Complex networks

Hierarchy, core-periphery

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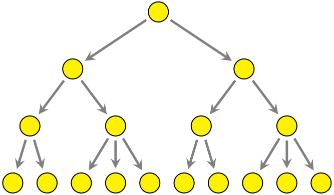
Mesoscopic structures

- Communities
- Core-periphery
- Layers

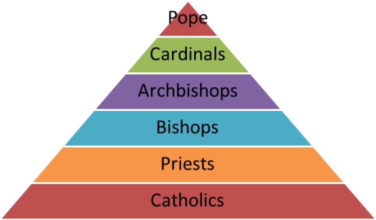


Hierarchy

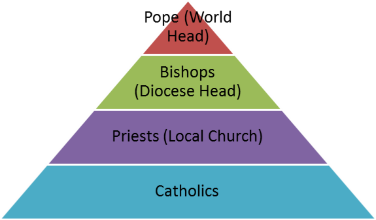
- Greek: hierarhia ($\iota\varepsilon\rho\alpha\rho\chi\acute{\iota}\alpha$) "rule of a high priest" from hierarkhes ($\iota\varepsilon\rho\alpha\rho\chi\eta\zeta$) "leader of the sacred rites"
- Used first for the word in the 5th–6th centuries for both celestial hierarchy and the ecclesiastical hierarchy by Pseudo-Dionysius the Areopagite
- In English 19th century

Catholic Church

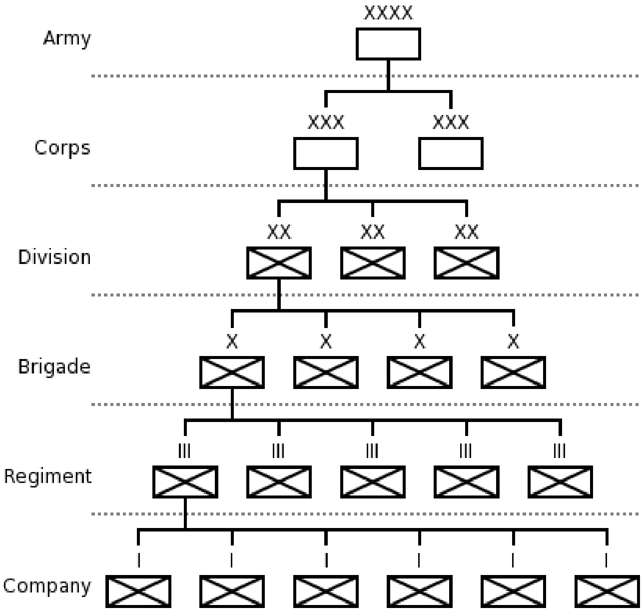
- Misconception:



- Actual hierarchy:

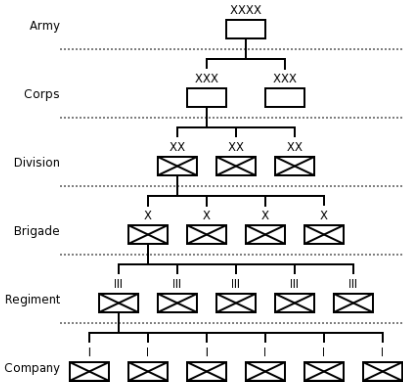


Army

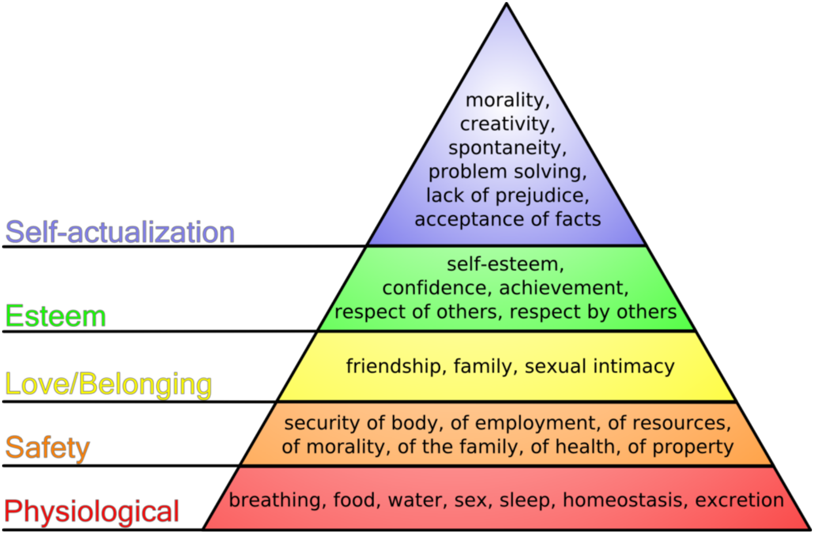

XXXX
Army
XXX
XXX
Corps
XX
XX
XX
Division
X
X
X
X
Brigade
III
III
III
III
III
Regiment
I
I
I
I
I
I
Company

Army

- This may be the flow of commands but does not represent interactions and flow of info.

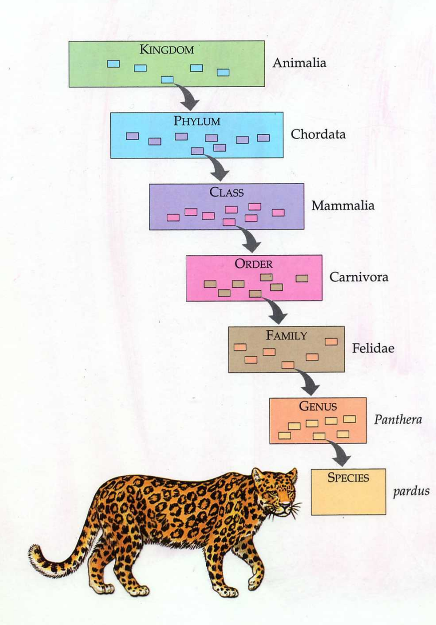


Hierarchy of needs

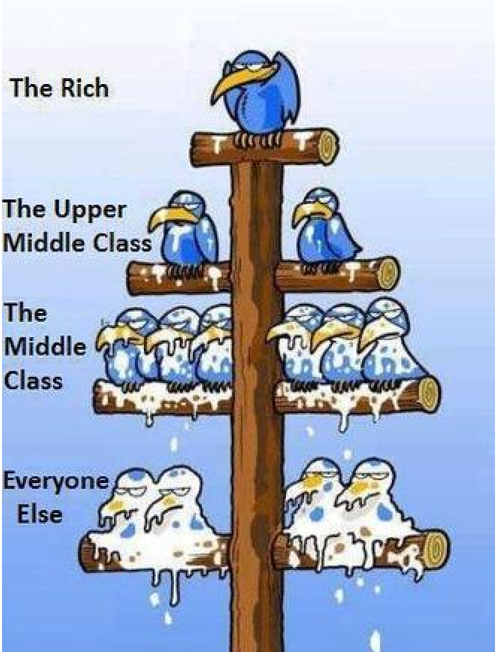


Nested hierarchy: Evolution

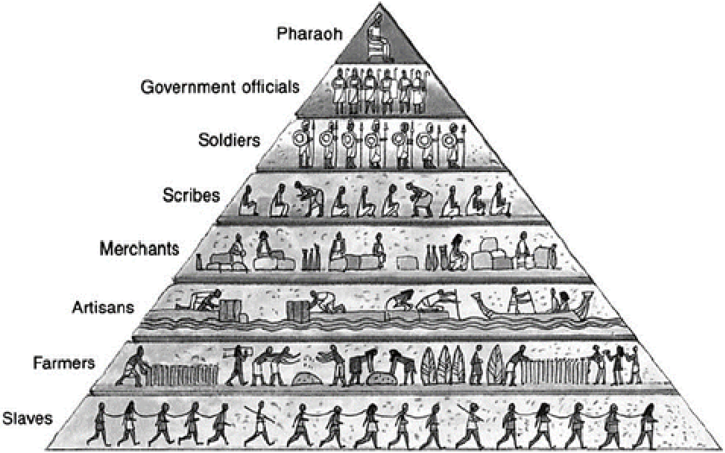
Classifying life (Figure 1.8)



