix $g_{lm}(t) = \partial_t c_{lm}(t)$ and write

$$\begin{aligned}\partial_t \rho &= \sum_{l,m=1}^{d^2-1} g_{lm} F_l^\dagger \rho F_m + \frac{1}{d} \sum_{l=1}^{d^2-1} g_{l0} F_l^\dagger \rho + \frac{1}{d} \sum_{l=1}^{d^2-1} g_{0l} \rho F_l + \frac{1}{d^2} g_{00} \rho \\ &= \sum_{l,m=1}^{d^2-1} g_{lm} F_l^\dagger \rho F_m + F^\dagger \rho + \rho F + \frac{1}{d^2} g_{00} \rho \qquad F = \frac{1}{d} \sum_{l=1}^{d^2-1} g_{0l} \rho F_l\end{aligned}$$

We now decompose

$$F = \frac{1}{2}(F + F^\dagger) + i\frac{1}{2i}(F - F^\dagger) = G + iH \qquad G = G^\dagger\,,\, H = H^\dagger$$

$$\partial_t\rho = -i[H,\rho] + \sum_{l,m=1}^{d^2-1} g_{lm}F_l^\dagger\rho F_m + \tilde{G}_2(\rho) \quad \tilde{G}_2(\rho) = G\rho + \rho G + \frac{1}{d^2}g_{00}\rho$$
$$= \left\{G + \frac{1}{d^2}g_{00}, \rho\right\} = \{G_2, \rho\}$$

Here we used the cyclic property of the trace Tr $(A_1 \dots A_{n-1}A_n) = \text{Tr}\,(A_nA_1 \dots A_{n-1})$

Now we use the trace preserving property:

$$0 = \partial_t(\text{Tr}\,\rho) = \sum_{l,m=1}^{d^2-1} g_{lm}\text{Tr}\left(F_l^\dagger\rho F_m\right) + \text{Tr}\{G_2,\rho\} = \sum_{l,m=1}^{d^2-1} g_{lm}\text{Tr}\left(\rho F_m F_l^\dagger\right) + 2\,\text{Tr}\,\rho G_2$$
$$= 0\ \forall\rho$$
$$\Rightarrow\ G_2 = -\frac{1}{2}\sum_{l,m=1}^{d^2-1} g_{lm}F_mF_l^\dagger$$

This leads to

$$\partial_t\rho = -i[H,\rho] + \sum_{l,m=1}^{d^2-1} g_{lm}\left(F_l^\dagger\rho F_m - \frac{1}{2}\{F_mF_l^\dagger,\rho\}\right)$$

We can now diagonalise the matrix g_{lm}, denoting the eigenvalues by Γ_k, and end up with

$$\partial_t\rho = -i[H,\rho] + \sum_k \Gamma_k\left(L_k^\dagger\rho L_k - \frac{1}{2}\{L_kL_k^\dagger,\rho\}\right) \qquad \textbf{Lindblad equation}$$

L_k: **jump operators** (linear combinations of the F_m resulting from diagonalisation, which can be chosen to be traceless by redefining H)

Therefore, the generic time evolution of any quantum system is described by the Lindblad equation provided it is Markovian.

The operators L_k and the coefficients Γ_k can be obtained directly by explicit computation. According to the Stinespring dilatation theorem, the generic time evolution of a quantum system can be obtained as the restriction of unitary time evolution on a larger quantum system. Provided this system and the full Hamiltonian is known, it is possible to compute a master equation governing the dynamics of the subsystem sometimes exactly or using some approximation e.g. perturbation theory. Assuming that the conditions of Markovian approximation hold, this equation then gives a Lindblad equation with explicit expressions for L_k and Γ_k.

Master equation for open quantum systems

Here we present a perturbative derivation of the master equation and its Markovian limit for a generic system-environment setting.

Assume we have a system plus environment, with total Hilbert space

$$\mathcal{H}_T = \mathcal{H}_S \otimes \mathcal{H}_E$$

described by unitary dynamics governed by the von Neumann equation

$$\partial_t \rho_T(t) = -i[H_T, \rho_T(t)]$$

$$H_T = H_S \otimes \mathbf{1}_E + \mathbf{1}_S \otimes H_E + \alpha H_I \quad H_I = \sum_i S_i \otimes E_i$$

Consider interaction picture using αH_I as the interaction term. Operators then evolve as

$$\hat{O}(t) = e^{i(H_S+H_E)t} O e^{-i(H_S+H_E)t}$$

The evolution of the states can be recast as an integral equation:

$$\partial_t \hat{\rho}_T(t) = -i\alpha\big[\hat{H}_I(t), \hat{\rho}_T(t)\big] \;\Rightarrow\; \hat{\rho}_T(t) = \hat{\rho}_T(0) - i\alpha \int_0^t ds\, \big[\hat{H}_I(s), \hat{\rho}_T(s)\big]$$

$$\partial_t \hat{\rho}_T(t) = -i\alpha\big[\hat{H}_I(t), \hat{\rho}_T(t)\big] = -i\alpha\left[\hat{H}_I(t), \hat{\rho}_T(0) - i\alpha \int_0^t ds\, \big[\hat{H}_I(s), \hat{\rho}_T(s)\big]\right]$$

$$= -i\alpha\big[\hat{H}_I(t), \hat{\rho}_T(0)\big] - \alpha^2 \int_0^t ds\, \Big[\hat{H}_I(t), \big[\hat{H}_I(s), \hat{\rho}_T(s)\big]\Big]$$

Repeating this step, we get

$$\partial_t \hat{\rho}_T(t) = -i\alpha\big[\hat{H}_I(t), \hat{\rho}_T(0)\big] - \alpha^2 \int_0^t ds\, \Big[\hat{H}_I(t), \big[\hat{H}_I(s), \hat{\rho}_T(t)\big]\Big] + O(\alpha^3)$$

Perturbative approximation

We now assume that the strength of interaction with the environment is small and drop the term $O(\alpha^3)$:

$$\partial_t \hat{\rho}_T(t) = -i\alpha\big[\hat{H}_I(t), \hat{\rho}_T(0)\big] - \alpha^2 \int_0^t ds\, \Big[\hat{H}_I(t), \big[\hat{H}_I(s), \hat{\rho}_T(t)\big]\Big]$$

Taking partial trace over the environment

$$\partial_t \hat{\rho}_S(t) = -i\alpha \mathrm{Tr}_E\big[\hat{H}_I(t), \hat{\rho}_T(0)\big] - \alpha^2 \int_0^t ds\, \mathrm{Tr}_E \Big[\hat{H}_I(t), \big[\hat{H}_I(s), \hat{\rho}_T(t)\big]\Big]$$

We now make some additional assumptions:

- The initial state is of the form $\rho_T(0) = \rho_S(0) \otimes \rho_E(0)$
  This means that initially the system is assumed to be uncorrelated with the environment.

- The environment is in thermal equilibrium (this is just to make the environmental state specific, other choices are also possible):
  $\rho_E(0) = \frac{1}{Z_E} e^{-H_E/T} \quad Z_E = \mathrm{Tr}_E e^{-H_E/T}$

Therefore

$$Tr_E[\hat{H}_I(t), \hat{\rho}_T(0)] = \sum_i Tr_E[\hat{S}_i(t) \otimes \hat{E}_i(t), \rho_S(0) \otimes \rho_E(0)]$$
$$= \sum_i \left(\hat{S}_i(t)\rho_S(0)\mathrm{Tr}_E[\hat{E}_i(t)\rho_E(0)] - \rho_S(0)\hat{S}_i(t)\mathrm{Tr}_E[\rho_E(0)\hat{E}_i(t)]\right)$$

We can always redefine the environmental operators E_i so that $\langle E_i \rangle = Tr_E[\rho_E(0)E_i] = 0$:

$$H_T = H_S + H_E + \alpha \sum_i S_i \otimes E_i = \underbrace{H_S + \alpha \sum_i \langle E_i \rangle S_i}_{H_S'} + H_E + \alpha \sum_i S_i \otimes \underbrace{(E_i - \langle E_i \rangle)}_{E_i'}$$

Therefore, we reduce the equation to

$$\partial_t \hat{\rho}_S(t) = -\alpha^2 \int_0^t ds \, \mathrm{Tr}_E \left[\hat{H}_I(t), [\hat{H}_I(s), \hat{\rho}_T(t)]\right]$$

Markovian approximation

New assumption: the environment thermalizes very fast compared to the system dynamics so we can always write

$$\hat{\rho}_T(t) = \hat{\rho}_S(0) \otimes \rho_E(0) \quad \rho_E(0) = \frac{1}{Z_E} e^{-H_E/T} \quad Z_E = \mathrm{Tr}_E e^{-H_E/T}$$

This means that the environment does not keep a memory of the system: system-environment interactions do not depend on the interactions that happened before. This is essentially the same as the Stosszahlansatz (molecular chaos assumption) used to derive irreversibility in classical statistical mechanics (Boltzmann's H-theorem).

