Problem set 3 for the course "Many -body physics 1", 2022

Rules: You can solve any number of problems. You can collect 30 points at maximum, and you must collect 10 points to pass. You can help each-other and discuss, give hints to each-other (this is even encouraged), but you are not allowed to copy...
Deadline: midnight, June 10.

3.1. (30) Friedel oscillations

Consider a localized charge with $n_{ext} = Z\delta(r)$ in 2D, interacting with the surrounding interacting electron gas through the Coulomb interaction. The electrons are also interacting via the long range Coulomb interaction in 2D. Calculate the resulting Friedel oscillations around the impurity, use the results of arxiv.org/abs/1111.5337. Pay special attention to separate the Thomas-Fermi contribution, dominating at short distances, from the oscillating part of the charge density. (15)

3.2. (30) Hartree-Fock energy in Jellium model

Calculate the ground state energy per particle of the Jellium model in the (Hartree-)Fock approximation in 2D! Determine the optimal interparticle distance or Fermi wavenumber, which minimizes this energy. When do you expect the perturbation theory to be valid?

3.3. (35) Polarons with longitudinal optical (LO) phonons.

In this problem, you will study how the mass of the electrons is influenced by couplings to longitudinal optical (LO) phonons in an insulator or semiconductor. Longitudinal phonons can create a polarization density, the divergence of which corresponds to a charge density. This charge density couples strongly to charge carriers through a Coulomb interaction, and ultimately gives rise to a coupling of the form

$$H_{\rm int} = g \sum_\sigma \int \mathrm{d}^3\mathbf{r}\, \varphi(\mathbf{r})\, \psi^\dagger_\sigma(\mathbf{r})\psi_\sigma(\mathbf{r}) \,, \tag{1}$$

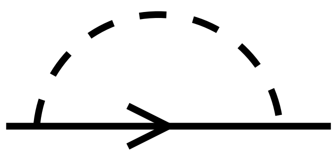
$$\varphi(\mathbf{r}) = \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\, i\gamma(\mathbf{q})\, e^{i\,\mathbf{r}\,\mathbf{q}}\, (b^\dagger(-\mathbf{q}) - b(\mathbf{q}))\,, \tag{2}$$

where for LO optical phonons $\gamma(\mathbf{q}) \approx C/|\mathbf{q}|$. Here the field $\psi_\sigma(\mathbf{r}) = \int e^{i\,\mathbf{r}\,\mathbf{k}}\, c_\sigma(\mathbf{k})\, \mathrm{d}^3\mathbf{k}/(2\pi)^3$ describes one conduction electron band with a dispersion $\xi(\mathbf{k}) = \mathbf{k}^2/2m - \mu$. [The creation operators satisfy the commutation/anticommutation relations $\{c_\sigma(\mathbf{k}), c^\dagger_{\sigma'}(\mathbf{k})\} = (2\pi)^3\, \delta_{\sigma\sigma'}\delta(\mathbf{k}-\mathbf{k}')$ and $[b_\sigma(\mathbf{q}), b^\dagger(\mathbf{q}')] = (2\pi)^3\, \delta(\mathbf{q}-\mathbf{q}')$].

a. **(5p)** Assume that phonos propagate very slowly, and use the equations of electrostatics (in Fourier space) to determine the potential felt by the electrons. Assume some ionic crystal like NaCl. (You do not need to give a very precise derivation, you can just assume that you have three optical modes but only one of them has a longitudinal polarization, $\mathbf{e_q}||\mathbf{q}$.)

b. **(5 p)** Now derive the Fourier transform of the phonons' unperturbed Green's function, $D_0(x - x') = -i\langle \mathrm{T}\varphi(x)\varphi(x')\rangle_0$, assuming a dispersionless phonon mode, with $\omega(\mathbf{q}) = \omega_0$. (Show the details of the calculation, follow the steps at class!)

c. **(5 p)** Compute the electron self-energy, $\Sigma_{\rm ph}(\omega, \mathbf{q})$, shown in the figure below. (Hints: Use the formalism in Fourier space, just as at class, and evaluate the frequency integral. Discuss the details of each step. Use the fact that you have a semiconductor, and therefore $\xi_\mathbf{k} = \mathbf{k}^2/(2m) - \mu$ is always positive, since μ is inside the gap.)



