Complex networks

Preferential attachment, scale free networks, configuration model

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Previously on Complex networks

- Erdős-Rényi model
- Small world

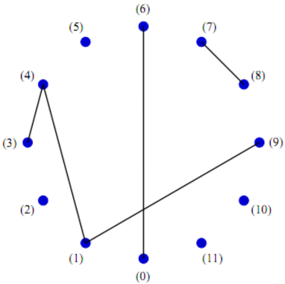
Erdős-Rényi model, versions

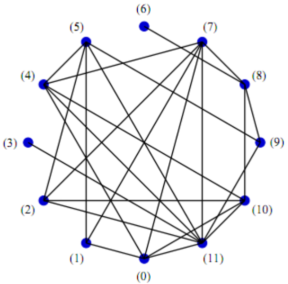
- N nodes and L links placed randomly: (N, L)
- N nodes and links with probability p: (N, p)
  - Number of links in a complete graph:

$$L_c = \binom{N}{2} = \frac{N(N-1)}{2}$$

  - Relation between p and L:

$$p = \frac{2L}{N(N-1)}$$





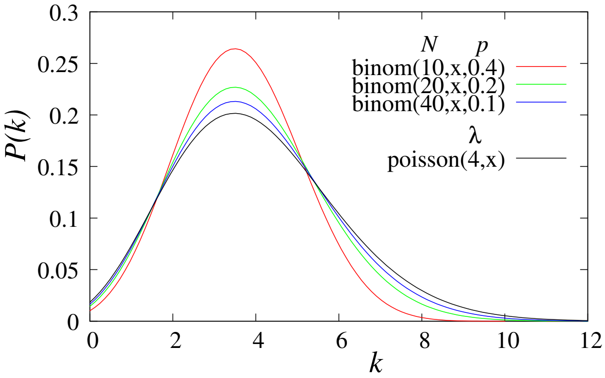
$N = 12$ $p = 0.3788$ $p = 0.758$

Erdős-Rényi: degree distribution

- Poisson limit theorem, $\lambda \equiv pN$

$$\lim_{N\to\infty} \binom{N}{k} p^k (1-p)^{N-k} = e^{-\lambda} \frac{\lambda^k}{k!}$$

- Poisson distribution: mean: λ, variance: λ



Erdős-Rényi: clustering coefficient

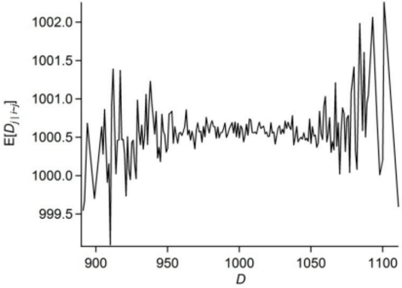
- Clustering coefficient:
- Let us consider two links of a node. The probability that it is a triangle is proportional to the probability that the missing link exists
- Thus

$$C = p = \frac{\langle k \rangle}{N - 1}$$

- In large ER graphs the clustering coefficient is almost zero
- There are hardly any triangles in the ER graphs

Erdős-Rényi: assortativity

- The Erdős-Rényi graphs should be non-assortative:
- Reasoning: The link between nodes are established in an independent way without any correlation so the actual node with degree k randomly samples the graph, thus the average degree of the friends is also k
- Funnily the probability to be connected to a node with degree k is not proportional to k.



Noldus,Mieghem 2015

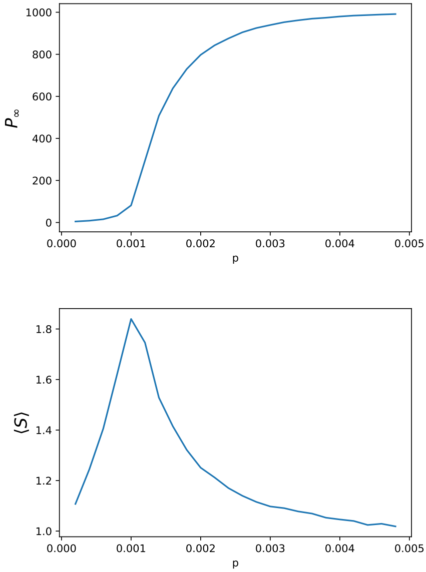
Erdős-Rényi: Percolation

- **Connected components**: There is a path between any two node of a connected component.
- **Percolation**: The network percolates if

$$\lim_{N\to\infty} |S_\infty|/N = \lim_{N\to\infty} P_\infty > 0,$$

  where S_∞ is the largest connected component and $|S_\infty|$ is its size, and P_∞ is the probability of node belonging to the largest connected component.
- Which means that macroscopic fraction of the nodes belongs to the largest connected components.
- Importance: functioning system cannot fall into (infinitely) many pieces

Erdős-Rényi: Percolation



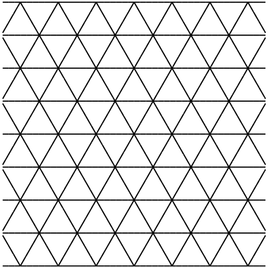
$N = 1000$

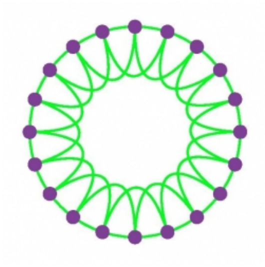
Erdős-Rényi: Summary

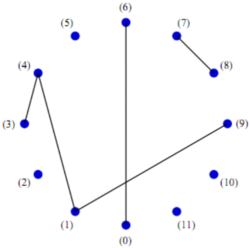
- Ensemble of random graphs
- No correlations
- Sharp degree distribution (Poisson)
- Small clustering coefficient
- Non-assortative
- Percolation threshold at $\langle k \rangle = 1$
- Small world: average path length $\langle s \rangle \sim \log N$

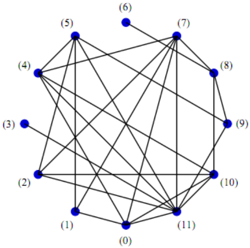
Small World and clustering

- Erdős-Rényi networks are small words with low clustering
- Triangle lattices are large words with high clustering