As a result, we get

$$\partial_t \hat{\rho}_S(t) = -\alpha^2 \int_0^t ds \, \mathrm{Tr}_E \left[\hat{H}_I(t), [\hat{H}_I(s), \hat{\rho}_S(t) \otimes \rho_E(0)]\right]$$

This is now a closed system for the density matrix of the system!

We can now change the variable $s \to t - s$:

$$\partial_t \hat{\rho}_S(t) = -\alpha^2 \int_0^t ds \, \mathrm{Tr}_E \left[\hat{H}_I(t), [\hat{H}_I(t-s), \hat{\rho}_S(t) \otimes \rho_E(0)]\right]$$

and also extend the integral to infinity, reasoning that it does not matter if we include the far past since the memory of the environment is short. What we say here essentially is that that the temporal autocorrelation of the environment decays very fast, i.e. on a time scale much faster than the scale on which the system's dynamics takes place:

$$\partial_t \hat{\rho}_S(t) = -\alpha^2 \int_0^\infty ds \, \mathrm{Tr}_E \left[\hat{H}_I(t), [\hat{H}_I(t-s), \hat{\rho}_S(t) \otimes \rho_E(0)]\right] \quad \textbf{Redfield equation}$$

Rotating wave approximation

We can define the following "superoperator"

$A \in \mathcal{B}(\mathcal{H}_S): A \to \widetilde{H}A = [H_S, A]$

We expand the operators S_i in the eigenbasis of $\widetilde{H}$:

$$S_i = \sum_\omega S_i(\omega) \qquad [H_S, S_i(\omega)] = -\omega S_i(\omega) \qquad \left[H_S, S_i^\dagger(\omega)\right] = -\omega S_i^\dagger(\omega)$$

Recalling that

$$H_I = \sum_i S_i \otimes E_i \text{ but since } H_I = H_I^\dagger \Rightarrow H_I = \sum_i S_i^\dagger \otimes E_i^\dagger$$

we can compute

$$\widehat{H}_I(t) = e^{i(H_S+H_E)t} H_I e^{-i(H_S+H_E)t} = \sum_{k,\omega} e^{-i\omega t} S_k(\omega) \otimes \hat{E}_k(t) = \sum_{k,\omega} e^{+i\omega t} S_k^\dagger(\omega) \otimes \hat{E}_k^\dagger(t)$$

Expanding the commutators:

$$\begin{aligned}\partial_t \hat{\rho}_S(t) &= -\alpha^2 \int_0^\infty ds\, \mathrm{Tr}_E \left[\widehat{H}_I(t), \left[\widehat{H}_I(t-s), \hat{\rho}_S(t) \otimes \rho_E(0)\right]\right] \\ &= -\alpha^2 \int_0^\infty ds\, \mathrm{Tr}_E[\widehat{H}_I(t)\widehat{H}_I(t-s)\hat{\rho}_S(t) \otimes \rho_E(0) - \widehat{H}_I(t)\hat{\rho}_S(t) \otimes \rho_E(0)\widehat{H}_I(t-s) \\ &\qquad - \widehat{H}_I(t-s)\hat{\rho}_S(t) \otimes \rho_E(0)\widehat{H}_I(t) \\ &\qquad + \hat{\rho}_S(t) \otimes \rho_E(0)\widehat{H}_I(t-s)\widehat{H}_I(t)]\end{aligned}$$

Substituting the expression for $\widehat{H}_I(t)$ and collecting the terms gives

$$\begin{aligned}\partial_t \hat{\rho}_S(t) = \sum_{\substack{\omega,\omega' \\ k,l}} \Big(& e^{i(\omega'-\omega)t} \Gamma_{kl}(\omega)\left[S_l(\omega)\hat{\rho}_S(t), S_k^\dagger(\omega')\right] \\ &+ e^{-i(\omega'-\omega)t} \Gamma_{lk}^*(\omega)\left[S_l(\omega), \hat{\rho}_S(t) S_k^\dagger(\omega')\right]\Big)\end{aligned}$$

$$\Gamma_{kl}(\omega) = \alpha^2 \int_0^\infty ds\, e^{i\omega s} \mathrm{Tr}_E\left[\hat{E}_k^\dagger(t)\hat{E}_l(t-s)\rho_E(0)\right]$$

Rotating wave approximation: environment is fast - keep only resonant terms with $\omega = \omega'$ (all other terms oscillate too fast, average out on time scale of evolution of the system)

$$\partial_t \hat{\rho}_S(t) = \sum_{\substack{\omega \\ k,l}} \left(\Gamma_{kl}(\omega)\left[S_l(\omega)\hat{\rho}_S(t), S_k^\dagger(\omega)\right] + \Gamma_{lk}^*(\omega)\left[S_l(\omega), \hat{\rho}_S(t) S_k^\dagger(\omega)\right]\right)$$

Now we decompose

$$\Gamma_{kl}(\omega) = \frac{1}{2}\gamma_{kl}(\omega) + i\pi_{kl}(\omega) \qquad \pi_{kl}(\omega) = -\frac{i}{2}\left(\Gamma_{kl}(\omega) - \Gamma_{kl}^*(\omega)\right)$$

$$\gamma_{kl}(\omega) = \Gamma_{kl}(\omega) + \Gamma^*_{kl}(\omega) = \alpha^2 \int_{-\infty}^{\infty} ds\, e^{i\omega s} \mathrm{Tr}_E\left[\hat{E}_k^\dagger(t)\hat{E}_l\rho_E(0)\right]$$

and rewrite

$$\partial_t\hat{\rho}_S(t) = -i[H_{Ls}, \hat{\rho}_S(t)] + \sum_{\substack{\omega \\ k,l}} \gamma_{kl}(\omega)\left(S_l(\omega)\hat{\rho}_S(t)S_k^\dagger(\omega) - \frac{1}{2}\left\{S_k^\dagger(\omega)S_l(\omega), \hat{\rho}_S(t)\right\}\right)$$

$$H_{Ls} = \sum_{\substack{\omega \\ k,l}} \pi_{kl}(\omega)S_k^\dagger(\omega)S_l(\omega)$$

Back to Schrödinger picture

$$\partial_t\rho_S(t) = -i[H_S + H_{Ls}, \rho_S(t)] + \sum_{\substack{\omega \\ k,l}} \gamma_{kl}(\omega)\left(S_l(\omega)\rho_S(t)S_k^\dagger(\omega) - \frac{1}{2}\left\{S_k^\dagger(\omega)S_l(\omega), \rho_S(t)\right\}\right)$$

Markovian master equation

Diagonalising the matrix $\gamma_{kl}(\omega)$:

$$\gamma_{kl}(\omega) = U\begin{pmatrix} \gamma_1(\omega) & 0 & \dots & 0 \\ 0 & \gamma_2(\omega) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma_N(\omega) \end{pmatrix}U^\dagger$$

we can write our equation in a diagonal form

$$\partial_t\rho_S(t) = -i[H_S + H_{Ls}, \rho_S(t)] + \sum_{\omega,k}\left(L_k(\omega)\rho_S(t)L_k^\dagger(\omega) - \frac{1}{2}\left\{L_k^\dagger(\omega)L_k(\omega), \rho_S(t)\right\}\right)$$

This is a **Lindblad equation**, the $L_k(\omega)$ are the **jump operators** and the term H_{Ls} is called **Lamb shift**.

Short notation:

$\partial_t\rho_S(t) = \tilde{L}\rho_S(t)$ $\quad\tilde{L}$: **Lindblad superoperator, also called the Liouville operator**

Note that

$$\partial_t \mathrm{Tr}_S\rho_S(t) = \mathrm{Tr}_S\left(\tilde{L}\rho_S(t)\right) = 0$$

So, this time evolution preserves the trace of the density matrix - i.e., probability.

