Problem set 12 for Quantum Field Theory course

2019.05.07.

Topics covered

- Bubble sum in QED
- Renormalization of composite operators
- $O(N)$ structure at two-loop level
- Power counting

Recommended reading
Peskin–Schroeder: An introduction to quantum field theory

- Sections 7.5
- Sections 10.1
- Sections 12.4

Problem 12.1 Bubble sum in QED

Renormalization of the electric charge is connected to renormalization of the photonic propagator in QED. Let us call the sum of all 1PI insertions to the propagator $\Pi^{\mu\nu}$. A visual depiction is given on Fig. 1.

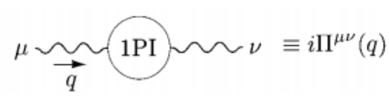


Figure 1: 1PI "bubble" of the photon propagator.

In the lecture it was shown that the Ward–Takahashi identity leads to a constraint on the tensor structure:

$$\Pi^{\mu\nu} = \left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right)\Pi(q^2)\,. \tag{1}$$

(a) Write the fully dressed photonic propagator as a sum of 1PI insertions (starting from order 0). Bring it to the form

$$\frac{-i\eta_{\mu\nu}}{q^2} + \frac{-i\eta_{\mu\rho}}{q^2}\Delta^\rho_\nu \Pi(q^2) + \frac{-i\eta_{\mu\rho}}{q^2}\Delta^\rho_\sigma\Delta^\sigma_\nu\Pi^2(q^2) + \ldots\,. \tag{2}$$

What is Δ^ρ_ν?

(b) Show that Δ^ρ_ν is a projector:

$$\Delta^\rho_\sigma\Delta^\sigma_\nu = \Delta^\rho_\nu\,, \tag{3}$$

and write the fully dressed propagator as a geometrical series. Perform the sum to obtain

$$\frac{-i}{q^2(1-\Pi(q^2))}\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{q^2}\frac{q_\mu q_\nu}{q^2}\,. \tag{4}$$

Utilize the Ward identity to show that (at least in the case of S-matrix calculations) terms proportional to q^μ or q^ν vanish, so we are left with

$$\frac{-i\eta_{\mu\nu}}{q^2(1-\Pi(q^2))} . \tag{5}$$

Hint: photon propagators end in a vertex in case of S-matrix calculations. The vertex involves the matrix element of a Noether current j^μ that has zero divergence.

(c) Note that expression (5) has a pole at $q^2 = 0$ which implies that the photon mass remains zero at all orders. Denote its residue coefficient with Z_3. Write the amplitude of a low-q^2 scattering of two fermions to show that the physical charge can be expressed with the bare one as:

$$e = \sqrt{Z_3} e_0 , \tag{6}$$

where e_0 is the bare charge.

Remark: using a nonzero q^2 also shows that the effective electric charge (and so the fine structure constant) obtains a momentum dependence.

(d) Draw the leading order diagram that contributes to $\Pi^{\mu\nu}(q^2)$ and write it as a momentum integral using momentum-space Feynman rules.

Problem 12.2 Renormalization of ϕ^2

So far we treated renormalisation of the propagator and the vertex. This exercise involves similar calculation for a composite operator, ϕ^2 of the massless ϕ^4-theory.

We aim to compute correlation functions involving ϕ^2, and consider

$$\langle\Omega| T\phi(x_1)...\phi(x_n)\phi^2(x)|\Omega\rangle \propto \langle 0| T\phi(x_1)...\phi(x_n)\phi^2(x) e^{-\frac{i\lambda}{4!}\int \mathrm{d}^4 y \phi^4(y)}|0\rangle . \tag{7}$$

Let us restrict the problem to a two point-function i.e. $n = 2$.

(a) Use Wick's theorem and draw the tree-level ($\lambda = 0$) diagrams for the matrix element above. Perform a Fourier transform in the spatial variables

$$\left\langle T\phi(p)\phi(q)\phi^2(k)\right\rangle = \int \mathrm{d}^4 x e^{ipx} \int \mathrm{d}^4 y e^{iqy} \int \mathrm{d}^4 z e^{ikz} \left\langle T\phi(x)\phi(y)\phi^2(z)\right\rangle \tag{8}$$

to show that the ϕ^2 insertion corresponds to an external momentum insertion to the propagator. Note that the diagram containing the pairing of the two legs of the ϕ^2 insertion vanishes for $k \neq 0$ due to momentum conservation.

Remark: note that $\phi^2(k)$ here is the Fourier transform of $\phi^2(x)$, not $[\phi(k)]^2$!