d. **(5 p)** Now derive and investigate the selfconsistency equation, which describes the renormalization of the quasiparticle dispersion, $\xi_\mathbf{k} \to \tilde{\xi}_\mathbf{k} \equiv \xi_\mathbf{k} - \Delta_\mathbf{k}$:

$$\Delta_\mathbf{k} = \int_\mathbf{q} g^2 |\gamma(\mathbf{q})|^2 \frac{1}{\omega_0 + \Delta_\mathbf{k} + \xi_{\mathbf{k}+\mathbf{q}} - \xi_\mathbf{k}}.$$

(Hint: calculate the electron's self-energy, and from that, the (inverse) Green's function up to second order. Remember that $\tilde{\xi}_{\mathbf{k}}$ is determined by the condition that the Green's function has a pole, i.e., its inverse vanishes at $\omega = \tilde{\xi}_{\mathbf{k}}$.) Construct the self-consistency equation for Δ_0, and compute the approximate value of Δ_0 by iterating the integral equation.

e. **(5 p)** Expand the previous equation around $\mathbf{k} = 0$ by assuming a small quasiparticle momentum, $\mathbf{k}$, and a dispersion $\Delta_{\mathbf{k}} \approx \Delta_0 + \alpha \mathbf{k}^2/2m$, and obtain an expression for the effective mass. Evaluate the appearing integrals. [Hint: You will have to expand the denominator up to second order, since you have cross terms, $\sim kq\cos(\theta)$, too, the square of which will also give a k^2 contribution.] Evaluate the mass and show that it gets heavier. Can it be infinite?

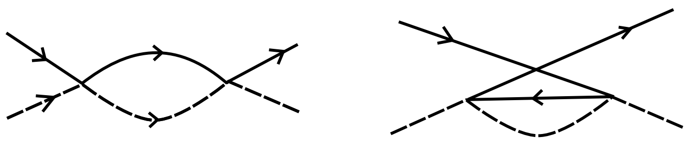
3.4. (30) Vertex function and renormalization group equations of 1D interacting electrons
Study the interacting electron system also studied at class, described by the interaction term

$$H_{\text{int}} = \sum_{\{\sigma_i\}} \frac{1}{L} \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} V_{\sigma_1\sigma_2;\sigma_3\sigma_4} c^\dagger_{L\mathbf{k}\sigma_1} c^\dagger_{R\mathbf{k}'\sigma_2} c_{R\mathbf{k}'+\mathbf{q}\sigma_4} c_{L\mathbf{k}-\mathbf{q}\sigma_3}$$

with the tensor $V_{\sigma_1\sigma_2;\sigma_3\sigma_4}$ now incorporating both forward (V_2) and backward (V_1) scattering,

$$V_{\sigma_1\sigma_2;\sigma_3\sigma_4} = V_2 \delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4} - V_1 \delta_{\sigma_1\sigma_4}\delta_{\sigma_2\sigma_3} \rightarrow V_2\,\mathbf{I} - V_1\,\mathbf{X}.$$

Compute the vertex diagram (b) below using the diagram rules discussed at class for the special case, where the external momenta are at the Fermi surface, but the incoming and outgoing frequencies are all different.



(a) (b)

a. **(5p)** First construct the diagram's contribution using the diagram rules discussed at class.

b. **(10p)** Evaluate the frequency integral first and then carry out the summation (converted into an integral) over the internal momentum to show that this diagram's contribution to the vertex function is

$$\delta\Gamma_b \approx \frac{1}{L}\frac{1}{2\pi v_F}\left[V_2^2\mathbf{I} - 2(V_1V_2 - V_1^2)\mathbf{X}\right] \ln(2\Lambda v_F/|\omega_2 - \omega_3|),$$

with Λ denoting the momentum cut-off around the Fermi energy.

c. **(5p)** Introduce the dimensionless couplings, $g_{1,2} \equiv V_{1,2}/(2\pi v_F)$. Using the result obtained at class for the other vertex diagram show that the reduction of the cut-off $\Lambda \rightarrow \tilde{\Lambda}$ can be compensated by small changes of the dimensionless couplings, $g_{1,2} \rightarrow \tilde{g}_{1,2}$ such that the constant part of the vertex function Γ remains invariant for any value of the external frequencies. Derive from that the renormalization group equations (scaling equations).