An example of a Lindblad equation: quantum Brownian motion

System: quantum particle of mass M moving in a potential $V(x)$
Environment: harmonic oscillators at temperature T

Master equation (simplified version)

$$\dot{\rho} = -\frac{i}{\hbar}[H_S, \rho] - \frac{i}{\hbar}\gamma[x, \{p, \rho\}] - D\big[x, [x, \rho]\big]$$

γ: dissipation $\quad D = \dfrac{2\gamma M k_B T}{\hbar^2}$: position localisation coefficient

This is the high-T limit of the master equation: (W.H. Zurek, arXiv: quant-ph/0105127)

Dual Liouville operator

$$\dot{\rho} = \tilde{L}\rho \ \Rightarrow \ \partial_t\langle A\rangle = \big\langle \tilde{L}^{\#}(A)\big\rangle \ \text{with} \ \langle A\rangle = \mathrm{Tr}(\rho A)$$

Essentially this means that

$$\mathrm{Tr}\big(\tilde{L}(\rho)A\big) = \mathrm{Tr}\left(\rho\tilde{L}^{\#}(A)\right)$$

i.e. $\tilde{L}^{\#}$ is the adjoint of $\tilde{L}$ under the trace inner product.

For the above master equation, we have

$$\tilde{L}^{\#}(A) = -\frac{i}{\hbar}[A, H_S] - \frac{i}{\hbar}\gamma\{p, [A, x]\} - D\big[x, [x, A]\big]$$

Example: the dual of the term $[x, \{p, \rho\}]$ is $[p, \{A, x\}]$:

$$\begin{aligned}\mathrm{Tr}\left([x, \{p, \rho\}]A\right) &= \mathrm{Tr}\left((xp\rho + x\rho p - p\rho x - \rho p x)A\right) = \mathrm{Tr}\left(\rho(Axp + pAx - xAp - pxA)\right)\\ &= \mathrm{Tr}\left(\rho\{p, [A, x]\}\right)\end{aligned}$$

Homework: derive the other two terms!

Ehrenfest equations

$$\tilde{L}^{\#}(x) = \frac{p}{m} \qquad \Rightarrow \frac{d\langle x\rangle}{dt} = \frac{1}{m}\langle p\rangle$$

$$\tilde{L}^{\#}(p) = -V'(x) - 2\gamma p \quad \Rightarrow \frac{d\langle p\rangle}{dt} = -\langle V'(x)\rangle - 2\gamma\langle p\rangle$$

Problem with Ehrenfest equations in the case with no environment:

- wave packet spreads in time, so its width in position space grows
- $\langle V'(x)\rangle \neq V'(\langle x\rangle)$

Decoherence term

In position representation $\rho(x, x') = \langle x|\rho|x'\rangle$

$$\langle x|[\hat{x},[\hat{x},\rho]]|x'\rangle = \langle x|\hat{x}^2\rho - 2\hat{x}\rho\hat{x} + \rho\hat{x}^2|x'\rangle = (x-x')^2\rho(x,x')$$

resulting in

$$\partial_t\rho(x,x') = -\underbrace{D(x-x')^2}_{\gamma_D}\rho(x,x') + \text{other terms} \quad \gamma_D\text{: decoherence rate}$$

This leads to a quick suppression of position interference terms:

$\rho(x,x') \propto \exp(-\gamma_D t)$ for $|x-x'|$ large

$$\frac{\gamma_D}{\gamma} = \frac{(x-x')^2}{\lambda_T^2} \qquad \lambda_T = \frac{\hbar}{\sqrt{2Mk_BT}} \text{ : thermal de Broglie wavelength}$$

It keeps the wave packet localised to scales $|x-x'| \lesssim \lambda_T$!

Also: at macroscopic separations $|x-x'| \gg \lambda_T$ the decoherence time scale is very fast compared to dissipation - in the limiting case it is possible for the system to be classical with negligible dissipation, and so its dynamics can be described by **conservative Newtonian dynamics**. To get that we need to replace $\langle V'(x)\rangle$ by $V'(\langle x\rangle)$, so the distance over which the potential varies should be much longer than λ_T.

The decoherence term $[x,[x,\rho]]$ can also be interpreted as diffusion in momentum space, with interesting applications to quantum chaos (cf. W.H. Zurek, arXiv: quant-ph/0105127).

Loss of purity

$$\dot{\rho}\rho = -\frac{i}{\hbar}[H_S,\rho]\rho - \frac{i}{\hbar}\gamma[x,\{p,\rho\}]\rho - D[x,[x,\rho]]\rho$$

Take trace and evaluate terms using cyclic property of trace and $[x,p] = i\hbar$

$$\Rightarrow \frac{d}{dt}\text{Tr}\,\rho^2 = -4D\;\text{Tr}(\rho^2x^2 - (\rho x)^2) + 2\gamma\text{Tr}\,\rho^2$$

The second term is usually unimportant (except for very sharply localised states with $D\Delta x^2 < \gamma$ i.e. $\Delta x < \lambda_T$), so starting from a pure state with $\rho = |\Psi\rangle\langle\Psi|$ purity generally decreases according to

$$\frac{d}{dt}\text{Tr}\,\rho^2 = -4D\,(\langle x^2\rangle - \langle x\rangle^2)$$

Classical and quantum information

By measuring or observing a system, we extract information about it. Therefore, it is interesting to consider the information theory description of what we can learn about physical states.

Classical (Shannon) information

If an event happens with some probability p, we can associate to its observation a *surprise* $I(p)$:

1. $I(p)$ is monotonically decreasing with p
2. $I(p) \geq 0$
3. $I(1) = 0$
4. $I(p_1 p_2) = I(p_1) + I(p_2)$

(4) has the unique solution $I(p) = -k \log p$ and (2) implies $k > 0$, with (1) and (3) then automatically true. Choice of k corresponds to the choice of units: $k = 1/\log 2$ gives $I(p) = -\log_2 p$ in bits.

Shannon entropy and its relation to Gibbs entropy

If querying a source of information can result in N outcomes $X_1, \dots, X_N$ of a random variable X with probabilities $p_1, \dots, p_N$, then the information content of the source is given by the average surprise

$$H(X) = \sum_{i=1}^{N} p_i I(p_i) = -\sum_{i=1}^{N} p_i \log p_i$$

Maximum possible entropy is for equiprobable events

$$p_i = \frac{1}{N} \;\Rightarrow H = \log N$$

Note that (up to a factor k_B which specifies the units) this is exactly the thermodynamics Gibbs entropy associated to a system which can be in N states of a statistical ensemble with probabilities $p_1, \dots, p_N$. Therefore, the Gibbs entropy just quantifies our ignorance about the state of the system i.e. how much we learn if we ascertain its true state.

Remark: the Gibbs entropy can be defined far away from thermal equilibrium, and can be shown to agree to the classical Clausius definition

$$\delta S = \frac{\delta Q}{T}$$

Relative entropy

Taking two random variables X and Y, suppose we learn that $Y = y_j$. Then the remaining information we can extract from measuring X is given by

$$H(X|y_j) = -\sum_i p(x_i|y_j) \log p(x_i|y_j)$$

Averaging over all outcomes for Y gives the relative entropy

$$H(X|Y) = -\sum_{i,j} p(y_j)p(x_i|y_j)\log p(x_i|y_j) = -\sum_{i,j} p(x_i, y_j)\log \frac{p(x_i, y_j)}{p(y_j)}$$

which quantifies the information in X which is independent of Y.

Properties:

5. $H(X, Y) = H(X|Y) + H(Y) = H(Y|X) + H(X)$

   where the entropy of the joint system is
   $H(X, Y) = -\sum_{i,j} p(x_i, y_j)\log p(x_i, y_j)$
6. If $Y = f(X)$ then $H(f(X)|X) = 0$ so we get

   $$H(X|f(X)) + H(f(X)) = H(f(X)|X) + H(X) \Rightarrow H(f(X)) \leq H(X)$$

   i.e. the entropy of a variable can only decrease when it is passed through a function.

7. X and Y are independent $\Rightarrow H(X|Y) = H(X)$

8. $H(X, Y) \leq H(X) + H(Y)$

Mutual information

How much information does X contain besides the one not determined by Y?

$$I(X; Y) = H(X) - H(X|Y) = H(X) + H(Y) - H(X, Y)$$

It is non-negative: $I(X; Y) \geq 0$ and symmetric: $I(X; Y) = I(Y; X)$

Mutual information quantifies the information contained in both X and Y i.e. it is a measure of mutual dependence (correlation) between the variables.