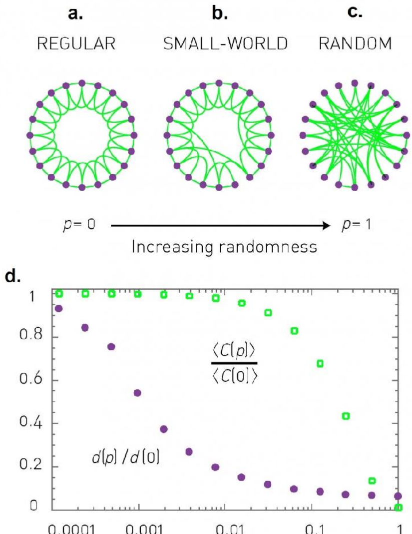






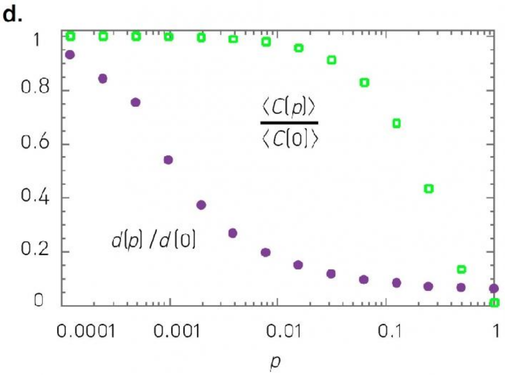
Watts-Strogatz model

- Take a lattice with high clustering
- Introduce shortcuts (rewire)
- Parameter p fraction of rewired links



Watts-Strogatz model

- Take a lattice with high clustering
- Introduce shortcuts (rewire)
- Parameter p fraction of rewired links
- Can be high clustering and small world



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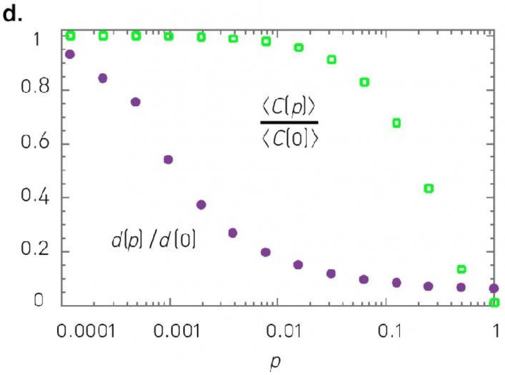
Watts-Strogatz model

- Degree distribution: shifted Poisson
- This is the major criticism towards the model
- On the other hand tunable randomness.



Scale-free function

- What does it mean?
- Must not have scale included
- Problem: most mathematical functions require dimensionless arguments, e.g. $\exp(x/x_0)$, $\log(x/x_0)$, $\sin(x/x0)$
- Single exception: power law x^α
- Mathematically: scale invariance

$$f(\alpha x) = \alpha^k f(x)$$

- Solution:

$$f(x) = Ax^k$$

Scale-freeness

$$P(x) \sim x^{-\gamma}$$

- What does it mean?
- Normalization? Must have minimum, or maximum value depending on γ (or both!)
- Very uneven distribution: High probability of small value, but very large values are also possible
- Few very rich and a lot of poor
- Origin? Bible: Matt. 25:29, *For whoever has will be given more, and they will have an abundance. Whoever does not have, even what they have will be taken from them.*

Power law distribution

$$P(x) = Cx^{-\gamma}$$

- Two cutoffs: $x \in [a, b]$, C is set to

$$\int_a^b P(x)dx = 1$$

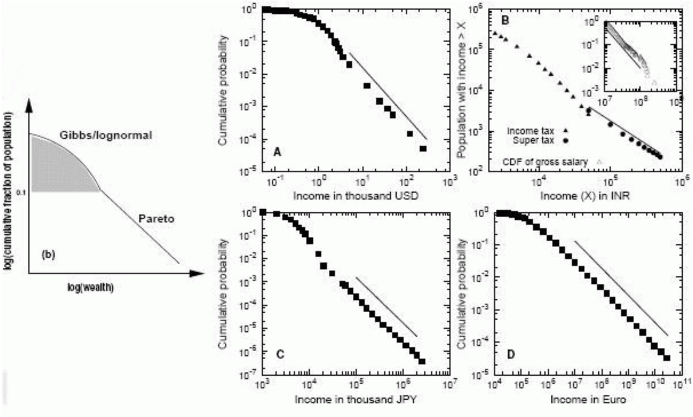
- Cumulative distribution:

$$P(x' > x) = \int_x^b P(x')dx = \frac{C}{\gamma - 1}x^{-(\gamma-1)}$$

- The cumulative distribution decays with a smaller $\gamma - 1$ exponent

Scale-freeness

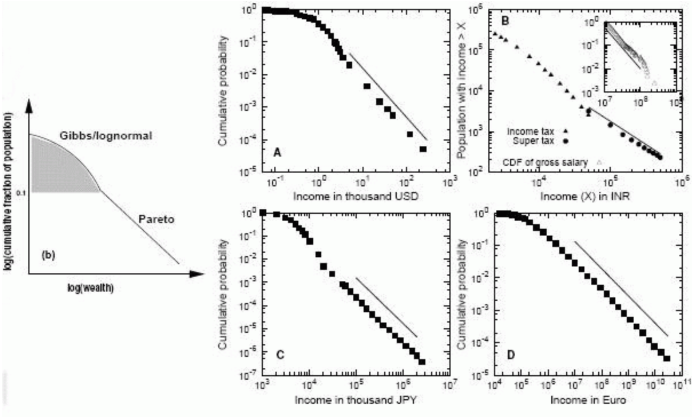
- Economic inequality, Pareto (1890) distribution $P(x) \sim x^{-\alpha}$, $a \simeq 2.5$