Social hierarchy

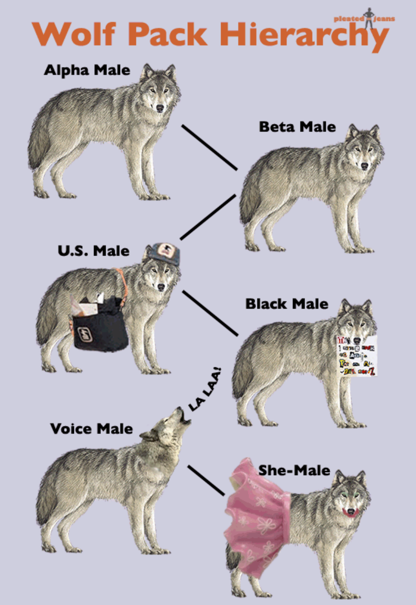


Social hierarchy

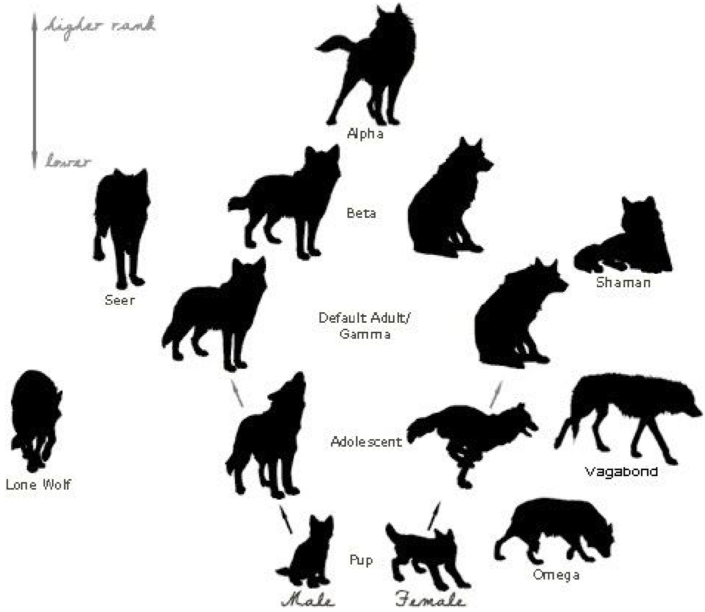


Source: Barry K. Beyer et al., *The World Around Us: Eastern Hemisphere*, MacMillan Publishing (adapted)

Animal hierarchy: Sorry :-)



Animal hierarchy



Benefit of hierarchy

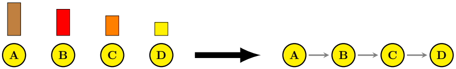
- Evolutionary benefit:
  - Selects the fittest and gives highest chance to reproduce
  - Increases efficiency to solve tasks for the group
  - Positive feedback loop: Getting better access to resources strengthens position
  - Complex tasks need organization of work (c.f. flight of a swarm, attacking a big animal, etc.)
  - The drive of lower rank to become upper rank gives a vivid dynamics, which accelerates natural selection.
- Dominance vs. prestige
- Power vs. status
- Status hierarchy vs. decision hierarchy
- Hierarchies are ubiquitous in human society

Hierarchy

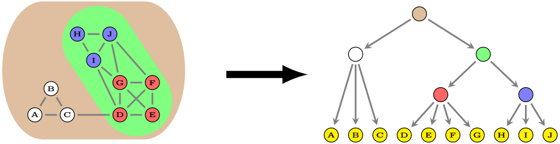
- Is hierarchy a widespread feature of complex systems organization?
- What types of hierarchies do exist?
- Are hierarchies the result of selection pressures or, conversely, do they arise as a by-product of structural constraints?
- How to detect hierarchy?

Types of hierarchy

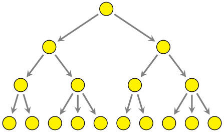
- Order hierarchy



- Nested hierarchy

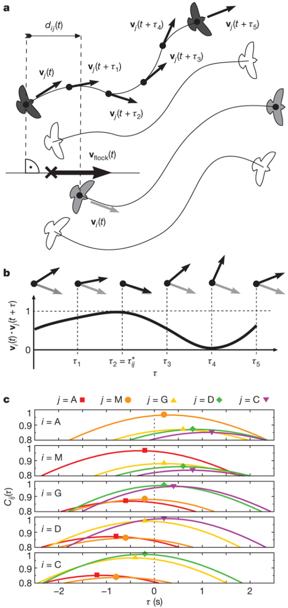


- Flow hierarchy



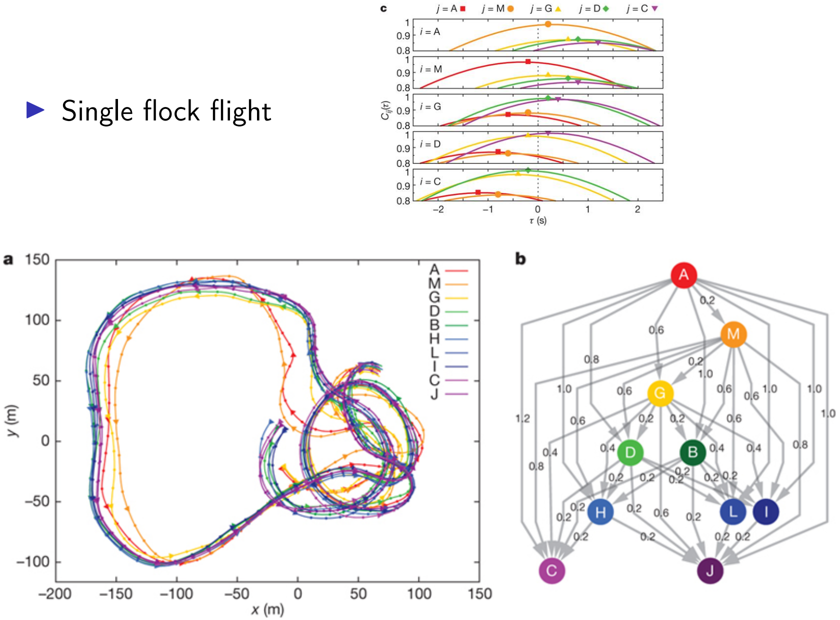
Direction of hierarchy in real systems

- Pigeon flocks (Nagy, ...Vicsek Nature, 2013)
- 10 birds with GPS recorder
- Free flight and homing
- Correlation in pairwise velocity offset by τ
- Time delay of maximal correlation indicates who is following who



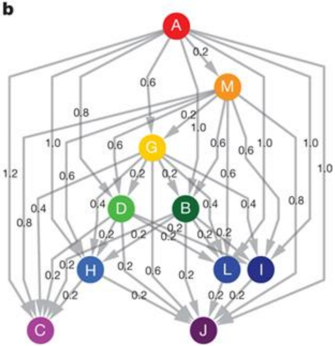
Pigeon flocks

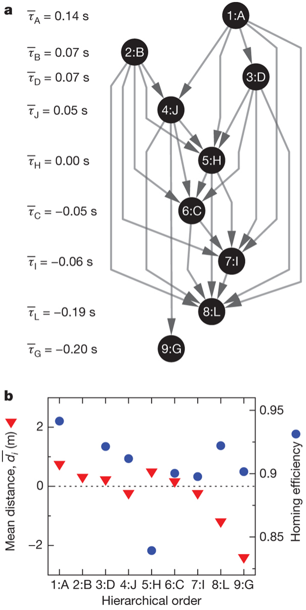
- Single flock flight



Pigeon hierarchy: homing efficiency?