(b) In the massive ϕ^4-theory ϕ^2 appears in the mass term, so insertion of ϕ^2 in a massless theory corresponds to mass renormalization (this is our motivation for this setting). Use your tree-level result to set the renormalization condition

$$\left\langle \phi(q)\phi(p)\phi^2(k)\right\rangle = 2\frac{i}{p^2}\frac{i}{q^2}\bigg|_{p^2=q^2=k^2=-M^2} , \tag{9}$$

where M is just a reference point.

Remark: $i\varepsilon$-s are dropped for brevity.

(c) Do an $O(\lambda)$ calculation of (7) in momentum space. Note that it involves a loop integral which diverges. Derive the result

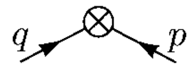
$$-\frac{i}{p^2}\frac{i}{q^2}\frac{\lambda}{(4\pi)^{d/2}}\int_0^1 \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\mathrm{d}x , \tag{10}$$

where Δ is dependent on k and x. What is $\Delta(k, x)$?

(d) The operator is renormalised by adding a counter term so that the relation between the bare and the renormalised operator becomes

$$\phi^2(x)_R = (1+\delta_{\phi^2})\phi^2(x)_0 \tag{11}$$

This generates the counter term diagram



with the contribution

$$2\frac{i}{p^2}\frac{i}{q^2}\delta_{\phi^2} \tag{12}$$

Show that condition (9) yields

$$\delta_{\phi^2} = \frac{\lambda}{2(4\pi)^{d/2}}\int_0^1 \frac{\Gamma(2-d/2)}{(x(1-x)M^2)^{2-d/2}}\mathrm{d}x. \tag{13}$$

Problem 12.3 $O(N)$ **structure of renormalization**

In this exercise we consider the $O(N)$ "extension" of two bosonic problems from earlier: setting sun graph and ϕ^2 renormalization.

(a) Generalise the calculation in Problem 12.2 using

$$-i\lambda(\delta_{kl}\delta_{mn}+\delta_{km}\delta_{ln}+\delta_{kn}\delta_{ml}) \tag{14}$$

for the vertex and

$$\frac{i\delta_{kl}}{p^2+i\varepsilon} \tag{15}$$

for the propagator.
Hint: one can easily read off effect of the $\phi^j\phi^j$ insertion on the indices by looking at the pairings. Note that you only need to consider the extra factors due to the $O(N)$ structure and borrow the results of 12.2 regarding the result of the loop integral.

(b) Show that the result for the counter-term is

$$\delta_{\phi^2} = (N+2)\frac{\lambda}{2(4\pi)^{d/2}}\int_0^1 \frac{\Gamma(2-d/2)}{(x(1-x)M^2)^{2-d/2}}\mathrm{d}x. \tag{16}$$

(c) Write down the momentum-integral for the $O(N)$ setting sun (see Problem set 11, Figure 2). What is the dependence on N?
Hint: here again one can simply superpose the index structure on the expressions in Problem 11.2.

Problem 12.4 Power counting in QED and Yukawa theory

In general, the degree of divergence for a given graph in any theory can be computed as

$$D = d - \sum_i V_i[\lambda_i] - \sum_i N_i[\phi_i], \tag{17}$$

where d is the number of spacetime dimensions, V_i is the number of vertices involving a coupling λ_i. N_i is the number of external legs of a field ϕ_i, and the braces denote the energy dimension of couplings and fields.

(a) Using the Yukawa Feynman rules, compute the degree of divergence in $d=4$ in a graph with N_ϕ external and I_ϕ internal scalar, N_ψ external and I_ψ internal fermion lines and V vertices following the arguments at the lecture. Check that the result is the same as from the expression (17) i.e.

$$D_Y = 4 - N_\phi - 3/2N_\psi \tag{18}$$

Do the same for QED and show that

$$D_{QED} = 4 - N_A - 3/2N_\psi \tag{19}$$

where N_A is the number of external photon lines.

(b) Draw the possible divergent graphs of QED. What is their "naive" degree of divergence obtained by applying (19) to them?

Hint: graphs with an odd number of fermion legs are zero, and those with $N_\psi = 0$ *and an odd number of photonic legs are excluded by the Ward identity.*

(c) For the photon polarisation expression (19) yields a higher degree of divergence D than the actual graphs. Also, photon-photon scattering appears to be logarithmically divergent, although it is actually convergent. What could be the reason behind these phenomena? (Hint: consider the implications of the Ward identity.)