Zyga Phys.org 2007

Scale-freeness

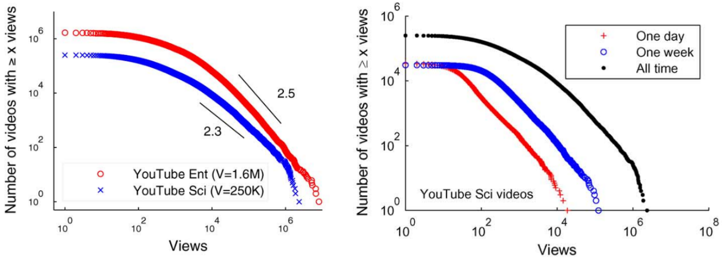
- Pareto principle: 20-80 rule:
- 80% of wealth is in the hands of 20% of the population
- 80 % of land is owned by 20% of people
- 80% of the sales is due to 20% clients



Zyga Phys.org 2007

Scale-freeness

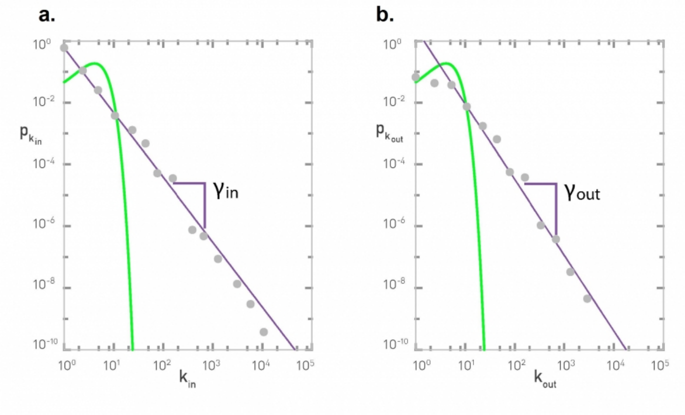
- Views of youtube videos



Cha et al. 2009

Scale-freeness

- WWW page popularity
- Exponents are $\gamma_{\text{in}} \simeq 2.1$ $\gamma_{\text{out}} \simeq 2.45$



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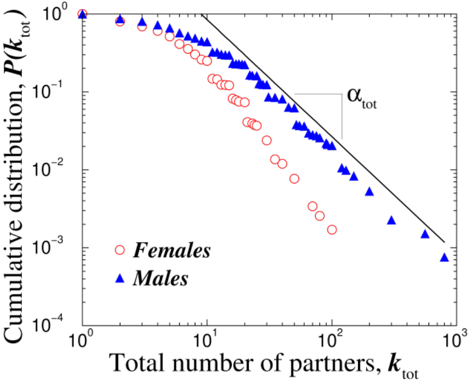
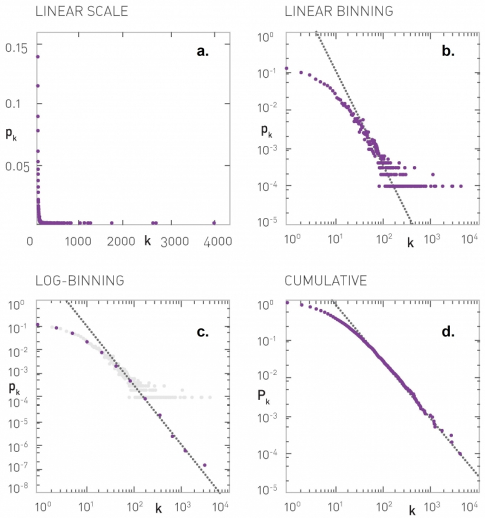
Scale-freeness

- Number of sexual partners in Sweden



Power law: plotting



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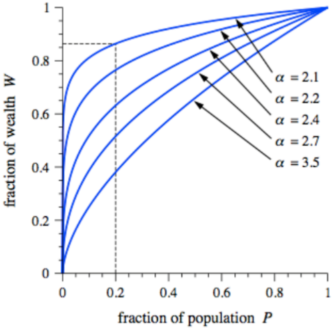
Pareto principle

- Cumulative distribution is:

$$P_>(x) = \int_x^\infty P(x')dx' = \left(\frac{x}{x_{min}}\right)^{-\gamma+1}$$

- For $\gamma > 2$ the fraction of wealth larger than x is

$$W(x) = \frac{\int_x^\infty x'P(x')dx'}{\int_{x_min}^\infty x'P(x')dx'} = \left(\frac{x}{x_{min}}\right)^{-\gamma+1} = P_>(x)^{\frac{\gamma-2}{\gamma-1}}$$



Zipf plots

- George K. Zipf linguist
- Ordered the words according to their occurrence frequency (1935)
- Plotted the frequency against the rank
- Zipf plot

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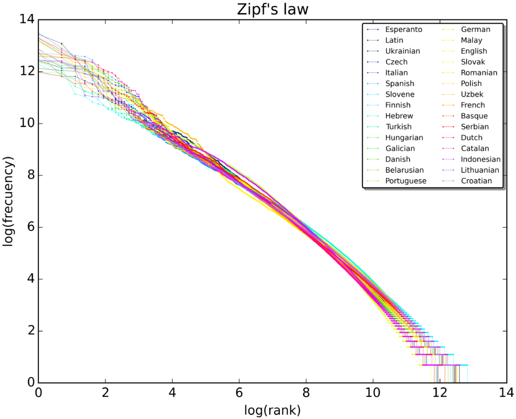