Von Neumann entropy

Mixed state: we are ignorant about true state of system
$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Von Neumann entropy:
$$S(\rho) = -\mathrm{Tr}\,\rho\log\rho = -\sum_i p_i \log p_i$$
Let us assume we have a bipartite system $\mathrm{H}_A \otimes \mathrm{H}_B$ which is in a pure state $|\Psi\rangle$. Any pure state can be written using the Hilbert-Schmidt decomposition
$$|\Psi\rangle = \sum_i \sqrt{p_i}|\psi_i\rangle_A \otimes |\phi_i\rangle_B \quad \langle\psi_i|\psi_j\rangle = \delta_{ij} = \langle\phi_i|\phi_j\rangle \;\; p_i \geq 0$$

$\langle \Psi | \Psi \rangle = 1 \Rightarrow \sum_i p_i = 1$

Partial states:

$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi| = \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad \rho_B = \mathrm{Tr}_A |\Psi\rangle\langle\Psi| = \sum_i p_i |\phi_i\rangle\langle\phi_i|$

Entropies of the full system and the subsystems:

$S_{AB} = S(|\Psi\rangle\langle\Psi|) = 0 \quad S_A = S(\rho_A) = -\sum_i p_i \log p_i \quad S_B = S(\rho_B) = -\sum_i p_i \log p_i$

Mutual information:

$I_{AB} = S_A + S_B - S_{AB} = 2S_A = 2S_B$!!

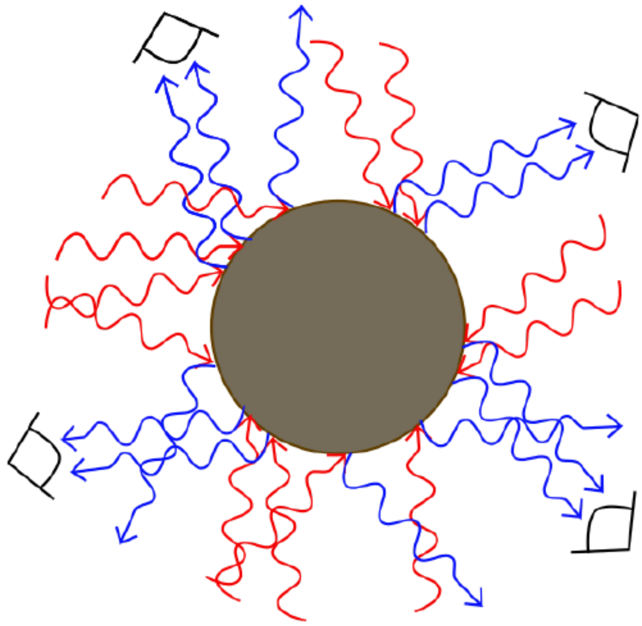
The composite system has twice as much mutual information as can be stored in either of the subsystems! This is much different from the classical case.

In fact, the von Neumann entropy measures entanglement between the subsystems - which is a highly quantum mechanical phenomenon.

Note that by measuring any of the subsystems only half of the mutual information can be extracted - i.e. we can only access as much as allowed by classical theory. The rest of the correlations is irreducibly quantum - it is related to the fact that albeit the subsystems are in a mixed state, the full system is in a pure state (cf. also purification).

Lecture 13

Branching states



Typically, the environment consists of many subsystems, and observers intercept only a small part of the environment to learn about the state of physical objects (they do not need to interact with the object directly at all!).

Considering the wave-function of the system with the environment, it is a mathematical theorem that is can always be written as a Hilbert-Schmidt decomposition

$$|\Psi\rangle_{SE} = \sum_i \lambda_i |s_i\rangle|\mathcal{E}_i\rangle \quad \lambda_i \geq 0 \quad \sum_i \lambda_i^2 = 1$$

where $\langle s_i|s_j\rangle = \delta_{ij} = \langle \mathcal{E}_i|\mathcal{E}_j\rangle$.

But this is only guaranteed for a bi-partitioned system. If the Hilbert space is a product of more than 2 factors, it is only mathematically guaranteed that a general state can be decomposed into a multiple sum such as

$$|\Psi\rangle_{SE_1\ldots E_n} = \sum_{i,i_1,\ldots,i_N} C_{ii_1\ldots i_N} |s_i\rangle \left|\varepsilon_{i_1}^{(1)}\right\rangle \ldots \left|\varepsilon_{i_N}^{(N)}\right\rangle$$

Digression: why is the Hilbert-Schmidt decomposition guaranteed for bi-partite systems? We can start from the guaranteed decomposition

$$|\Psi\rangle_{SE} = \sum_{i,j} C_{ij} |\sigma_i\rangle|\epsilon_j\rangle$$

Using the theorem of singular value decomposition in linear algebra, there always exist unitary matrices U and V such that the matrix C can be written as

$$C = U \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_d \end{pmatrix} V \quad \text{with} \quad \lambda_i \geq 0$$

so

$$|\Psi\rangle_{SE} = \sum_i \lambda_i |s_i\rangle|e_i\rangle \quad |s_i\rangle = \sum_k U_{ki}|\sigma_k\rangle \quad |e_i\rangle = \sum_k V_{ik}|\epsilon_k\rangle$$

However, we can proceed by subdividing the environment into many pieces. Then decoherence results in a state of the form

$$|\Psi\rangle_{SE_1\ldots E_n} = \sum_i \lambda_i |s_i\rangle \left|\varepsilon_i^{(1)}\right\rangle \ldots \left|\varepsilon_i^{(N)}\right\rangle$$

This is a very special sort of state in the multi-factor Hilbert space called a **branching state**: in such a state the wave function decomposes into branches that are orthogonal to each other to a very high precision.

Branching states and (proto-)locality

Why does decoherence result in a branching state? The reason is that the Hamiltonians describing fundamental physics are in a special class: they have a "proto-local" structure.

Proto-local structure: denote the total Hilbert space by $\mathcal{H}_U$ (U for Universe) and the Hamiltonian by H_U. It has a proto-local structure if there exists a factorisation of the Hilbert space

$$\mathcal{H}_U = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots$$

such that when expanding the Hamilton into terms which depend on 1, 2, etc. factors as

$$H = \sum_i H_i^{(1)} + \sum_{i_1,i_2} H_{i_1 i_2}^{(2)} + \sum_{i_1,i_2,i_3} H_{i_1 i_2 i_3}^{(3)} + \ldots$$

with $H_{i_1\ldots i_n}^{(N)} \in \mathcal{B}\left(\mathcal{H}_{i_1} \otimes \ldots \otimes \mathcal{H}_{i_n}\right)$ corresponding to "n-body" interactions, the expansion terminates after the first few terms. Many fundamental Hamiltonians have only terms up to $n = 2$ so elementary interaction steps only involve subsystems in a pairwise fashion. For the argument it is probably enough that the terms decrease fast enough with n although I know no theorems in this regard.

In addition, observations show that the Universe satisfies what is called the Past Hypothesis, which means that entropy was low in the past, i.e. for the initial conditions at the time of the Big Bang. In our Universe, this is the origin of the global arrow of time.

For us this can be stated by assuming that the system starts from a state that has a product form in the above decomposition i.e.

$$|\Psi_0\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \ldots \quad \text{with} \quad |\psi_i\rangle \in \mathcal{H}_i$$

so that the initial state has a low entanglement between the subsystems. Prominent examples of decoherence such as position localisation is a time evolution which proceeds in individual scattering events which always only entangle the system with a given subsystem of the environment. Since every step happens in a bi-partite subsystem, we can always re-diagonalise the state in that subsystem and so the structure of the wave function becomes

$$|\Psi\rangle_{SE_1\ldots E_n} = \sum_i \lambda_i |s_i\rangle \left|\varepsilon_i^{(1)}\right\rangle \ldots \left|\varepsilon_i^{(N)}\right\rangle$$

Note that here the product structure is

$$\mathcal{H}_S \otimes \mathcal{H}_{E_1} \otimes \dots \otimes \mathcal{H}_{E_n}$$

and each environmental factor is composed of many elementary "proto-local" Hilbert space factors. This requires an appropriate coarse graining.

Such a time evolution can be verified in detail in concrete models of decoherence. In fact, this is probably what we mean by decoherence in the first place! There are many physical processes we do not call decoherence as they do not lead to a branching wave function.

In my view, decoherence is a special class of physical processes for which the result is a branching wave function. Positional localisation is one example: in each individual scattering event, the position wave function is entangled with a single photon, and it is rapidly entangled with a huge number of independent photons in the environment, each being a part of a different sub-environment E_i which label different spatial regions in which the scattered photons end up.

