Many-body problem 1. Exercises
(**Deadline**: 29. March 2022.)

1. Prove that the second quantized form of a single particle operator, $\sum_{j=1}^{N} V(r_j)$ for bosons is

$$H = \sum_{k,p} f_{k,p} a_k^+ a_p, \quad f_{k,p} = \int dr \psi_k^+(r) V(r) \psi_p(r),$$

where $[a_k, a_p^+] = \delta_{k,p}$, and the single particle eigenfunctions are $\psi_p(r)$'s. (20)

2. Go beyond the realm of linear response and determine the the second order correction to the expectation value of a physical observable, A, when a weak external field, $h_B(t)$ is applied by coupling it to operator B. Pay attention to the normalization of the wavefunction! (20)

(a) Write down the corresponding Schrödinger equation, and try to approximate its solution by a 2 step iteration!

(b) Normalize the wavefunction!

(c) Evaluate the expectation value of operator A to second order in the external field.

3. A spin-1/2 particle is placed in an external magnetic field as $\mathbf{B} = (B_1 \cos(\omega t), B_1 \sin(\omega t), B_0)$, where $B_0 \gg B_1$. (20)

(a) Treating the oscillating part of the Hamiltonian (B_1) as the interaction, write down the Schrödinger equation in the interaction representation.

(b) Find the time evolution operator, $U(t) = T_t \exp\left[-i \int_0^t H_{int}(t')dt'\right]$ by solving the corresponding Schrödinger equation (hint: use Fourier transformation).

(c) If the particle starts out at time $t = 0$ from the eigenstate $S_z = -1/2$, what is the probability to find it in the very same state at time t later?

4. Consider an interacting fermi gas as $H = H_0 + V$ with $H_0 = \sum_{k,\sigma} \xi_k a_{k,\sigma}^+ a_{k,\sigma}$ and

$$V = \sum_{k,k'q,\sigma,\sigma'} v(q) a_{k+q,\sigma}^+ a_{k'-q,\sigma'}^+ a_{k',\sigma'} a_{k,\sigma}.$$

Demonstrate that the ground state energy can be expressed in terms of the single particle Green's function! (20)

(a) Calculate $i\partial_t a_{k,\sigma} = [a_{k,\sigma}, H]$ using the above Hamiltonian!

(b) Multiply the above equation with $\sum_{k,\sigma} a_{k,\sigma}^+$ from the left and take its expectation value. Express $\langle a_{k,\sigma}^+ \partial_t a_{k,\sigma} \rangle$ using the single particle Green's function (i.e. by taking the time derivative of $G(k,t)$ at negative vanishing times), and determine $\langle V \rangle$.

(c) Calculate the total ground state energy by introducing $G(k,\omega)$!