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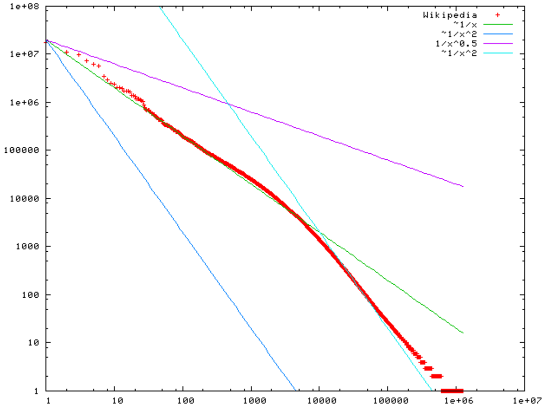
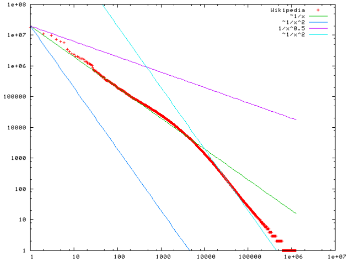
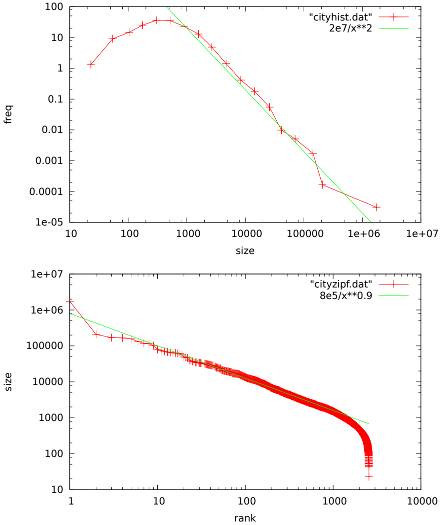


Zipf plots

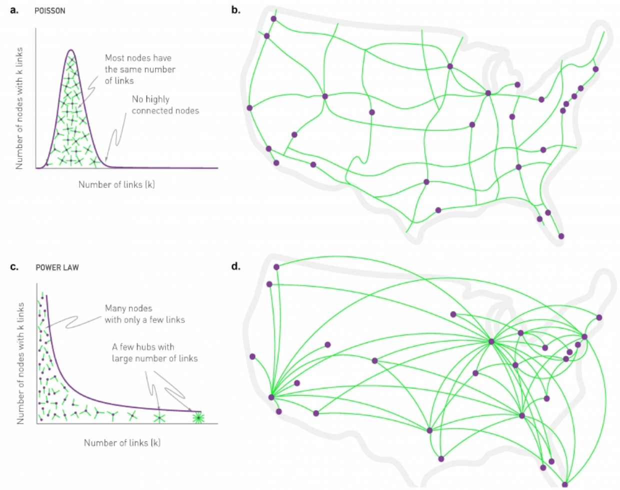
- Meaning of Zipf plot
- Rank n with frequency $f(n) = n^{-\beta}$
- There are n more frequent words than $f^{-1}(n)$
- In other words $f^{-1}(n)$ is equivalent to the cumulative frequency distribution $\beta = 1/(\gamma - 1)$



Hungarian cities



Inhomogeneities in networks



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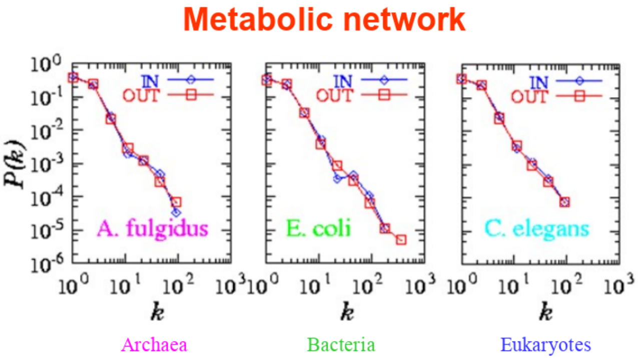
Scale free networks



Organisms from all three domains of life are **scale-free** networks!

H. Jeong, B. Tombor, R. Albert, Z.N. Oltvai, and A.L. Barabasi, Nature, **407** 651 (2000)

Scale free networks

Network	***N***	***L***	**⟨*k*⟩**	**⟨*kin2*⟩**	**⟨*kout2*⟩**	**⟨*k2*⟩**	***γin***	***γout***	***γ***
Internet	192244	609066	6.34	-	-	240.1	-	-	3.42*
WWW	325729	1497134	4.6	1546	482.4	-	2	2.31	-
Power Grid	4941	6594	2.67	-	-	10.3	-	-	Exp.
Mobile-Phone Calls	36595	91826	2.51	12	11.7	-	4.69*	5.01*	-
Email	57194	103731	1.81	94.7	1163.9	-	3.43*	2.03*	-
Science Collaboration	23133	93437	8.08	-	-	178.2	-	-	3.35*
Actor Network	702388	29397908	83.71	-	-	47353.7	-	-	2.12*
Citation Network	449673	4689479	10.43	971.5	198.8	-	3.03*	4.00*	-
E. Coli Metabolism	1039	5802	5.58	535.7	396.7	-	2.43*	2.90*	-
Protein Interactions	2018	2930	2.9	-	-	32.3	-	-	2.89*-

Scale free networks: moments

- Moments of power law distribution

$$\langle k^m \rangle = \int_{k_{min}}^{\infty} k^m P(k) dk$$

- Normalization ($\gamma > 1$)

$$P(k) = (\gamma - 1) k_{min}^{\gamma - 1} k^{-\gamma}$$

- Moments, if $1 + m < \gamma$:

$$\langle k^m \rangle = \frac{k_{min}^m (\gamma - 1)}{\gamma - 1 - m}$$

- If $m \geq \gamma - 1$ the moment diverges

Scale free networks: moments

- Moments diverge for $m \geq \gamma - 1$
- $\gamma \leq 2 \rightarrow$ No average
- $\gamma \leq 3 \rightarrow$ No variance
- Many networks fall in this category

Network	***N***	***L***	**⟨*k*⟩**	**⟨*kin2*⟩**	**⟨*kout2*⟩**	**⟨*k2*⟩**	***γin***	***γout***	***γ***
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Distances in scale free networks

Average distance scale with node number N as

- $\langle l \rangle \sim const.$ for $\gamma = 2$ Size of the biggest hub is of order $\mathcal{O}(N)$
- $\langle l \rangle \sim \frac{1}{\log(\gamma - 1)} \log \log N$ for $2 < \gamma < 3$. Path length increases slower than logarithmically, ultra-small world
- $\langle l \rangle \sim \log N / \log \log N$ for $\gamma = 3$. Some key models produce $\gamma = 3$
- $\langle l \rangle \sim \log N$ for $\gamma > 3$. The second moment of the degree distribution is finite, similar to random network. Small world.

