Problem set 10 for Quantum Field Theory course

2019.04.23.

Topics covered

- Renormalisation: one loop structure of ϕ^4
- Counter terms in Yukawa theory: corrections to fermion and boson propagators, and the vertex term
- Renormalisation with tensor indices: $O(N)$ theory

Recommended reading
Peskin–Schroeder: An introduction to quantum field theory

- Sections 10.2, 11.2

Problem 10.1 Interaction counter term in ϕ^4-theory

For the interacting boson field, the counter terms are:

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi_R)^2 - \frac{1}{2}m^2\phi_R^2 - \frac{\lambda}{4!}\phi_R^4 + \frac{1}{2}\delta_Z(\partial_\mu \phi_R)^2 - \frac{1}{2}\delta_m \phi_R^2 - \frac{\delta_\lambda}{4!}\phi_R^4\,, \tag{1}$$

where ϕ_R is the rescaled field

$$\phi = Z^{1/2}\phi_R \tag{2}$$

with the field strength renormalisation Z. λ and m are the physically measured values and δ_λ, δ_m, δ_Z are the counter terms fixed by the renormalisation conditions. In this exercise we are going to calculate δ_λ up to leading order in perturbation theory.

(a) Draw the diagrams up to second order in λ for a $p_1, p_2 \to p_3, p_4$ process. Bear in mind that δ_λ term from (1) also generates such a graph.

(b) The renormalisation condition determining δ_λ

$$i\mathscr{M}(p_1, p_2 \to p_3, p_4)\Big|_{s=4m^2,\,t=u=0} = -i\lambda \tag{3}$$

at any order of perturbation theory. The contributions of the three channels can be written as

$$i\mathscr{M} = -i\lambda + (-i\lambda)^2[iV(s) + iV(t) + iV(u)] - i\delta_\lambda + O(\lambda^3)\,, \tag{4}$$

with

$$iV(p^2) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\varepsilon}\frac{i}{(k+p)^2 - m^2 + i\varepsilon}\,. \tag{5}$$

Perform the momentum integral using the methods we learnt in Problems 5.1-5.3. Derive the result

$$V(p^2) = -\frac{1}{2}\int_0^1 dx \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}\frac{1}{\left[m^2 - x(1-x)p^2\right]^{2-d/2}}\,. \tag{6}$$

(c) Express δ_λ using (6). Take the limit $\varepsilon = 4 - d \to 0$ to get rid of the divergent terms and write $i\mathscr{M}$ up to $O(\lambda^3)$.

Problem 10.2 Counter term for the Yukawa boson's mass

In this exercise we consider Yukawa theory:

$$\mathscr{L} = \mathscr{L}_{KG} + \mathscr{L}_D - g_0 \overline{\Psi}\Psi\phi\,, \tag{7}$$

with

$$\mathscr{L}_{KG} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2\,, \tag{8}$$

and

$$\mathscr{L}_D = \overline{\Psi}(i\partial\!\!\!/ - M_0)\Psi\,. \tag{9}$$

(a) Rescale the fields with their field strength renormalisation (Z for ϕ and Z_2 for Ψ) and write the Lagrangian as a sum of renormalised Lagrangian and counter terms. What are the counter-terms $\delta_g, \delta_Z, \delta_m, \delta_M, \delta_{Z_2}$ in terms of the bare and physical parameters?

(b) Draw the one-loop contribution to the boson propagator and show that it leads to the following momentum integral:

$$-i\Sigma(p^2) = -(-ig)^2\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left[\frac{i(k\!\!\!/ + p\!\!\!/ + M)}{(k+p)^2 - M^2 + i\varepsilon}\,\frac{i(k\!\!\!/ + M)}{k^2 - M^2 + i\varepsilon}\right] + \text{counter terms.} \tag{10}$$

Hint: watch out for the minus sign coming from the fermion loop.