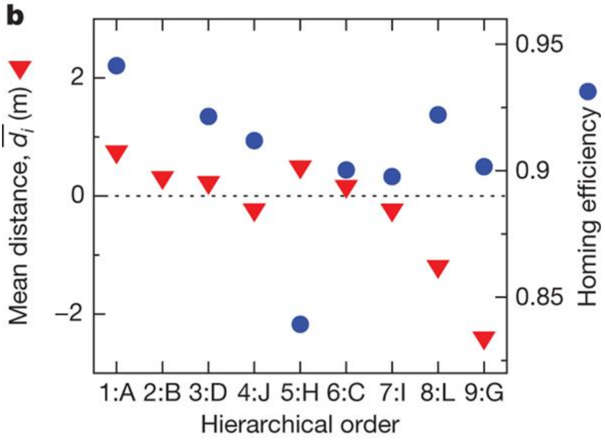
- Multiple flock flight





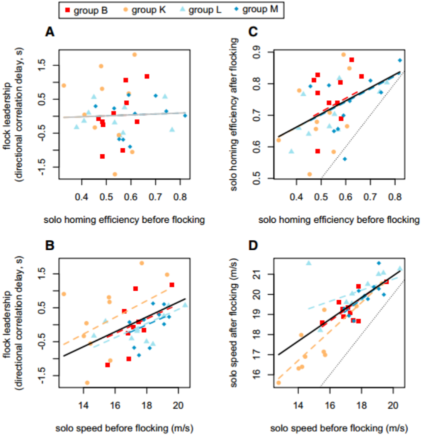
Pigeon hierarchy: homing efficiency?

- Is there any correlation between homing efficiency and leadership?



Pigeon hierarchy: homing efficiency or speed?

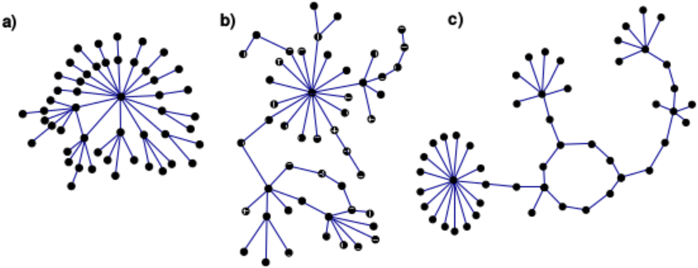
- Is there any correlation between homing efficiency/speed and leadership?



Random scale-free graph as null model

- Start with a configuration model with given degree distribution
- *Maximally hierarchical*: Connect nodes with decreasing k
- *Maximal anti-hierarchical*: Node with the highest degree is connected to the lowest degree, observing that the network should remain connected
- *Random*: What the word says

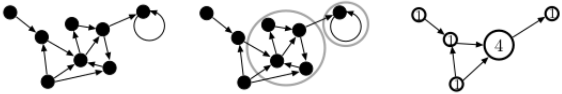
Maximally hierarchical   Random   Maximal anti-hierarchical



Directed Acyclic Graphs (DAG)

- Directed graph
- Contains no cycles (no path returns to the same node)
- Obtaining a DAG from any directed graph
- Replace each cycle with a single node

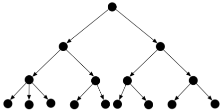
- Node can be weighted α_i sum of the nodes in the cycle (node weighted condensed graph, or DAG)

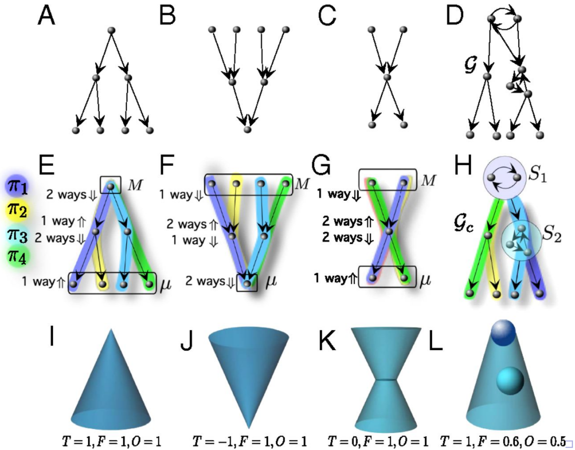


Direction of hierarchy: Main measures

- Treeness: T
- Feedforwardness: F
- Orderability: O

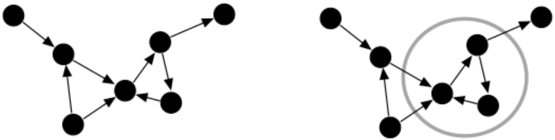
Deviation from perfect hierarchy





Orderability

- Fraction of nodes not part of a cycle
- $O = 4/7$ for the following example:

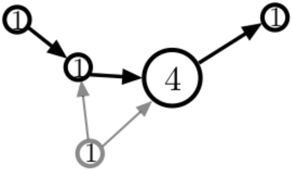


- Other def: fraction of nodes with weight 1 in the *node weighted condensed graph*
- $O = 4/8$ for the following example:

Feedforwardness

- M: set of *maximal nodes*, without incoming links in DAG
- μ: set of *minimal nodes*, without outgoing links in DAG
- $\pi_i \in \Pi$: (set of) path(s) going from a maximal node to a minimal one
- Calculate the ratio of the length of the path and the sum of node strength along it:

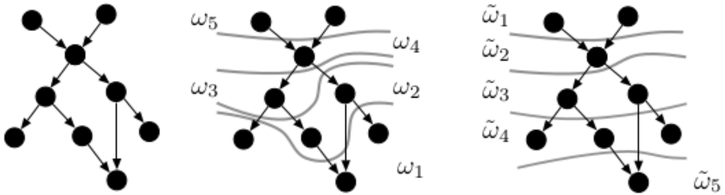
$$F(\pi_i) = \frac{\sum_{v_i \in \pi_i} 1}{\sum_{v_i \in \pi_i} \alpha_i}$$



- For the sample $F(\pi_i) = 4/7$
- Feedforwardness F average for all paths for all layers. (Path is always starting from a maximal node)

Layers

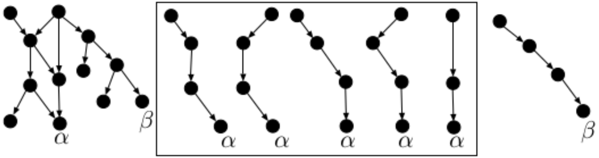
- Start from DAG
- Remove all nodes with no outgoing nodes. They are layer one
- Continue. This is normal (backward) layering
- Forward layering is by removing nodes with no incoming links



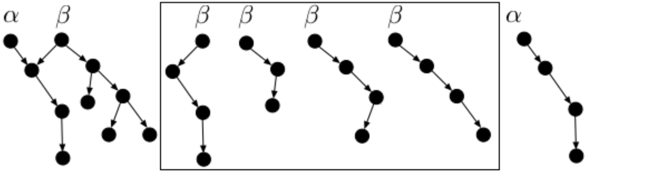
- Feedforwardness is defined on the backward layering (good question why?)
- O and F are not independent

Treeness

- Calculate uncertainty of paths both forwards and backwards
- endpoint α has an uncertainty of 5, β has 1