Network	***N***	***L***	**⟨*k*⟩**	**⟨*kin2*⟩**	**⟨*kout2*⟩**	**⟨*k2*⟩**	***γin***	***γout***	***γ***
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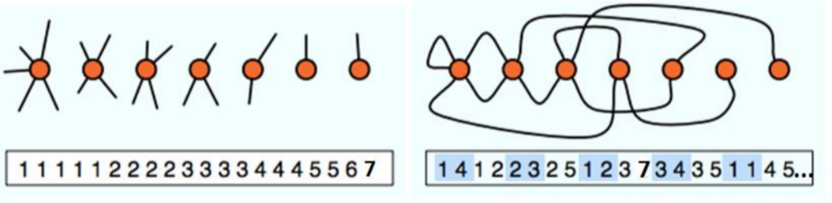
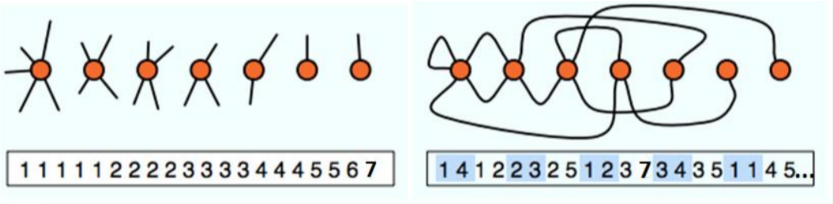
Configuration model

- How to generate random uncorrelated networks with given degree distribution.
- E.g. Random regular graph: all nodes have exactly k links
- Idea: Generate nodes with given distribution of stubs (half links) and connect the links randomly.
- Below is an example with the algoritm



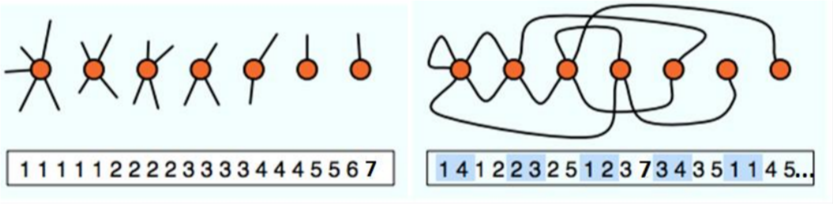
Configuration model

- Idea: Generate nodes with given distribution of stubs (half links) and connect the links randomly.
- This is a model for degree sequence (for large N it will be representative for the distribution)
- For a given degree sequence all possible pairings have the same probability
- Above can be proven from the construction algorithm



Configuration model: Problems

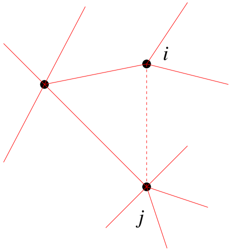
- Self loops, multiple links
- Graphs with these objects will be over represented in the ensemble
- Prohibiting such pairings will mess up distribution (though sometimes a necessity)
- For large N and sparse ($k_i/L \to 0$) networks their probability is negligible.
- Problems with power law distributions especially with $\gamma < 3$



Configuration model: Clustering

- A node with at least two links connect to nodes i and j.
- A link connects to a node with degree k with probability proportional to $p(k)k$, where $p(k)$ is the probability of finding a node with degree k
- Excess degree distribution is the probability distribution, for a vertex reached by following an edge, of the number of other edges attached to that vertex:

$$q_k = \frac{(k+1)p(k+1)}{\langle k \rangle}$$



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- The global clustering coefficient:

$$C = \sum_{k_i, k_j}^{\infty} q_{k_i} q_{k_j} \frac{k_i k_j}{2L} = \frac{1}{2L} \left(\sum_{k=0}^{\infty} k q_k \right)^2 = \cdots = \frac{1}{N} \frac{\left(\langle k^2 \rangle - \langle k \rangle \right)^2}{\langle k \rangle^3}$$

- Vanishes for large N unless $\langle k^2 \rangle$ diverges.

Configuration model: Assortativity

- A node with at least two links connect to nodes i and j.
- A link connects to a node with degree k with probability proportional to $p(k)k$, where $p(k)$ is the probability of finding a node with degree k
- Probability of connecting to a node with degree k is

$$p_{nn}(k) = \frac{kp(k)}{\langle k \rangle}$$

- Average degree of neighbors:

$$\langle k \rangle_{nn} = \sum_k k p_{nn}(k) = \sum_k \frac{k^2 p(k)}{\langle k \rangle} = \frac{\langle k^2 \rangle}{\langle k \rangle}$$

- The configuration model is thus non-assortative
- However $\langle k \rangle_{nn} > k$

$$\langle k \rangle_{nn} - \langle k \rangle = \frac{\langle k^2 \rangle}{\langle k \rangle} - \langle k \rangle = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} = \frac{\sigma_k^2}{\langle k \rangle} > 0$$

Configuration model: Percolation

- Let us do the same as for the Erdős-Rényi.
- u is the probability of a node *not* belonging to the giant component
- For any node the links should go to nodes *not* belonging to the giant component

$$u = \sum_{k=1}^{\infty} p_{nn}(k) u^{k-1} = \sum_{k} \frac{kp(k)}{\langle k \rangle} u^{k-1} \equiv g(u)$$

here we used that p_{nn} depends on k and some result from above.
- $u = 1$ is the trivial solution: no giant component!
- Similarly to ER we must have $g'(u)|_{u=1} > 1$ for this

Configuration model: Percolation

- The function we had

$$g(u) = \frac{1}{\langle k \rangle} \sum_k k p(k) u^{k-1}$$

- We must have $g'(u)|_{u=1} > 1$

$$g'(u) = \frac{1}{\langle k \rangle} \sum_k k(k-1) p(k) u^{k-2}$$

- At $u = 1$

$$g'(u)|_{u=1} = \frac{1}{\langle k \rangle} \left(\sum_k k^2 p(k) - \sum_k k p(k) \right) = \frac{1}{\langle k \rangle} \left(\langle k^2 \rangle - \langle k \rangle \right) > 1$$