Why proto-local?

This is my terminology ☺ In the real world, the natural index in which the Hilbert space factorizes is position, because interactions usually fall off with distance. So, one can think of a structure

$$\mathcal{H}_U = \bigotimes_{V_i} \mathcal{H}_{V_i}$$

where the V_i are small volume cells and $\mathcal{H}_{V_i}$ are the degrees of freedom localised in them. To keep things safe, I do not assume factorisation to individual points is possible.

It is a very interesting thought, however, to start from a form

$$\mathcal{H}_U = \bigotimes_{q \in M} \mathcal{H}_q$$

and ask whether the index set M can be endowed with a geometry. One needs a distance notion here; such a notion is provided by entanglement: the distance between "proto-points" q_1 and q_2 can be chosen as a monotonically decreasing function of the entanglement between $\mathcal{H}_{q_1}$ and $\mathcal{H}_{q_2}$. One can therefore attempt to derive geometry from entanglement, and then even the Einstein equations. Those interested are referred to S.M. Carroll and A. Singh, arXiv: 1801.08132 and references therein, and also M. Van Raamsdonk, arXiv: 1005.3035 and 1809.01197. Similar thoughts can also be formulated in the framework of tensor networks (used in simulating quantum many-body systems), see e.g. B. Swingle arXiv: 1209.3304.

Pointer states, quasi-classical variables and redundant records

Using a suitable coarse-graining of the environment into subsystems, under which the decomposition of the wave-function has the branching form

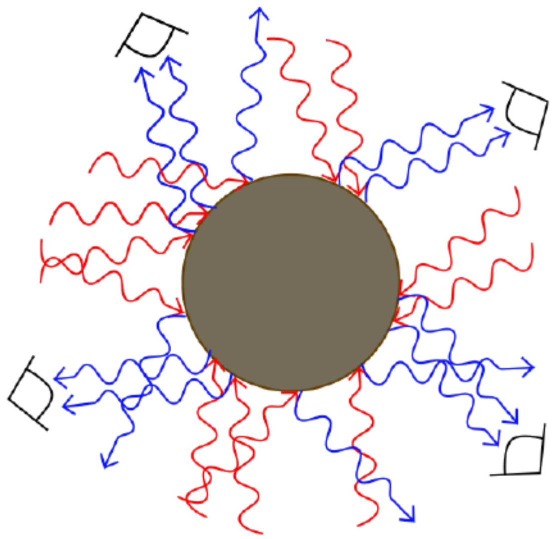
$$|\Psi\rangle_{SE_1\dots E_n} = \sum_i \lambda_i |s_i\rangle \left|\varepsilon_i^{(1)}\right\rangle \dots \left|\varepsilon_i^{(N)}\right\rangle$$

with N sufficiently large, the states $|s_i\rangle$ are pointer states of the system S which survive the constant "monitoring" by the environment: "quantum darwinism" (Zurek).

Quasi-classical variables: observables built from projectors on pointer states.

For such observables (apart from extremely short time scales) the system does not exist in superpositions corresponding to values distinguishable with any reasonable finite precision. Therefore, their dynamics is classical, and deterministic - apart from the environmental noise which leads to small fluctuations and also dissipation.

Measuring a quasi-classical observable



- It is to determine which branch of the wave-function is realised.
- This can be decided by measuring any one of the environmental "record keeping states" $\left|\varepsilon_i^{(a)}\right\rangle$, so there is no need for direct interaction with S and this is what happens in practice.
- Although we destroy one particular record, other observers can verify our observations by checking another record $\left|\varepsilon_i^{(b)}\right\rangle$ and we are going to agree.

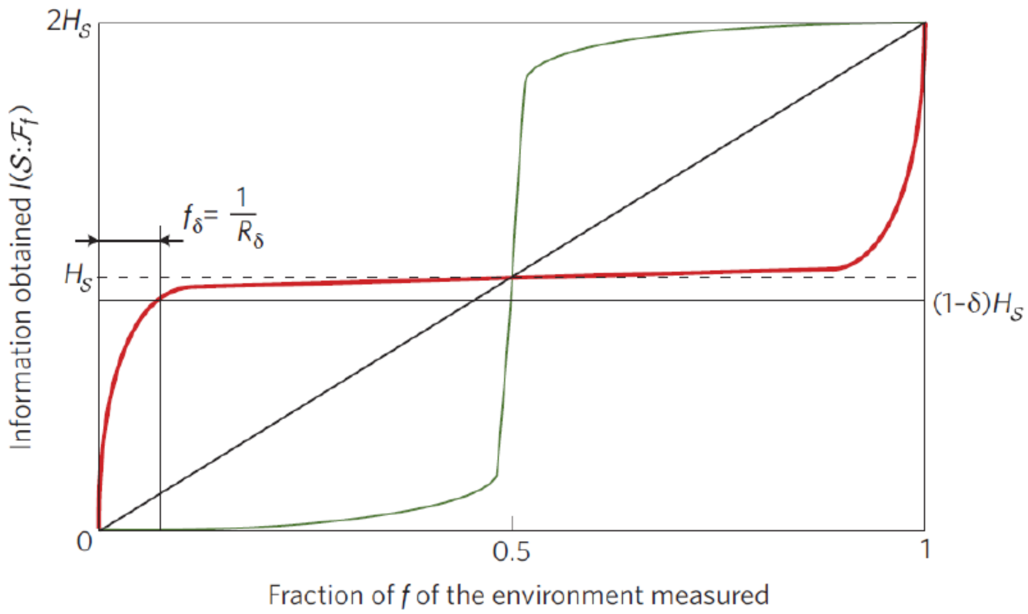
We (observers) and all the other sub-environments are an (astronomically) large number of Wigner's friends which always agree about the result for quasi-classical observables. This is guaranteed by the branching form

$$|\Psi\rangle_{SE_1\dots E_n} = \sum_i \lambda_i |s_i\rangle \left|\varepsilon_i^{(1)}\right\rangle \dots \left|\varepsilon_i^{(N)}\right\rangle$$

Therefore, quasi-classical observables are those for which multiple records exist. Being quasi-classical is a matter of degree - the more records it has, the more classical the observable is. The most classical ones are the "Darwinian" winners of the environment-induced superselection - called **einselection** by Zurek.

Partial information plot

We can compute the mutual information between the system S and n subsystems of the environment as a function of the fraction $f = n/N$ of the environment we use to get information about the system. Typical result:



(from W.H. Zurek, Nature Physics 5: 181-188, 2009.)
Redundancy: $R = 1/f$

Green line: pure state picked out in random from $\mathcal{H}_S$ - necessary to measure half the environment to get any information, but once we get halfway, we get the rest very soon.
Red line: decohered state - a small fraction of the environment gives the classically available info $H(S)$, but the full info $2H(S)$ can only be extracted if we measure all the environment - the quantum correlations carried by entanglements are delocalised globally around the environment.

E.g. for quantum Brownian motion: $I\left(S{:}E_f\right) \approx H(S) + (1/2)\log f/(1-f)$

Environment as both censor and witness:

- Selects the states we have verifiable information about. This selection is dynamical and depends on the Hamiltonian of the full system (Universe);
- Creates multiple records so that for these degrees of freedom we can cross-check and compare notes with others!

Quantum theory: the fundamental postulates

Quantum postulates:

(i) States are vectors in Hilbert space
(ii) Time evolution is unitary
(iii) Immediate repetition of measurement gives the same result
(iv) Outcomes correspond to eigenstates of Hermitian operators
(v) Probabilities are given by the Born's rule.

Zurek: (iii-v) can be derived from decoherence and einvariance (environment assisted invariance).

(iii): guaranteed by decoherence, but one can only choose from a limited set of observables (environment censorship) and follows from existence of (redundant) environmental records.