- endpoint β has an uncertainty of 5, α has 1



Treeness

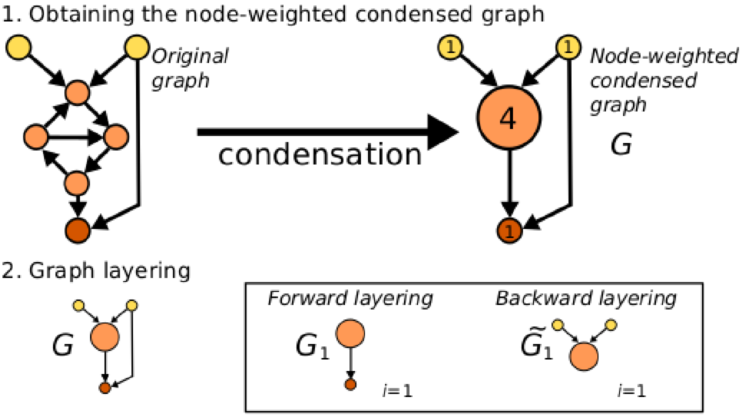
- Calculate uncertainty of paths both forwards and backwards
- Forward entropy: For each starting node
- Calculate the number of probability of starting towards a given neighbor $(1/k_i^{out})$
- Calculate the the uncertainty from the from the chosen neighbor node.
- Multiply the two, this is $P(\pi_k|v_i)$
- Sum up the entropy normalized by the number of starting points

$$H_f = -\frac{1}{|M|}\sum_{\pi_k, v_i} P(\pi_k|v_i)\log P(\pi_k|v_i)$$

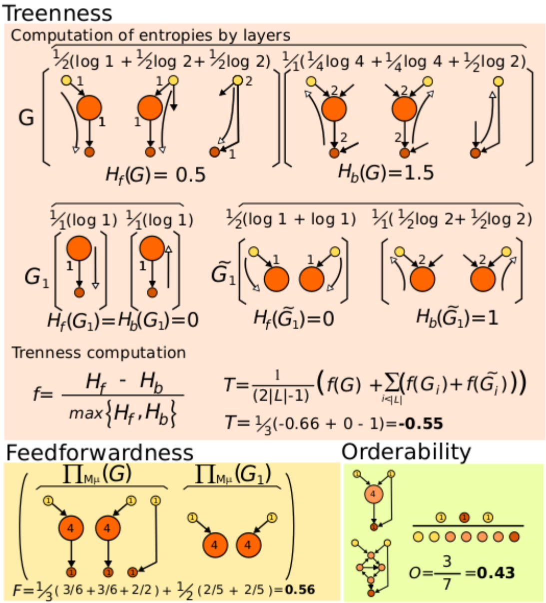
- Do it also for backwards

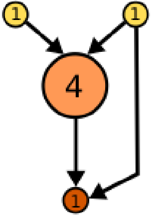
Calculating measures: an example

- DAG condensation and layers:

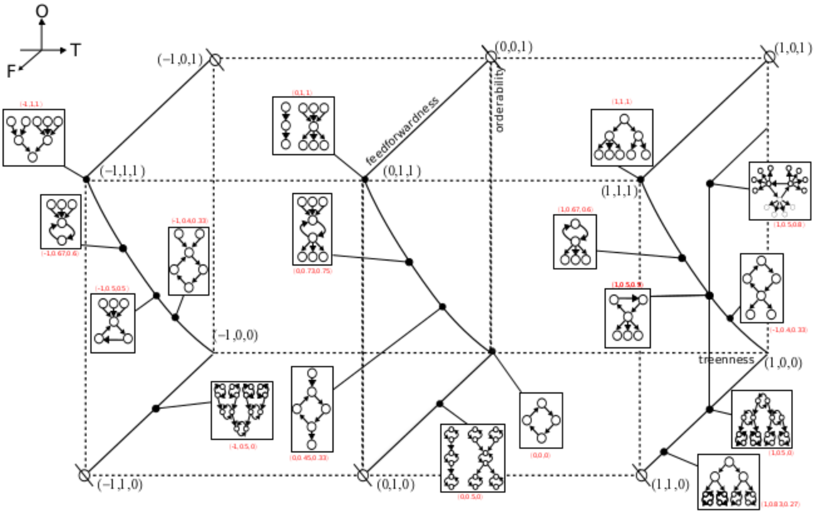


Calculating measures: an example



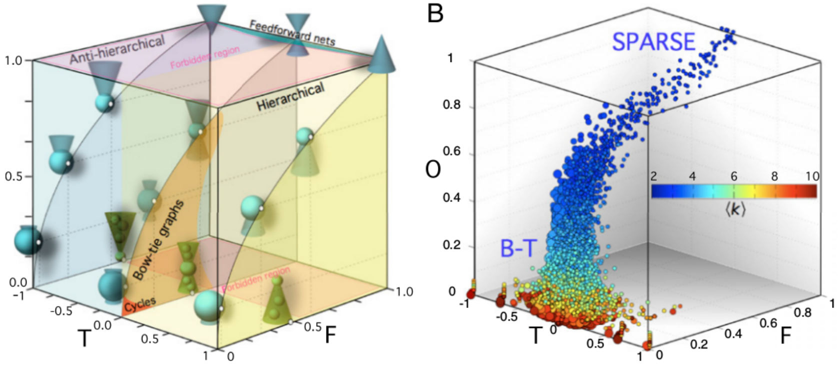


Hierarchy measures: The *TOF* space



Hierarchy measures: The *TOF* space

Random network results



The *TOF* space: examples

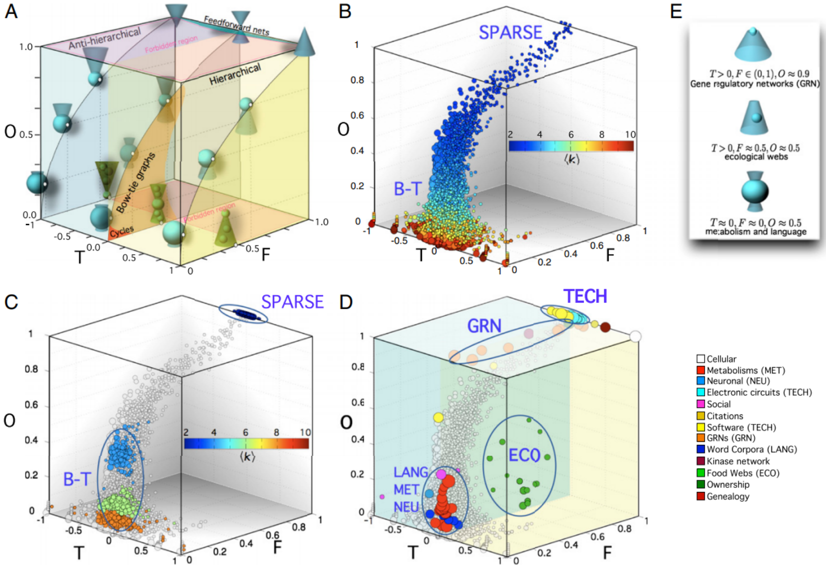


Fig. 2. The morphospace of possible hierarchies Ω. (*A*) Different morphologies and their respective location withins Ω (Fig 1). Green icons represent unlikely configurations (see *SI Appendix* for more information). (*B* and *C*) The occupation of Ω by an ensemble of random models. This set includes Erdös–Rényi (ER) graphs with different sizes (100, 250, and 500) and average degrees $\langle k \rangle$ (see color bar). Symbols are proportional to network size. (*C*) Morphospace occupation of Callaway's network model overlapped with the ER ensemble as a reference. Three network sizes (100, 250, and 500) and four connectivities ($\langle k \rangle = 2, 4, 6, 8$)

Other methods

- Example: email network
  1. number of emails
  2. average response time
  3. response score
  4. number of cliques
  5. raw clique score
  6. weighted clique score
  7. degree centrality
  8. clustering coefficient
  9. mean of shortest path length from a specific vertex to all vertices in the graph
  10. betweenness centrality
  11. Hubs-and-Authorities importance
- Give score

Other methods

- Example: email network
- Give score
    1. Rank users from most important to least important
    2. Group users which have similar social scores and clique connectivity
    3. Determine n different levels (or echelons) of social hierarchy within which to place all the users. This is a clustering step, and n can be bounded.