- This gives

$$\langle k^2 \rangle - 2\langle k \rangle > 0$$

Configuration model: Percolation

- The Molloy-Reed criterion for existence of a giant component

$$\langle k^2 \rangle - 2\langle k \rangle > 0 \quad \text{or} \quad \kappa \equiv \frac{\langle k^2 \rangle}{\langle k \rangle} > 2$$

- **Erdős-Rényi**: $\langle k^2 \rangle = \langle k \rangle (1 + \langle k \rangle)$

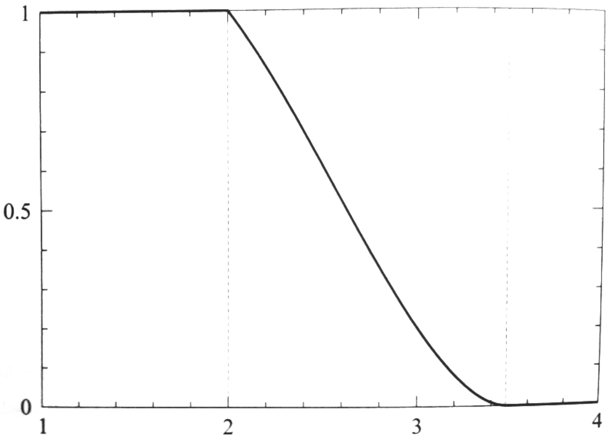
$$\kappa = \frac{\langle k \rangle (1 + \langle k \rangle)}{\langle k \rangle} = 1 + \langle k \rangle > 2$$

  From which we get back $\langle k \rangle > 1$
- **Random regular graph**: $\langle k \rangle = k$, $\langle k^2 \rangle = k^2$:

$$k - 2 > 0 \quad \rightarrow \quad k > 2$$

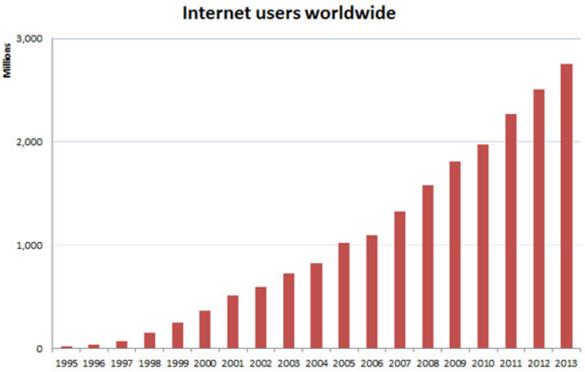
Configuration model: Percolation

- Power law distribution with γ
- Relative size of the giant component



Network models

- Up to now: geometric (static) models
- Reality: growth



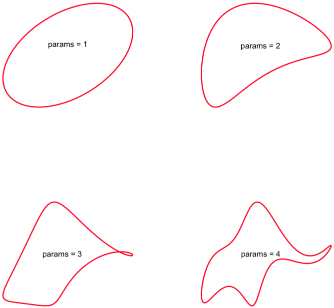
Network models

- Lewis Carroll - The Complete Illustrated Works. Gramercy Books, New York (1982). Page 727

"That's another thing we've learned from your Nation," said Mein Herr, "map-making. But we've carried it much further than you. What do you consider the largest map that would be really useful?"
"About six inches to the mile."
""Only six inches!"exclaimed Mein Herr. "We very soon got to six yards to the mile. Then we tried a hundred yards to the mile. And then came the grandest idea of all! We actually made a map of the country, on the scale of a mile to the mile!"
"Have you used it much?" I enquired.
"It has never been spread out, yet," said Mein Herr: "the farmers objected: they said it would cover the whole country, and shut out the sunlight! So we now use the country itself, as its own map, and I assure you it does nearly as well.

Network models: number of parameters?

- Physicist folklore:
- Two parameters: linear
- Three parameters: parabola
- *With four parameters I can fit an elephant, and with five I can make him wiggle his trunk* - Von Neumann
- Make it as simple as possible



Preferential attachment

- Start with a seed of small network (e.g. clique)
- Attach new nodes to the existing network.
- If attached randomly, random network with exponential degree distribution
- Popular ones have higher chance to get new connections
- New ones attach with probability proportional to existing degree
- This is preferential attachment
- In networks it is called the Barabási-Albert model

Barabási-Albert model

- Probability that a node connects to a node is proportional to the degree of the target node:

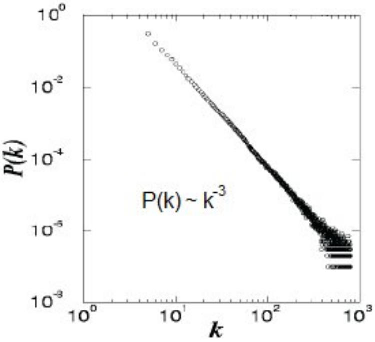
$$\Pi(i) = \frac{k_i}{\sum_j k_j}$$

- Parameter m number of links the new node makes
- Published in 1999
- Extensive impact on science

Barabási-Albert model

- Empirical degree distribution: power law
- Exponent independent of m
- $\gamma = 3$



Preferential attachment

Similar models:

- György Pólya (1887-1985) 1923: *Polya process* in the mathematics literature
- George Udmy Yule (1871-1951) in 1925: the number of species per genus of flowering plants; *Yule process* in statistics
- Robert Gibrat (1904-1980), 1931: rule of proportional growth *Gibrat process* in economics
- George Kinsley Zipf (1902-1950), 1949: the distribution of wealth in the society.
- Herbert Alexander Simon (1916-2001), 1955, the distribution of city sizes and other phenomena
- Derek de Solla Price (1922-1983), 1976, used it to explain the citation statistics of scientific publications, "cumulative advantage"
- Robert Merton (1910-2003), 1968: Matthew effect