Argument for (iv)

Assume there are two pointer states of S $|u\rangle$ and $|v\rangle$ which get entangled with the environment which starts in an initial state $|\mathcal{E}_0\rangle$:

$$|u\rangle|\mathcal{E}_0\rangle \to |u\rangle|\mathcal{E}_u\rangle \quad |v\rangle|\mathcal{E}_0\rangle \to |v\rangle|\mathcal{E}_v\rangle$$

If the environment stores any information at all about these vectors, then $|\langle\mathcal{E}_u|\mathcal{E}_v\rangle| < 1$. On the other hand, since the time evolution of the system + environment is unitary, the scalar product is preserved:

$$\langle u|v\rangle \underbrace{\langle\mathcal{E}_0|\mathcal{E}_0\rangle}_{1} = \langle u|v\rangle\langle\mathcal{E}_u|\mathcal{E}_v\rangle \;\Rightarrow \langle u|v\rangle = 0 \text{ !!!}$$

So, states corresponding to different outcomes are orthogonal therefore there exists a Hermitian operator of which they are the eigenstates. (The eigenvalues of the Hermitian operator can be chosen by us - they are simply labelling the different outcomes!)

Note: despite the no-cloning theorem, the information about the pointer states can be copied without limitation.
The no-cloning theorem only forbids unitary operation that can copy any state (even superpositions) in the Hilbert space, while here copying is restricted to particular elements which are orthogonal to each other.

Argument for (v): deriving Born's rule

Let us assume that in our branching state we have n terms have the same weight

$$|\Psi\rangle_{SE_fE_{1-f}} = \sum_{i=1}^{n} \frac{1}{\sqrt{n}} |s_i\rangle \left|\mathcal{E}_i^{(f)}\right\rangle \left|\mathcal{E}_i^{(1-f)}\right\rangle$$

Here E_f is the fraction $f \ll 1$ of the environment that we intercept to do our manipulations, and E_{1-f} is the rest (that we do not even know about anything at all). Now we cannot predict the outcome of the time evolution which we call measurement, since it is only the total wave-function that obeys a deterministic (unitary) evolution, but not separately any subsystem of it.

Given our ignorance about the environment, how are we to assign probabilities to the outcomes then? We could try to argue by taking the partial trace over the environment, but in fact that is circular: the partial trace prescription was derived from Born's rule itself!

Note: we can consider a unitary "swap" σ_{jk} operation on $S \otimes E_f$ which exchanges the outcomes j and k. Then we have

$$
\begin{aligned}
\left(\sigma_{jk} \otimes \mathbf{1}\right)|\Psi\rangle_{SE_f E_{1-f}} = &\sum_{i \neq j,k} \frac{1}{\sqrt{n}}|s_i\rangle \left|\mathcal{E}_i^{(f)}\right\rangle \left|\mathcal{E}_i^{(1-f)}\right\rangle + \frac{1}{\sqrt{n}}|s_j\rangle \left|\mathcal{E}_j^{(f)}\right\rangle \left|\mathcal{E}_k^{(1-f)}\right\rangle \\
&+ \frac{1}{\sqrt{n}}|s_k\rangle \left|\mathcal{E}_k^{(f)}\right\rangle \left|\mathcal{E}_j^{(1-f)}\right\rangle
\end{aligned}
$$

Now the probabilities p_j and p_k are swapped since their association to the ignored environment E_{1-f} is swapped. However, this swap can be undone by a unitary swap Σ_{jk} performed on the ignored environment which swaps $\left|\mathcal{E}_k^{(1-f)}\right\rangle$ and $\left|\mathcal{E}_j^{(1-f)}\right\rangle$:

$$
\left(\mathbf{1}_{E_f} \otimes \Sigma_{jk}\right)\left(\sigma_{jk} \otimes \mathbf{1}_{E_{1-f}}\right)|\Psi\rangle_{SE_f E_{1-f}} = |\Psi\rangle_{SE_f E_{1-f}}
$$

However, since we by assumption do not know anything about the environment, this operation is totally unobservable for us. Therefore $p_j = p_k$, i.e. all outcomes are equiprobable and so

$$
p_j = \frac{1}{n}
$$

Analogy: this is the way we get the probabilities for an unbiased coin or die from symmetry considerations. However, here the symmetry is a quantum symmetry, and our ignorance is quantum ignorance (i.e. the unknown information is not purely classical, but also involves entanglement with the ignored environment E_{1-f}.

Now what if the amplitudes are different? For simplicity, consider only two branches and also suppress the sub-environment notation. Assume that

$$
|\Psi\rangle_{SE} = \sqrt{\frac{m_1}{N}}|s_1\rangle|\mathcal{E}_1\rangle + \sqrt{\frac{m_2}{N}}|s_2\rangle|\mathcal{E}_2\rangle \quad m_1 + m_2 = N
$$

We can then subdivide the environment by introducing orthogonal ancilla states

$$
|\mathcal{E}_1\rangle = \sqrt{\frac{1}{m_1}}|\mathcal{E}_{11}\rangle + \cdots + \sqrt{\frac{1}{m_1}}|\mathcal{E}_{1m_1}\rangle \quad |\mathcal{E}_2\rangle = \sqrt{\frac{1}{m_2}}|\mathcal{E}_{21}\rangle + \cdots + \sqrt{\frac{1}{m_2}}|\mathcal{E}_{2m_2}\rangle
$$

This corresponds to fine graining the environment, which is really huge, so this is not a problem!

Now this is mapped to the case of equiprobable outcomes, each having probability $1/N$, m_1 of which corresponds to $|s_1\rangle$ and m_2 of which corresponds to $|s_2\rangle$. We conclude that the probabilities of the outcomes are

$$
|s_1\rangle\text{: } \frac{m_1}{N} \quad |s_2\rangle\text{: } \frac{m_2}{N}
$$

Again, Born's rule rulez!

Now the only case is when the coefficients are not square roots of rational numbers, but then we can use that rational numbers are dense among reals and that the probability assignment is expected to be continuous.

So, we have derived Born's rule… or have we?

My take on this: yes, we have, but we must keep track of the assumptions:

1. We assume the quantum postulates (i) and (ii)

2. We assume that the Hamiltonian is proto-local, the number of subsystems is very large, and that it allows for decoherence processes which result in branched wave-functions (this needs the Past Hypothesis as well).
   Note: this part is justified by our experimental knowledge of the world.

3. We use the ignorance derivation of probabilities, which reflects a particular philosophical approach to the meaning of probabilities.

4. We must also note that the information we ignore is not just classical correlations with E_{1-f} but also quantum, which is an extension of the ignorance derivation (however plausible, it is still an extension!)

Then we derive (iii) repetition; (iv) outcomes correspond to eigenstates of Hermitian operators and that the only consistent probability assignment is (v) Born's rule.

Quasi-classical realms in a quantum universe

1. The quantum Universe is huge - most of its degrees of freedom are hidden from us and play the role of the recording station - many little quantum scribes busily making notes and never being appreciated for their absolutely vital contribution to our existence :)

2. A small (but still huge) portion of degrees of freedom - determined by the Universe's Hamiltonian - leave redundant records via decoherence and together they form our quasi-classical realm. Are there multiple disconnected quasi-classical realms? Maybe… but the others are then really totally disconnected from ours.

3. The quasi-classical realm is necessary for faithful, repeatable copying of information - a necessary condition for the evolution of beings like us (think about genetic info in DNA). The ever-present noise supplies the necessary mutations, and because the dynamics is mostly predictable, adaptation is possible.

4. Therefore, living beings are configurations of quasi-classical degrees of freedom, and we only have access to the pointer variables - the rest is censored from us. (This does not exclude however that quantum effects are important for life at the molecular level, e.g. photosynthesis).

5. We can observe quantum phenomena by observing sufficiently isolated subsystems (atoms, nuclei etc.) for phenomena with time scales shorter than their decoherence time scale.

6. Even for these quantum phenomena, we can only observe selected observables - these are the ones which can be coupled to pointer variables by the design of the experiment. They result in outcomes observable, repeatable and verifiable for us, and so we can do science with them.

Even though the Quantum Sea is huge, the portion within our horizon it is still finite (bounded by the so-called de Sitter entropy) - which means that the available memory is finite. So, there are only finitely many details of finitely many events that can ever be recorded, and the Universe can only perform finitely many quantum operations, before reaching the maximum entropy state - cf. with 19th century "heat death".