Remark: a similar effect happens for the electron self-energy, which is predicted to be linearly divergent, but in fact diverges only logarithmically. This means that the mass counter term is only logarithmically divergent. This is due to the fact that for $m = 0$ *there is an enhanced chiral* $U(1)$ *symmetry (separate* $U(1)$ *transformation of left/right handed fermions). Therefore the fermion mass renormalisation vanishes for* $m = 0$*, and so the divergence can only depend on the cut-off* Λ *as*

$$\delta_m \propto m \log \frac{\Lambda}{m} \tag{20}$$

(d) Draw the possible divergent graphs for the Yukawa theory. How many counter-terms do we need to renormalize our theory?

(e) How do the above results change when we consider the pseudo-scalar Yukawa theory with interaction $\lambda\phi\bar{\psi}\gamma_5\psi$?

Hint: in this theory ϕ *is odd under parity, so the interaction term is parity invariant. This implies that graphs with no external fermion lines and odd number of external boson lines vanish.*

Problem 12.5 Power counting in other models

(a) Consider the Dirac fermion with four-fermion interaction

$$\mathscr{L} = \bar{\psi}(i\partial\!\!\!/ - M)\psi - g(\bar{\psi}\psi)^2 \tag{21}$$

In four dimensions, what is the degree of divergence of a diagram with N external fermion lines and V vertices? What counter-terms are required at order g^2? What about order g^3? Is the theory renormalisable?

(b) What about the four-fermion interaction in two dimensions i.e. $d = 2$?

(c) Consider a boson with general polynomial interactions in space-time dimension d:

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \sum_{k=2}^{\infty} \frac{g_k}{k!}\phi^k \tag{22}$$

First suppose that besides the mass term g_2, only a single interaction term g_k with $k > 2$ is present. What is the condition on d for the theory to be renormalisable? In particular, in what space time dimensions are the interactions ϕ^3, ϕ^4 and ϕ^6 renormalisable?

Is there a space-time dimension in which the theory including all powers is renormalisable?

Hint: use the general relation (17) *for power counting.*

Addendum Proof of the general power counting formula (17)

In general power counting can be done by simple dimensional analysis. Note that the degree of divergence of an integral

$$\int \mathrm{d}^n k \frac{1}{k^\alpha} \tag{23}$$

is the same as its energy dimension i.e. $n - \alpha$.

The dimension of the n-point function in momentum space

$$\tilde{G}_{\alpha_1 \ldots \alpha_n}(p_1, \ldots, p_n) = \int \mathrm{d}^d x_1 e^{ip_1x_1} \cdots \int \mathrm{d}^d x_n e^{ip_nx_n} \langle \phi_{\alpha_1}(x_1) \ldots \phi_{\alpha_n}(x_n) \rangle \tag{24}$$

is $[\tilde{G}] = -nd + \sum_{k=1}^{n} [\phi_{\alpha_k}]$. Due to translational invariance, one can always factor out a delta function

$$\tilde{G}_{\alpha_1 \dots \alpha_n}(p_1, \dots, p_n) = \bar{G}_{\alpha_1 \dots \alpha_n}(p_1, \dots, p_n)(2\pi)^d \delta^{(d)}(p_1 + \dots + p_n) \tag{25}$$

with $[\bar{G}] = -(n-1)d + \sum_{k=1}^{n} [\phi_{\alpha_k}]$.

Primitive divergences are one-particle irreducible, so the legs can be amputated to isolate them:

$$\bar{G}^{amp}_{\alpha_1 \dots \alpha_n}(p_1, \dots, p_n) = \frac{\bar{G}_{\alpha_1 \dots \alpha_n}(p_1, \dots, p_n)}{\bar{G}_{\alpha_1 \alpha_1}(p_1, -p_1) \dots \bar{G}_{\alpha_n \alpha_n}(p_n, -p_n)} \tag{26}$$

with

$$[\bar{G}^{amp}] = -(n-1)d + \sum_{k=1}^{n} [\phi_{\alpha_k}] - \sum_{k=1}^{n} (-d + 2[\phi_{\alpha_k}]) = d - \sum_{k=1}^{n} [\phi_{\alpha_k}] \tag{27}$$

Let us denote the number of vertices by V, and the coupling at the lth vertex by λ_l. The total dimension of a graph contributing to $\bar{G}^{amp}$ can also be obtained as the sum of the dimension D of the momentum integral plus the dimensions of the couplings

$$d - \sum_{k=1}^{n} [\phi_{\alpha_k}] = D + \sum_{l=1}^{V} [\lambda_l] \tag{28}$$

Since D is identical to the degree of divergence, relation (17) follows.