Barabási-Albert model: Degree distribution calculation

- Number of nodes in time t, $N(t) = t$
- Number of links at time t, $L(t) = mt$
- Average degree at time t, $\langle k \rangle(t) = 2m/N$
- Number of nodes with degree k at time t

$$N(k,t) = Np(k,t) = tp(k,t)$$

- Preferential attachment:

$$\Pi(k) = \frac{k}{\sum_j k_j} = \frac{k}{2mt}$$

- Number of links added to nodes of degree k after the arrival of a new node

$$\underbrace{\frac{k}{2mt}}_{\text{Preferential attachment}} \times \underbrace{tp(k,t)}_{\text{Total number of } k \text{ nodes}} \times \underbrace{m}_{\text{New links}} = \frac{k}{2}p(k,t)$$

Barabási-Albert model: Degree distribution calculation

- Number of links added to nodes of degree k after the arrival of a new node

$$\underbrace{\frac{k}{2mt}}_{\text{Preferential attachment}} \times \underbrace{tp(k,t)}_{\text{Total number of } k \text{ nodes}} \times \underbrace{m}_{\text{New links}} = \frac{k}{2}p(k,t)$$

- Discrete time Master equation

$$(t+1)p(k,t+1) - tp(k,t) = \frac{k-1}{2}p(k-1,t) - \frac{k}{2}p(k,t)$$

Barabási-Albert model: Degree distribution calculation

- Discrete time Master equation

$$(t+1)p(k,t+1) - tp(k,t) = \frac{k-1}{2}p(k-1,t) - \frac{k}{2}p(k,t)$$

- For $k = m$ it is different, the gain term is the newly arriving node:

$$(t+1)p(m,t+1) - tp(m,t) = 1 - \frac{m}{2}p(m,t)$$

Barabási-Albert model: Degree distribution calculation

- We are interested in the steady state

$$\lim_{t \to \infty} p(k, t) = p(k)$$

- Steady state solution of the Master equation:

$$p(k) = \frac{k-1}{2} p(k-1) - \frac{k}{2} p(k)$$
$$p(m) = 1 - \frac{m}{2} p(m)$$

- Recursive relations

$$p(k) = \frac{k-1}{k+2} p(k-1) \quad \text{for } k > m$$
$$p(m) = \frac{2}{m+2} \quad \text{otherwise}$$

Barabási-Albert model: Degree distribution calculation

- Solution

$$p(k) = \frac{2m(m+1)}{k(k+1)(k+2)}$$

- Asymptotically

$$p(k) \sim k^{-3}$$

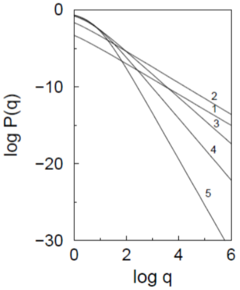
- Independent of m

Initial Attractiveness Model

- Even nodes without connections can be popular
- Often cited example: Citation networks (paper with no citation can be cited)

$$\Pi(k_i) = \frac{A + k_i}{A + \sum_j k_j}$$

- Asymptotically $p(k) \sim k^{-\gamma}$
- $\gamma = 2 + A/m$ tunable exponent



Dorogovtsev, Mendes, Samukhin, Phys. Rev. Lett. 85, 4633 (2000)

Distances in scale free networks

Average distance scale with node number N as

- $\langle l \rangle \sim const.$ for $\gamma = 2$ Size of the biggest hub is of order $\mathcal{O}(N)$
- $\langle l \rangle \sim \frac{1}{\log(\gamma - 1)} \log \log N$ for $2 < \gamma < 3$. Path length increases slower than logarithmically, ultra-small world
- $\langle l \rangle \sim \log N / \log \log N$ for $\gamma = 3$. Some key models produce $\gamma = 3$
- $\langle l \rangle \sim \log N$ for $\gamma > 3$. The second moment of the degree distribution is finite, similar to random network. Small world.

Assortativity in Barabási-Albert model

No calculations here :-)

- Disassortative regime $\gamma < 3$, $-m < A < 0$:

$$k_{nn} \sim m\frac{(m+A)^{1-A/m}}{2m+a}\zeta\left(\frac{2m}{2m+a}\right)N^{-A/(2m+A)}k^{A/m}$$

Only the k depdendence:

$$k_{nn}(k) \sim k^{-|A|/m}$$

- Neutra regime $\gamma = 3$, $A = 0$

$$k_{nn}(k) \sim \frac{M}{2}\log N$$

- Weak assortative regime $\gamma > 3$, $A > 0$

$$k_{nn}(k) \sim (m+A)\log\left(\frac{k}{m+a}\right)$$

Clustering in Barabási-Albert model

Calculations :-!

- Definition

$$C = \frac{2N(\Delta)}{k(k-1)}$$

- Probability that nodes i and j are connected: $P(i,j)$
- Probability that nodes i, j, l form a triangle

$$N_l(\Delta) = \sum_{i,j} P(i,j)P(j,l)P(l,i)$$

- We need to calculate $P(i,j)$
- For this we will need the time evolution of the degree of the nodes

Time evolution in Barabási-Albert model

- The time evolution of the degree of the nodes

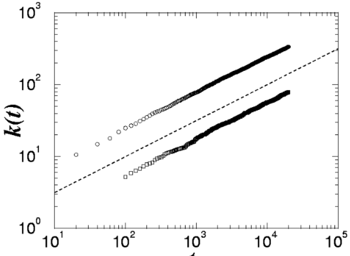
$$\frac{\partial k_i}{\partial t} \propto \Pi(k_i) = m\frac{k_i}{\sum_j k_j}$$

- Time is measured in units of nodes added, so at time t there are $N = t$ number of nodes and $L = mt$ number of links
- So

$$\frac{\partial k_i}{\partial t} \propto \frac{k_i}{2t}$$

- Solution

$$k_i(t) = m\sqrt{t/t_i} \sim t^{1/2}$$

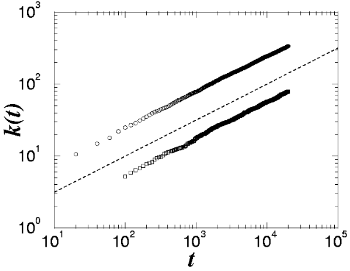


Time evolution in Barabási-Albert model

- The time evolution of the degree of the nodes

$$k_i(t) = m\sqrt{t/t_i} \sim t^{1/2}$$

- Advantage of the first comers!
- Very often one can take $t_i \equiv i$



Clustering in Barabási-Albert model

- Assume that $t_i < t_j$ (i came first)

$$P(i,j) = m\Pi(k_i(t_j)) = mk_i(t_j)/\left(\sum_l k_l\right) = m\frac{k_i(t_j)}{2mt_j}$$

- We know the time evolution of $k_i(t_j)$

$$k_i(t_j) = m\sqrt{t_j/t_i}$$

- From where we get

$$P(i,j) = \frac{m}{2}(t_i t_j)^{-1/2}$$

- Huhh... It is symmetric in i and j!