Consistent histories and redundant records

Recall quantum histories

$$\chi_\alpha = P_n^{(\alpha_n)}(t_n) \dots P_1^{(\alpha_1)}(t_1)$$

$$P_k^{(\alpha_k)}(t_k) = e^{-iHt_k} P_k^{(\alpha_k)} e^{iHt_k} \qquad \sum_{\alpha_k=1}^{d_k} P_k^{(\alpha_k)} = \mathbf{1} \quad \Rightarrow \quad \sum_\alpha \chi_\alpha = \mathbf{1}$$

Weak consistency: condition for probabilistic interpretation

$$\text{Re Tr } \chi_\alpha \rho \chi_{\alpha'}^\dagger = 0 \qquad \alpha \neq \alpha'$$

Probability of a given history is $p_\alpha = \text{Tr } \chi_\alpha \rho \chi_\alpha^\dagger$

Medium consistency:

$$D(\alpha, \alpha') = \text{Tr } \chi_\alpha \rho \chi_{\alpha'}^\dagger = 0 \qquad \alpha \neq \alpha'$$

This behaves well under composition of systems: if we have systems A and B with histories χ_α^A and χ_β^B and states ρ^A and ρ^B, then medium consistency in the subsystems implies medium consistensy for the product histories in the product state of the composite system, while this is not true for weak consistency:

$$\text{Tr}_{AB}(\chi_\alpha^A \otimes \chi_\beta^B)(\rho^A \otimes \rho^B)\left(\chi_{\alpha'}^A \otimes \chi_{\beta'}^B\right) = \text{Tr}_A\left(\chi_\alpha^A \rho^A \chi_{\alpha'}^A\right) \text{Tr}_B\left(\chi_\beta^B \rho^B \chi_{\beta'}^B\right)$$

So: $D^{AB}(\alpha\beta, \alpha'\beta') = D^A(\alpha, \alpha') D^B(\beta, \beta')$

But: $\text{Re } D^{AB}(\alpha\beta, \alpha'\beta') \neq \text{Re } D^A(\alpha, \alpha') \text{Re } D^B(\beta, \beta')$ in general!

From now on we use medium consistency and assume that the complete system + environment is in a pure state.

Consider history branches created by a (medium) consistent set of histories:

$$|\Psi_\alpha\rangle = \frac{1}{\sqrt{p_\alpha}} \chi_\alpha |\Psi\rangle$$

These can be interpreted as alternative histories of the system: they satisfy

$$|\Psi\rangle = \sum_\alpha \sqrt{p_\alpha}\, |\Psi_\alpha\rangle \quad \text{and} \quad \langle \Psi_\alpha | \Psi_\beta \rangle = 1$$

Consistent sets of histories in a system with a pure state have records: there exists a complete set of orthogonal projector operators R_α such that

$$|\Psi_\alpha\rangle = R_\alpha |\Psi\rangle$$

However, these records are not redundant and are also global in general, so to verify them one needs to perform global measurements on the Universe.

For mixed states the same is not true: medium decoherence

$$\mathrm{Tr}\, \chi_\alpha \rho \chi_{\alpha'}^\dagger = 0$$

does not imply the existence of records, which is the existence of a complete set of orthogonal projector operators R_α such that

$$\chi_\alpha \rho = R_\alpha \rho$$

although the reverse implication holds.

It can be shown that pure decoherence (with an undivided environment) leads to medium consistency for histories of the pointer variables, see C.J. Riedel, W.H. Zurek and M. Zwolak, arXiv: 1312.0331. This relies on the observation that the state resulting from pure decoherence

$$|\Psi\rangle_{SE} = \sum_i \lambda_i |s_i\rangle |\mathcal{E}_i\rangle$$

implies that the projectors on the environmental state $|\mathcal{E}_i\rangle$ serve as records to the system events corresponding to $|s_i\rangle$, i.e. the events $|s_i\rangle\langle s_i| \otimes \mathbf{1}_E$.

It is intuitively clear that if we build histories upon pointer states which leave multiple records in the environment, then this set of histories is objective in the sense that different observers can compare notes and find then consistent. They can also verify these histories by accessing only a small portion of the environment. This means that quasi-classical realms have an objective past which different observers can find out and on which they agree.

It is shown in arXiv: 1312.0331 that the existence of redundant records is a sufficient condition for redundant consistency, which then uniquely fixes the branch vectors. It selects, from the multitude of the alternative sets of consistent histories, a small subset endowed with redundant records characteristic of the objective classical past. The allowed sets only differ in minor details, and they lead to the same decomposition

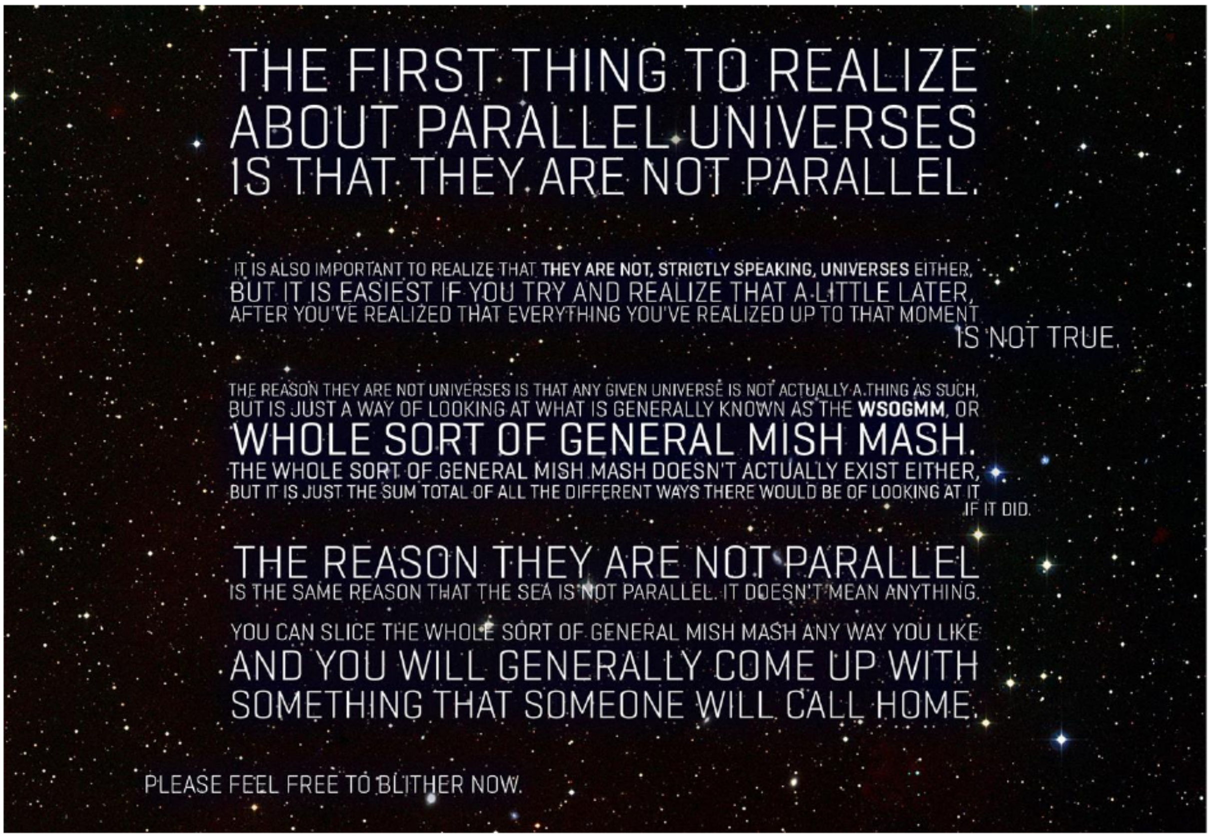
$$|\Psi\rangle = \sum_{\alpha} \sqrt{p_\alpha}\, |\Psi_\alpha\rangle \quad \text{and} \quad \langle \Psi_\alpha | \Psi_\beta \rangle = \delta_{\alpha\beta}$$

into branch vectors. The history of Universe (i.e. the branch actually realised) can then be found on by observations and different observers will agree on it.

Final note: quantum Darwinism seems to have passed its first experimental checks, see https://www.quantamagazine.org/quantum-darwinism-an-idea-to-explain-objective-reality-passes-first-tests-20190722/

Have we solved all problems?