(c) Utilize the trace identities and methods learned in Problems 5.1-5.3 to perform the integral. Write it in d dimensions as

$$-\frac{4ig^2}{(4\pi)^{d/2}}\int_0^1 dx\Delta^{d/2-1}(1-d)\Gamma(1-d/2)\,, \tag{11}$$

with

$$\Delta = M^2 - x(1-x)p^2\,. \tag{12}$$

(d) Now show that the divergent part can be cancelled by a

$$i(\delta_Z p^2 - \delta_m) \tag{13}$$

counter term. Determine δ_Z and δ_m from the conditions

$$\Sigma(p^2 = m^2) = 0\,, \qquad \left.\frac{\mathrm{d}\Sigma(p^2)}{\mathrm{d}p^2}\right|_{p^2=m^2} = 0\,. \tag{14}$$

Problem 10.3 Counter-term for the Yukawa fermion's mass

We start with the theory (7) and consider corrections to the mass of the Dirac fermion.

(a) Draw the one-loop diagram that contributes to the fermion propagator and show that it leads to the following momentum integral:

$$-i\Sigma(p\!\!\!/) = g^2\int\frac{d^4k}{(2\pi)^4}\frac{k\!\!\!/ + M}{k^2 - M^2 + i\varepsilon}\,\frac{1}{(k+p)^2 - m^2 + i\varepsilon} + \text{counter tems.} \tag{15}$$

(b) Perform this integral using the methods learnt in Problems 5.1-5.3. What is the divergent part?

(c) Determine the counter terms δ_M and δ_{Z_2} from the conditions

$$\Sigma(p\!\!\!/ = M) = 0\,, \qquad \left.\frac{\mathrm{d}\Sigma(p\!\!\!/)}{\mathrm{d}p\!\!\!/}\right|_{p\!\!\!/=M} = 0\,. \tag{16}$$

Problem 10.4 Renormalisation of the Yukawa vertex

In the model defined by Eq. (7) there is one last counter term to consider: δ_g.

(a) Draw the one-loop correction to the vertex function. Note that it contains three vertices and can be written as

$$g^3 \int \frac{d^4k}{(2\pi)^4} \frac{\not{k}+M}{k^2-M^2+i\varepsilon} \frac{\not{k}+M}{k^2-M^2+i\varepsilon} \frac{1}{k^2-m^2+i\varepsilon}\,. \tag{17}$$

(b) Perform the divergent part of the integral. Determine δ_g.

Hint: since the divergence is independent of outer momenta, we can set them all to 0. *Divergence appears only when* k^2 *shows up in the numerator, so it is sufficient to keep that term. Add a finite term to take out one of the factors from the denominator, then with two denominators all earlier tricks can be applied.*

Problem 10.5 Renormalisation and tensor structure: $O(N)$ theory.

The interacting boson field can be generalised to describe an N-component scalar field:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^k\partial^\mu\phi^k - \frac{1}{2}m^2\phi^k\phi^k - \frac{\lambda}{8}\left(\phi^k\phi^k\phi^l\phi^l\right), \tag{18}$$

with $k = 1,2,...,N$, and summation over repeated indices of the field.

Remark: the Lagrangian is invariant under rotation of ϕ^k *in the* N *vector space spanned by the fields, and consequently it is called the* $O(N)$ *model.*

(a) Write down the Feynman rules of this model.

Hint: assign and index to each external leg. Upon contraction this gives rise to Kronecker deltas in three possible arrangements. Check that the symmetry factors follow the usual rule of being the number of elements of the invariance group of the graph.

(b) Similarly to the case $N = 1$ there are three counter terms. Using the results of Problem 10.1. and also the ϕ^4 calculations from the lecture, show that only two of them are nonzero at one-loop level, and compute the corresponding counter terms.

Hint: note that the only new task is to compute the index expressions, as the Feynman integrals are identical to the $N = 1$ *case. Also remember that* $\delta^{kk} = N$.