The Enron email network example

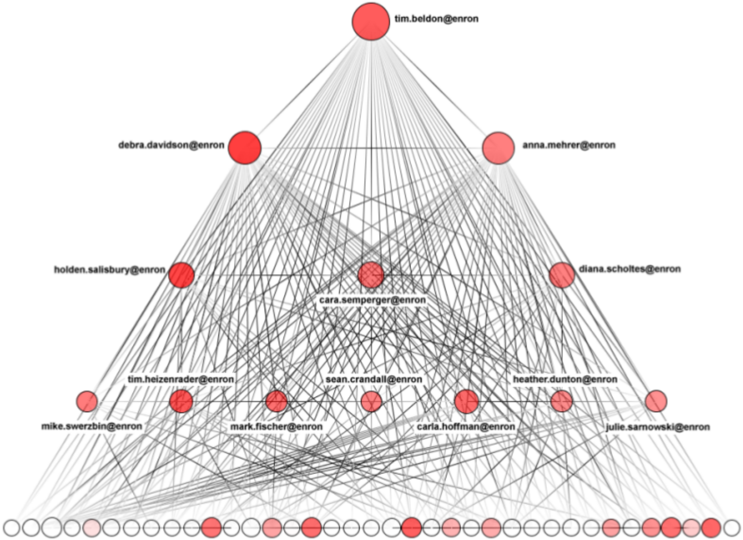
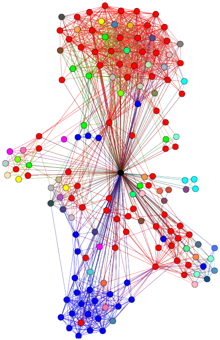
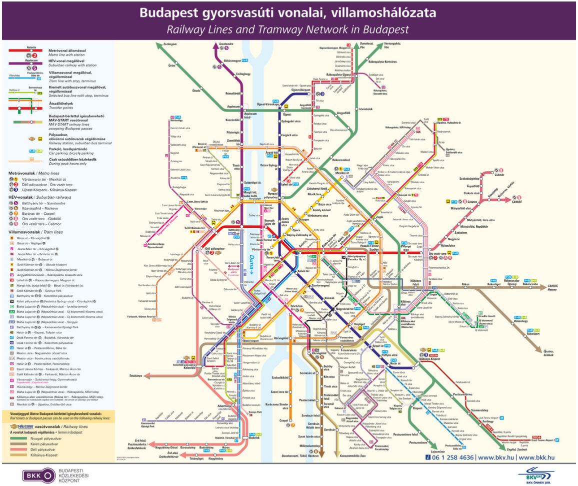


Figure 1: Enron North American West Power Traders Extracted Social Network

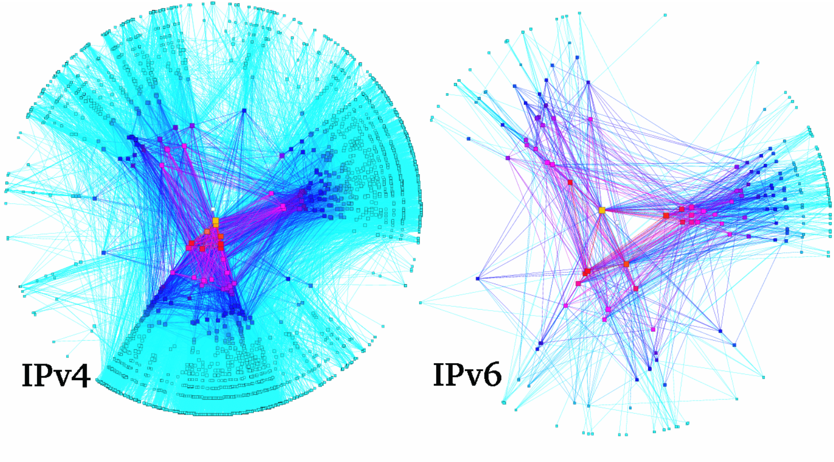
Mesoscopic structures

- Communities
- Core-periphery
- Layers



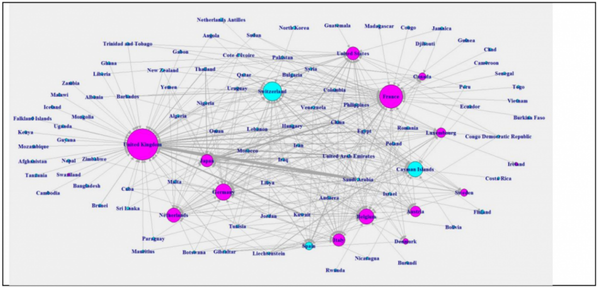


Core-periphery: E.g. Internet routing

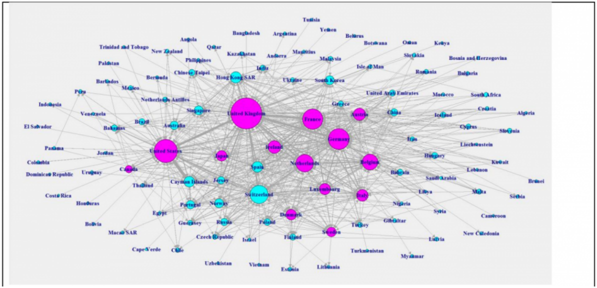


Core-periphery: E.g. Bank network

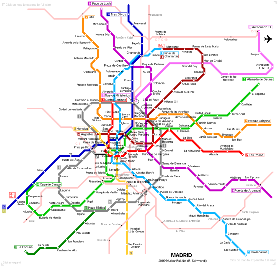
1990:



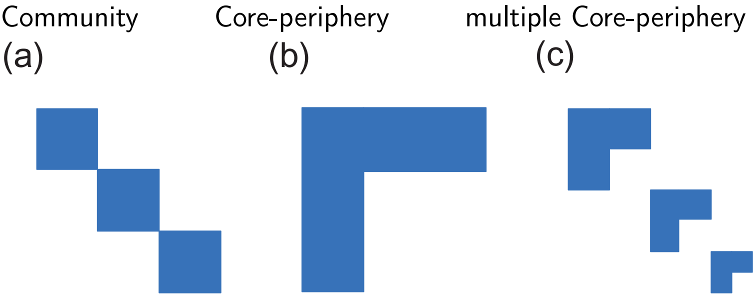
2007:



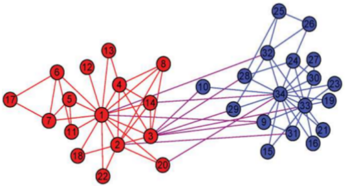
Core-periphery: E.g. Subway network

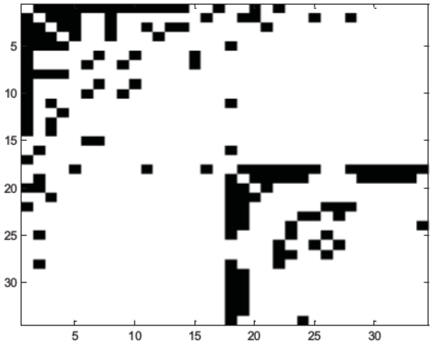


Core-periphery, onion structures: adjacency matrix

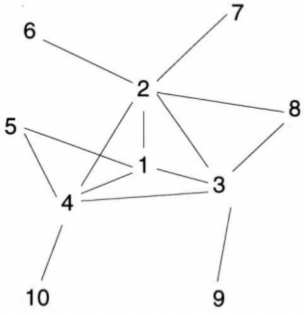


Multiple Core-periphery: e.g. Zakhary





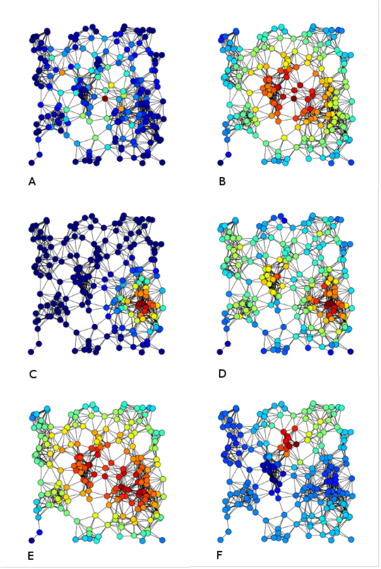
Core-periphery: Simple synthetic example



	1	2	3	4	5	6	7	8	9	10
1		1	1	1	1	0	0	0	0	0
2	1		1	1	0	1	1	1	0	0
3	1	1		1	0	0	0	1	1	0
4	1	1	1		1	0	0	0	0	1
5	1	0	0	1		0	0	0	0	0
6	0	1	0	0	0		0	0	0	0
7	0	1	0	0	0	0		0	0	0
8	0	1	1	0	0	0	0		0	0
9	0	0	1	0	0	0	0	0		0
10	0	0	0	1	0	0	0	0	0	

Core: Definition

- Core: part with high centrality
- Threshold on centrality
- Is it enough?