No: the wave function still contains the branches for all the alternate histories of the Universe! This is sometimes stated as follows: pure quantum mechanics does not contain any mechanism to choose between the alternate realities – such a (hypothetical) mechanism is sometimes called the "chooser". **Quantum theory does not have a "chooser".**



Douglas Adams: *The Hitchhiker's Guide to the Galaxy*

$|\Psi_U\rangle$:	WSOGMM
Choice of the history set χ_α	The way we slice up the WSOGMM. However, contrary to D.A.'s statement it is not up to us to choose it! The environment acts as censor, we do not have much say in this.
$|\Psi_\alpha\rangle$:	branches a.k.a. "parallel worlds", decohered by the Quantum Sea

It is still open to interpretation, some of the main options are:

- Everett type (relative state) interpretations: they are sometimes called many-world, but there are several flavours.
  - Note: in each of them the quantum world is unique with a deterministic evolution! It is the history of the quasi-classical realm that has branches.
  - They are very minimal in their set of quantum postulates: only accept (i) and (ii).
- Copenhagen type: still alive and kicking :)
- Ensemble interpretation: the wave function does not describe individual systems, only ensembles of systems – cannot be applied to individual systems, especially problematic for quantum cosmology.
- Relational interpretations: states are relative to observers and describe correlations between the system and the observer.
- Qbism: the wave function describes what observers know about the reality. When performing observations, their beliefs are updated by a quantum analogue of Bayesian inference.

This motivates some people to consider modifications of quantum mechanics.

- Hidden variables: Bohmian mechanics, cellular automatons
- Collapse models
- Nonlinear quantum mechanics
  - Problems with signalling
  - Results in influence without interaction between separate systems. See T.F. Jordan, arXiv: quant-ph/0702171, arXiv: 1002.4673

However, it seems to me that no matter what we do with this, the "good old" classical Universe won't come back.

In fact

Remarks on objective collapse models

Collapse models modify quantum mechanics and introduce mechanism so that macroscopic superpositions disappear. Some such models are:

- Ghirardi–Rimini–Weber (GRW) model: constituents of physical systems undergo spontaneous collapses distributed in time according to a Poisson process. The larger the system, the stronger the collapse of the centre-of-mass wave function.
- Continuous spontaneous localization (CSL) model: collapse in position occurs continuously in time.
- Diósi–Penrose (DP) model: collapse related to gravity.

In many cases (e.g. CSL and DP), the time evolution of the density matrix is governed by a Lindblad equation. Therefore, there exists an extension of the Hilbert space on which it can be simulated by unitary dynamics. So, these can be considered open quantum systems without an explicitly specified environment.

Main problems:

- Energy conservation is violated and in fact it turns out the energy increases with time due to collapse. This can be compensated by adding dissipation, but energy would still not be strictly conserved. The positive side of this is that it leads to experimentally observable signatures.

- Compatibility with relativity is hard to achieve, no Lorentz covariant formulation exists yet.

- Tail problem: wave function always has small, but non-vanishing tails, so the system is never fully localised. Eventually there are several levels of the tail problem, and it is hard to see how to address them in a satisfactory way.
  See e.g.
  - K.J. McQueen, Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics, Volume 49: 10-18, 2015.
  - Albert D.Z., Loewer B. (1996) Tails of Schrödinger's Cat

- Collapse theories do not get rid of the branches: they just do the same job as decoherence does, but we are still left with the problem of interpretation. There is still no "chooser".

In fact, given the above problems, and also the fact that the job is already performed by decoherence, I fail to be motivated towards these theories: only some strong experimental evidence would convince me.

No experiment found any evidence for these models so far. A recent experiment looking for energy violation predicted by the DP model has come up empty:

S. Donadi et al. Underground test of gravity-related wave function collapse. *Nat. Phys.* (2020). DOI: 10.1038/s41567-020-1008-4

Bibliography

This is not an exhaustive list of references by any means, it includes ones I found useful and/or interesting – it is a subjective selection. There are some more references interspersed in the text.

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- A highly accessible and funny book about EPR-Bell correlations, entanglement, polytopes and related stuff.

Bohmian mechanics with spin

T. Norsen: *The Pilot-Wave Perspective on Spin*, American Journal of Physics 82: 337-348, 2014.

Consistent histories

R.B. Griffiths: *Consistent Quantum Theory*, Cambridge University Press, 2002.
- Textbook, lots of detailed explanations, discusses many quantum paradoxes.

J.B. Hartle: *The Quasiclassical Realms of This Quantum Universe*, Found Phys (2011) 41: 982–1006.
- A nice review paper about the emergence of quasi-classical realms, combining histories, decoherence, hydrodynamics, thermodynamics and the emergence of a classical space-time.

J.B. Hartle: *Quantum Buzzwords*, arXiv: 2012.05229
- An overview of the histories approach placing it into the context of textbook (Copenhagen) quantum mechanics, Everett's approach and quantum cosmology.

A. Kent: *Consistent Sets Yield Contrary Inferences in Quantum Theory*, Phys. Rev. Lett. 78 (1997) 2874–2877.
- Paper showing that if one only requires (weak) consistency, it is possible to choose frameworks to arrive at contradictory inferences about the past.

Copenhagen interpretation using consistent histories

R. Omnés: *The Interpretation of Quantum Mechanics*, Princeton Univ. Press, 1994.
- A comprehensive monograph, very detailed.

R. Omnés: *Understanding Quantum Mechanics*, Princeton Univ. Press, 1999.
- A less technical, more introductory text.

Decoherence

E. Joos and H.D. Zeh: *The Emergence of Classical Properties Through Interaction with the Environment*, Z. Phys. B - Condensed Matter 59: 223-243, 1985.
- A classical paper, introduces positional localisation by decoherence.

C. Kiefer and E. Joos: *Decoherence: Concepts and Examples*, quant-ph/9803052
- A good short introduction to the fundamental concepts and examples.

M. Schlosshauer: *Decoherence and the Quantum-To-Classical Transition*, Springer-Verlag Berlin Heidelberg 2007
- Textbook, very detailed explanation and review, including an overview of the role of decoherence in different interpretations.

E. Joos, H.D. Zeh, C. Kiefer, D. Giulini, J. Kupsch and I.-O. Stamatescu: *Decoherence and the Appearance of a Classical World in Quantum Theory*, Springer-Verlag Berlin Heidelberg 1996, 2003
- Monograph, authors include the great classics.

Master equation for open quantum systems

D. Manzano: *A short introduction to the Lindblad Master Equation*, arXiv: 1906.04478
- A pedagogical introduction to the Lindblad equation.

H.-P. Breuer, E.-M. Laine, J. Piilo and B. Vacchini: *Non-Markovian dynamics in open quantum systems*, Rev. Mod. Phys. 88: 021002, 2016.
- Review of non-Markovian dynamics of open quantum systems.

Quantum Darwinism

W.H. Zurek: *Decoherence, einselection, and the quantum origins of the classical*, Rev. Mod. Phys. 75(3) 715-775, 2003.
- The definitive guide to einselection, einvariance and quantum Darwinism. It also contains a good exposition of decoherence in chaotic systems. However, it misses the latest developments of the last two decades.

W.H. Zurek: *Quantum theory of the classical: quantum jumps, Born's Rule and objective classical reality via quantum Darwinism*, Phil. Trans. R. Soc. A 376: 20180107, 2018.
- A very nice, concise review of quantum Darwinism and the emergence of the classical world, with a FAQ section.

W.H. Zurek: *Quantum Darwinism, Decoherence, and the Randomness of Quantum Jumps*, Physics Today 67: 44-50, 2014.
- A shorter review of quantum Darwinism and the emergence of the classical world.

C.J. Riedel, W.H. Zurek and M. Zwolak: *The Objective past of a quantum universe: Redundant records of consistent histories*, Phys. Rev. A 93: 032126, 2016.
- Quantum Darwinism applied to consistent histories.

Collapse models

M. Carlesso and S. Donadi: *Collapse models: main properties and the state of art of the experimental tests*, Springer Proceedings in Physics 237: 1 (2019)
- An up-to-date summary of the status of the main classes of spontaneous collapse models.

L. Diósi: *Models for universal reduction of macroscopic quantum fluctuations*, Phys. Rev. A40: 1165, 1989
R. Penrose: *On Gravity's Role in Quantum State Reduction*, General Relativity and Gravitation 8(5): 581-600, 1996.
- The Diósi-Penrose model of gravitational collapse.

Multiverses

R. Bousso and L. Susskind: *The Multiverse Interpretation of Quantum Mechanics*, Physical Review D85: 045007, 2012.
A. Aguirre and M. Tegmark: *Born in an Infinite Universe: a Cosmological Interpretation of Quantum Mechanics*, Physical Review Rev. D84: 105002, 2011.
- Two interesting, but **very** speculative takes on quantum mechanics and the multiverse of cosmology.

Online sources

There are some very good Wikipedia pages on several topics.

For those interested in philosophical aspects, the Stanford Encyclopedia of Philosophy is a very good start, containing reviews of all major interpretations and numerous relevant topics.