Clustering in Barabási-Albert model

- Back to the number of triangles:

$$N_l(\Delta) = \sum_{i,j} P(i,j)P(j,l)P(l,i) =$$

$$= \frac{m^3}{8} \sum_{t_i,t_j} (t_i t_j)^{-1/2}(t_j t_l)^{-1/2}(t_l t_i)^{-1/2}$$

$$= \frac{m^3}{8l} \sum_{t_i=1}^{N} \frac{1}{t_i} \sum_{t_j=1}^{N} \frac{1}{t_j}$$

- For large times $N \to \infty$

$$N_l(\Delta) = \frac{m^3}{8l} \log^2 N$$

Clustering in Barabási-Albert model

- So we have the number of triangles:

$$N_l(\Delta) = \frac{m^3}{8l}\log^2 N$$

- We also know that

$$k_l(t) = m\sqrt{N/t_l} \quad \text{so} \quad k_l(k_l - 1) \simeq m^2 N/t_l$$

- Finally the clustering coefficient is

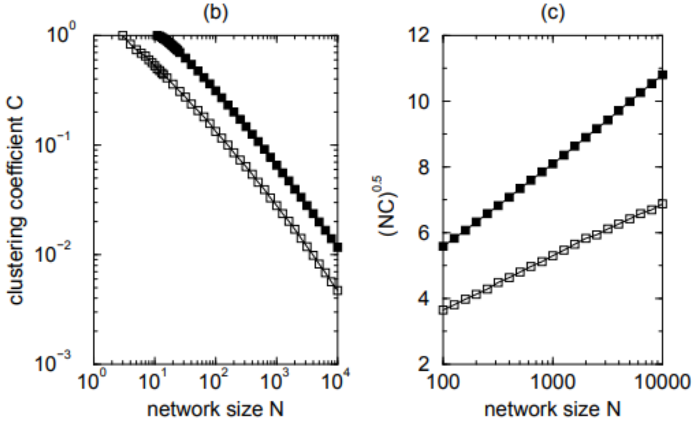
$$C = \frac{2\frac{m^3}{8l}\log^2 N}{k_l(k_l - 1)} = \frac{m}{8}\frac{\log^2 N}{N}$$

- For large networks $N \to \infty$ the clustering vanishes $C \to 0$

Clustering in Barabási-Albert model

- The clustering coefficient for BA networks

$$C = \frac{m}{8} \frac{\log^2 N}{N}$$



Other models

Linear growth, linear pref. attachment	$\gamma=3$	Barabási and Albert, 1999
Nonlinear preferential attachment $\Pi(k_i)\sim k_i^\alpha$	no scaling for $\alpha\neq 1$	Krapivsky, Redner, and Leyvraz, 2000
Asymptotically linear pref. attachment $\Pi(k_i)\sim a_\infty k_i$ as $k_i\to\infty$	$\gamma\to 2$ if $a_\infty\to\infty$ $\gamma\to\infty$ if $a_\infty\to 0$	Krapivsky, Redner, and Leyvraz, 2000
Initial attractiveness $\Pi(k_i)\sim A+k_i$	$\gamma=2$ if $A=0$ $\gamma\to\infty$ if $A\to\infty$	Dorogovtsev, Mendes, and Samukhin, 2000a, 2000b
Accelerating growth $\langle k\rangle\sim t^\theta$ constant initial attractiveness	$\gamma=1.5$ if $\theta\to 1$ $\gamma\to 2$ if $\theta\to 0$	Dorogovtsev and Mendes, 2001a
Internal edges with probab. p Rewiring of edges with probab. q	$\gamma=2$ if $q=\dfrac{1-p+m}{1+2m}$ $\gamma\to\infty$ if $p,q,m\to 0$	Albert and Barabási, 2000
c internal edges or removal of c edges	$\gamma\to 2$ if $c\to\infty$ $\gamma\to\infty$ if $c\to -1$	Dorogovtsev and Mendes, 2000c
Gradual aging $\Pi(k_i)\sim k_i(t-t_i)^{-\nu}$	$\gamma\to 2$ if $\nu\to -\infty$ $\gamma\to\infty$ if $\nu\to 1$	Dorogovtsev and Mendes, 2000b
Multiplicative node fitness $\Pi_i\sim\eta_i k_i$	$P(k)\sim\dfrac{k^{-1-C}}{\ln(k)}$	Bianconi and Barabási, 2001a
Edge inheritance	$P(k_{in})=\dfrac{d}{k_{in}^{\sqrt{2}}}\ln(ak_{in})$	Dorogovtsev, Mendes, and Samukhin, 2000c
Copying with probab. p	$\gamma=(2-p)/(1-p)$	Kumar *et al.*, 2000a, 2000b
Redirection with probab. r	$\gamma=1+1/r$	Krapivsky and Redner, 2001
Walking with probab. p	$\gamma\simeq 2$ for $p>p_c$	Vázquez, 2000
Attaching to edges	$\gamma=3$	Dorogovtsev, Mendes, and Samukhin, 2001a
p directed internal edges $\Pi(k_i,k_i)\propto(k_i^{in}+\lambda)(k_i^{out}+\mu)$	$\gamma_{in}=2+p\lambda$ $\gamma_{out}=1+(1-p)^{-1}+\mu p/(1-p)$	Krapivsky, Rodgers, and Redner, 2001