Core: Definition

Many ways to define densely connected parts in a network:

Name of dense network structure	**Definition**	**References**
Clique	a complete subgraph of size k, where complete means that any two of the k elements are connected with each other	[36,37]
***k*-clan**	a maximal connected subgraph having a subgraph-diameter $\leq k$, where the subgraph-diameter is the maximal number of links amongst the shortest paths *inside* the subgraph connecting any two elements of the subgraph	[37-39]
***k*-club**	a connected subgraph, where the distance between elements of the subgraph $\leq k$, and where no further elements can be added that have a distance $\leq k$ from all the existing elements of the subgraph	[37-39]
***k*-clique**	a maximal connected subgraph having a diameter $\leq$ k, where the diameter is the maximal number of links amongst the shortest paths (including those *outside* the subgraph), which connect any two elements of the subgraph	[37-40]

Core: Definition

***k*-clique community**	a union of all cliques with *k* elements that can be reached from each other through a series of adjacent cliques with *k* elements, where two adjacent cliques with *k* elements share *k*-1 elements *(please note that in this definition the term k-clique is also often used, which means a clique with k elements, and not the k-clique as defined in this set of definitions; the definition may be extended to include variable overlap between cliques)*	[41,42]
***k*-component**	a maximal connected subgraph, where all possible partitions of the subgraph must cut at least *k* edges	[43]
***k*-plex**	a maximal connected subgraph, where each of the *n* elements of the subgraph is linked to at least *n-k* other elements in the same subgraph	[37,44]
strong LS-set	a maximal connected subgraph, where each subset of elements of the subgraph (including the individual elements themselves) have more connections with other elements of the subgraph than with elements outside the subgraph	[37,45]

Core: Definition

LS-set	a maximal connected subgraph, where each element of the subgraph has more connections with other elements of the subgraph than with elements outside of the subgraph	[37,45,46]
lambda-set	a maximal connected subgraph, where each element of the subgraph has a larger element-connectivity with other elements of the subgraph than with elements outside of the subgraph (where element-connectivity means the minimum number of elements that must be removed from the network in order to leave no path between the two elements)	[37,47]
weak (modified) LS-set	a maximal connected subgraph, where the sum of the inter-modular links of the subgraph is smaller than the sum of the intra-modular edges	[37,45]
***k*-truss or *k*-dense subgraph**	the largest subgraph, where every edge is contained in at least (k–2) triangles within the subgraph	[48-51]
***k*-core**	a maximal connected subgraph, where the elements of the subgraph are connected to at least k other elements of the same subgraph; alternatively: the union of all k-shells with indices greater or equal k, where the k-shell is defined as the set of consecutively removed nodes and belonging links having a degree $\leq k$	[37,45,52]

Table after Csermely et al. 2013

Core: Discrete definition

- Borgatti-Everett
- Define *Core*: C_i, $i \in [1, N]$

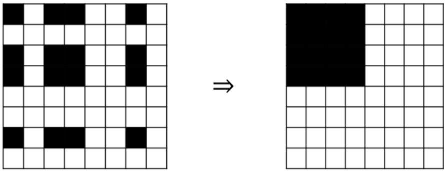
$$C_i = \begin{cases} 1 & \text{if } i \in \text{core} \\ 0 & \text{otherwise} \end{cases}$$

- e.g. $\mathsf{C} = \{0,1,0,1,1,1,0,0,0,0,0\}$, where 1 stands for core nodes
- $C_{ij} = C_i C_j$
- Maximize the overlap between C_{ij} and the adjacency matrix

$$\sum_{ij} A_{ij} C_{ij} = \max$$

Core: Discrete definition

- Borgatti-Everett
- Maximize the overlap between C_{ij} and A_{ij}



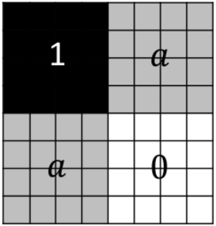
Core: Modified discrete definition

- Define *Core*: C_i, $i \in [1, N]$

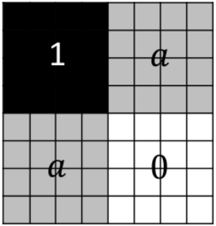
$$C_i = \begin{cases} 1 & \text{if } i \in \text{core} \\ 0 & \text{otherwise} \end{cases}$$

- Let s $= \{0, a, 1\}$ a three dimensional vector with $s(0)=0$, $s(1)=a$, $s(2)=1$
- $C_{ij} = s(C_i + C_j)$
- Minimize

$$\sum_{ij, i \neq j} (A_{ij} - C_{ij})^2 = \min$$



Core: Modified discrete definition



- Minimize

$$\sum_{ij,i\neq j}(A_{ij}-C_{ij})^2=\min$$

- Either use standard stochastic optimization
- Or use implicit iterative method:

$$C_i=\frac{\sum_{j,i\neq j}(A_{ij}-C_{ij})^2}{\sum_{ij,i\neq j}C_{ij}^2}$$

Core: Modified continuous method

- Rombach et al.
- Core is never so disjoint
- Instead of a step function use a smooth one: $g(i)$
- Use two parameters
  - α sharpness ($\alpha = 1$ previous case with step function)
  - β relative size of the core
- Many such functions e.g.

$$g(i) = \frac{1}{2}\mathrm{erf}\left(\frac{\beta - i/N}{1-\alpha}\right) + \frac{1}{2}$$

- Instead:

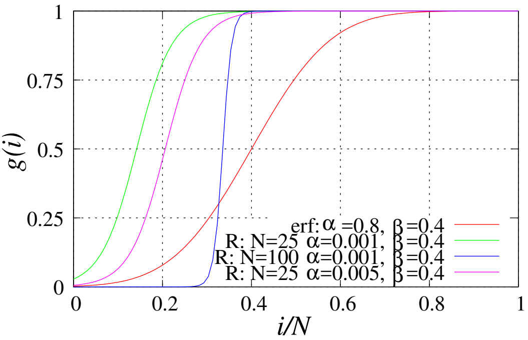
$$g(i) = \frac{1}{1 + \exp[-(i - N\beta)] \tan(\alpha\pi/2)}$$

- For latter, $g(i) = 0.5$ depends on N, α, (*No comment...*)

Core: Modified continuous method

$$g(i) = \frac{1}{2}\mathrm{erf}\left(\frac{\beta - i/N}{1-\alpha}\right) + \frac{1}{2}$$

$$g(i) = \frac{1}{1 + \exp[-(i - N\beta)]\tan(\alpha\pi/2)}$$



Core: Methods/score

1. method
   - Start from (α, β)
   - Maximize $R_{\alpha,\beta} = \sum_{ij} A_{ij} C_{ij}$
   - Then find optimal (α^*, β^*)
2. method
   - Use a two dimensional set of (α, β)
   - Maximize $R_{\alpha,\beta} = \sum_{ij} A_{ij} C_{ij}$ for each (α, β) pair
   - Aggreagate results

$$CS(i) = \frac{1}{Z} \sum_{\alpha\beta} R_{\alpha,\beta} C_{\alpha,\beta}(i)$$

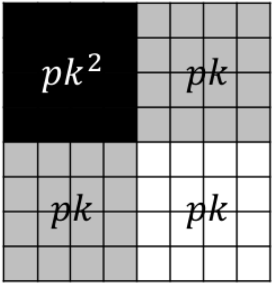
     where $Z = \max_i \sum_{\alpha\beta} R_{\alpha,\beta} C_{\alpha,\beta}(i)$
   - This gives a *core score* for each node

Core score example: network scientists

NNS2006 Node	Core score	NNS2010 Node	Core score
Barabási, A.-L.	1.00	Barabási, A.-L.	1.00
Oltvai, Z. N.	0.97	Newman, M. E. J.	0.94
Jeong, H.	0.96	Pastor-Satorras, R.	0.93
Vicsek, T.	0.95	Latora, V.	0.93
Kurths, J.	0.88	Arenas, A.	0.93
Neda, Z.	0.87	Moreno, Y.	0.92
Ravasz, E.	0.86	Jeong, H.	0.92
Newman, M. E. J.	0.86	Vespignani, A.	0.91
Pastor-Satorras, R.	0.85	Díaz-Guilera, A.	0.90
Schubert, A.	0.85	Guimerà, R.	0.90
Boccaletti, S.	0.85	Watts, D. J.	0.89
Vespignani, A.	0.84	Vazquez, A.	0.89
Farkas, I.	0.84	Viczek, T.	0.89
Derenyi, I.	0.83	Amaral, L. A. N.	0.89
Holme, P.	0.82	Solé, R. V.	0.88
Crucitti, P.	0.81	Albert, R.	0.87
Albert, R.	0.80	Kahng, B.	0.87
Schnitzler, A.	0.80	Boccaletti, S.	0.86
Solé, R.	0.80	Oltvai, Z. N.	0.86
Rosenblum, M.	0.79	Barthelemy, M.	0.85
Tomkins, A.	0.79	Kurths, J.	0.84
Moreno, Y.	0.78	Fortunato, S.	0.84
Latora, V.	0.78	Marchiori, M.	0.83
Rajagopalan, S.	0.78	Kertész, J.	0.83
Raghavan, P.	0.77	Caldarelli, G.	0.82
Pikovsky, A.	0.76	Dorogovtsev, S. N.	0.81
Kahng, B.	0.75	Boguñá, M.	0.80
Díaz-Guilera, A.	0.74	Goh, K. I.	0.80
Vazquez, A.	0.74	Crucitti, P.	0.80
Kim, B.	0.74	Strogatz, S. H.	0.80

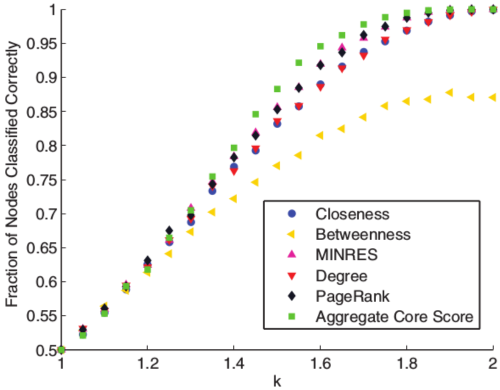
Core score: Benchmarks

- The good old Block model:
- $CP(N, d, p, k)$: N number of nodes, dN in the core, and other parameters such that:



Core score: Comparison

- Yet another centrality measure?

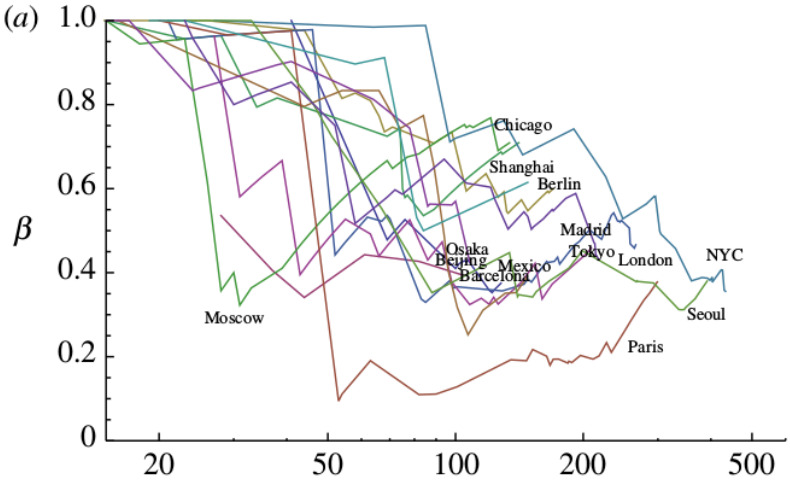


$CP(100, 0.5, 0.25, k)$

Metro lines

city	P (millions)	N_{L}	N	β (%)
Beijing	19.6	9	104	39
Tokyo	12.6	13	217	43
Seoul	10.5	9	392	38
Paris	9.6	16	299	38
Mexico City	8.8	11	147	39
NYC	8.4	24	433	36
Chicago	8.3	11	141	71
London	8.2	11	266	47
Shanghai	6.9	11	148	61
Moscow	5.5	12	134	71
Berlin	3.4	10	170	60
Madrid	3.2	13	209	46
Osaka	2.6	9	108	43
Barcelona	1.6	11	128	38

Metro lines



Number of stations

Random walk based methods

- The probability that a walker at node i jumps to j

$$m_{ij} = \frac{w_{ij}}{\sum_k w_{ik}}$$

  with w_{ij} being the strength of link ij
- The probability of finding the walker in node i at time t is $\pi_i(t)$
- The stationary solution for the probability distribution of finding a walker at node i is

$$\pi_i = \frac{\sigma_i}{\sum_j \sigma_j},$$

  where $\sigma_i = \sum_j w_{ij}$

Random walk based methods: persistence probability

- Let s be a partition of the network, then the probability that the walker is in part d if it was in part c the step before:

$$u_{cd} = \frac{\sum_{i \in s_c, j \in s_d} \pi_i m_{ij}}{\sum_{i \in s_c} \pi_i}$$

$$m_{ij} = \frac{w_{ij}}{\sum_k w_{ik}}$$

- Let us define $\alpha_r = u_{rr}$ the persistence probability, as $\tau_r = (1 - \alpha_r)^{-1}$ is the escape time from part r.

Random walk based method: Algorithm

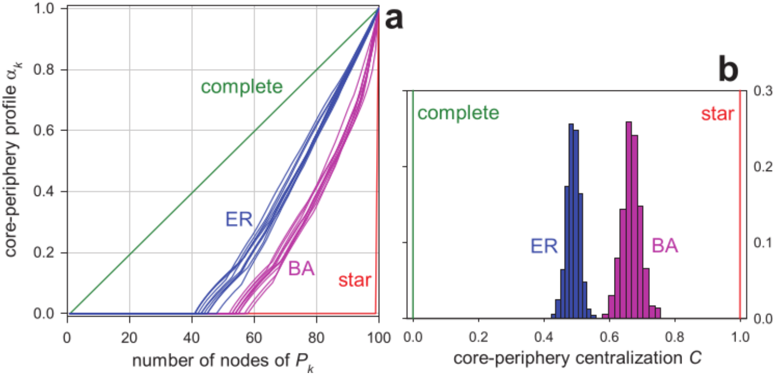
- Select at random a node i among those with minimal strength
- Set $P_1 = \{1\}$ and hence $\alpha_1 = 0$
- In the following steps chose the node (or random one from nodes) having

$$\alpha_k = \min_{h \in N \setminus P_{k-1}} \frac{\sum_{ij \in P_{k-1}} \pi_i m_{ij} + \sum_{i \in P_{k-1}} (\pi_i m) ih + \pi_h m_{hi})}{\sum_{i \in P_{k-1}} \pi_i + \pi_h}$$

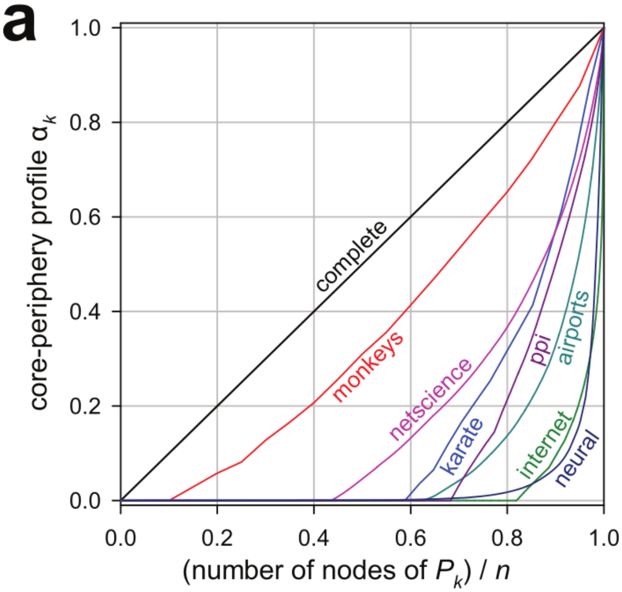
- The resulting α_i is the CP profile.
- For complete graph $\alpha_i = (k-1)/(N-1)$
- For star graph: $\alpha_i = 0$ for $i \in [1, N-1]$, and $\alpha_N = 1$
- Centralization:

$$C = 1 - \frac{2}{N-2} \sum_{k=1}^{N-1} \alpha_k$$

Core-periphery profile, centralization

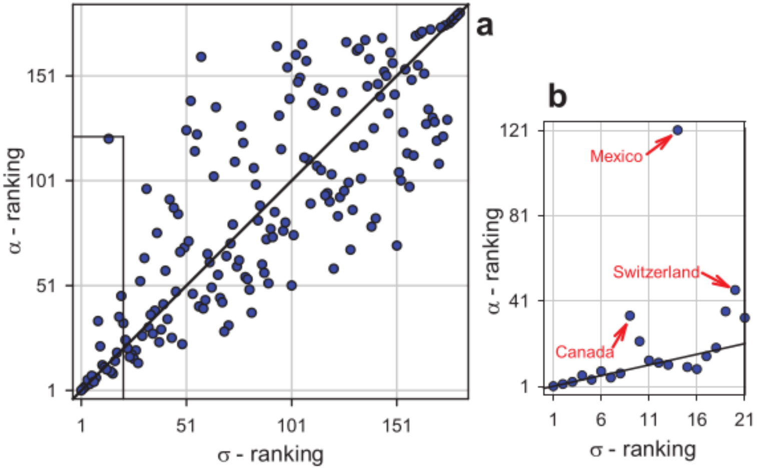


Core-periphery profile, centralization: Example



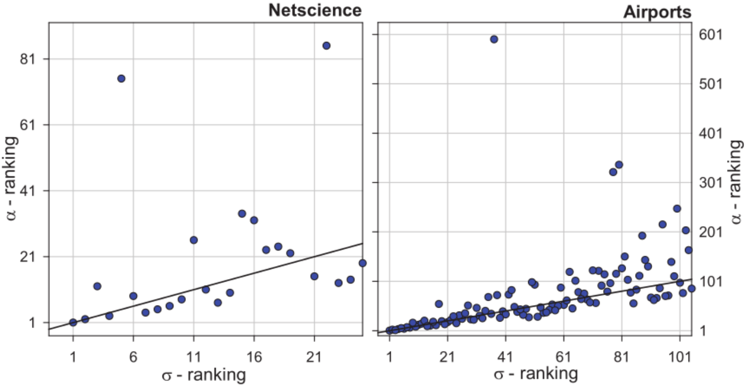
Comparison

- Biggest outlier: Mexico, huge amount of trade but only with USA $\to$ not central.



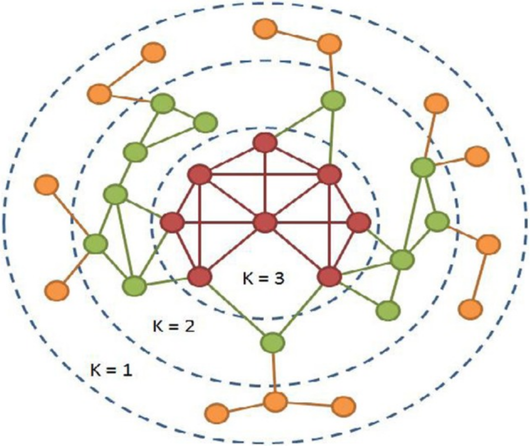
Comparison

- Strong correlation with some outliers
- This anomaly indicates peculiarities of some specific nodes



K-shell decomposition

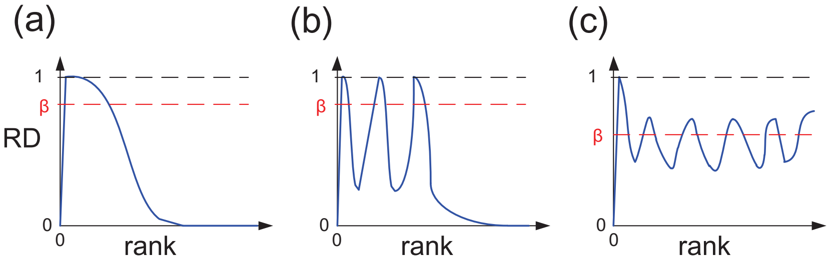
- Remove nodes having one link
- Repeat until there are no nodes with $k=1$
- Do it now with $k=2$
- Now with three, etc. *Four shalt thou not count, nor either count thou two, excepting that thou then proceed to three.*



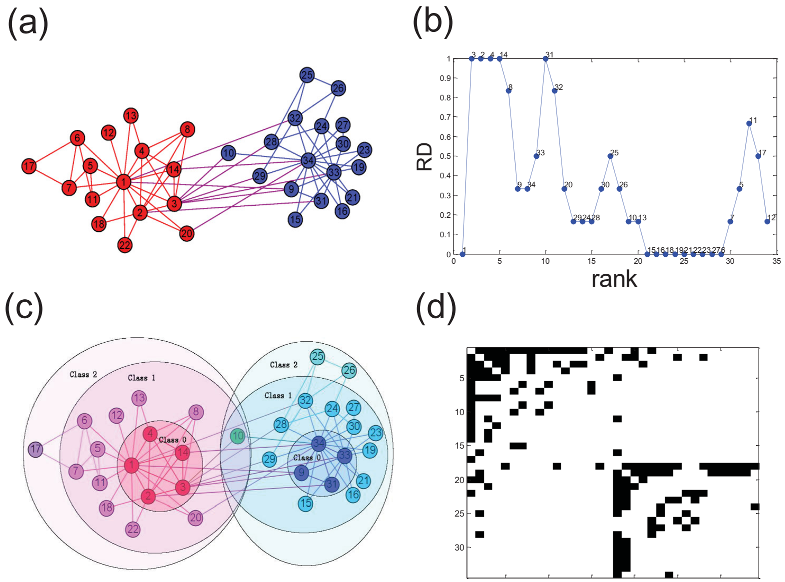
Multiple cores: Algorithm

- Rerank nodes based on local connection to existing core
- Calculate region density for each node
- Find core sets based on thresholds
- Look for periphery classes

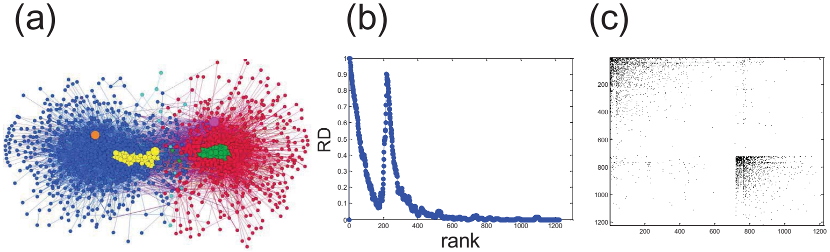
Single core Multiple core Community structure



Multiple cores: Karate club



Multiple cores: US polblogs



Examples from: Xiang et al. 2016