Problem set 9 for Quantum Field Theory course

2019.04.16.

Topics covered

- Path integrals at finite temperature
- Fermionic and bosonic path integrals
- Euclidean path integral and Feynman rules
- Casimir effect
- Convergence of perturbative series

Recommended reading

- Peskin–Schroeder: An introduction to quantum field theory
  Section 9.5
- R.J. Rivers: Path integral methods in quantum field theory
  Sections 1.8, 5.3

Problem 9.1 Finite temperature path integral: harmonic oscillator

Systems at finite temperature are described by the quantum statistical partition function:

$$Z = \mathrm{Tr}\left[e^{-\beta H}\right] = \int D\Psi \left\langle\Psi\right| e^{-\beta H} \left|\Psi\right\rangle , \tag{1}$$

with $\beta = 1/k_B T$ is the inverse temperature and the trace is performed by integrating over the continuum space of eigenfunctions $\Psi(\mathbf{x})$.

(a) Write the above expression as a functional integral of a field $\Phi(\mathbf{x},\tau)$ by observing that the inverse temperature can be thought of as imaginary time. What are the boundary conditions?

(b) This procedure of working with imaginary time is the Wick rotation at the level of path integrals. It is instructive to perform this rotation on a simple problem:

$$\int DqDp e^{i\int_{t_i}^{t_f} dt \sum_a [\dot{q}(a)p(a) - H(p(a),q(a))]}, \tag{2}$$

where

$$H(p,q) = \frac{p^2}{2m} + V(q), \tag{3}$$

and a indices the discretization. Use $\tau = it$, Hamilton's equations and the derivation from the lecture to bring (2) to the form below:

$$\mathcal{N}\int_{q(\beta)=q(0)} Dq(\tau) e^{-\int_0^\beta d\tau L_E(q)} . \tag{4}$$

What is the Euclidean Lagrangian L_E?

(c) Calculate $S_E = \int_0^\beta d\tau L_E$ for the quantum harmonic oscillator with unit mass

$$L_E = \frac{1}{2}\left(\dot{q}^2 + \omega^2 q^2\right). \tag{5}$$

Use a Fourier decomposition that satisfies the periodic boundary conditions:

$$q(\tau) = \sum_{n=-\infty}^{\infty} \frac{c_n}{\sqrt{\beta}} e^{2\pi i n\tau/\beta}. \tag{6}$$

Hint: q is real so there is a relation between the complex c_n coefficients.

(d) Write the result in terms of the real parameters a_n and b_n:

$$c_n = \frac{a_n + i b_n}{\sqrt{2}}, \tag{7}$$

and perform the Gaussian integrals. By using the identity

$$\frac{\sinh x}{x} = \prod_{n=1}^{\infty}\left(1 + \frac{x^2}{n^2\pi^2}\right) \tag{8}$$

show that this path integral (up to a β-dependent normalization factor) coincides with the partition function of the quantum harmonic oscillator.
Hint: from statistical physics we know it is given by $Z = \sum_{n=0}^{\infty} e^{-\beta\omega(n+1/2)}$.

Problem 9.2 Finite temperature path integral: free boson and fermion

In the previous exercise we have seen that we can use the path integral formalism for finite temperature systems by performing a Wick rotation. We get for the partition function

$$Z = \int D\phi e^{-S_E}, \tag{9}$$

where S_E is called the Euclidean action. For a free scalar field it reads

$$S_E = \int d^d x \int_0^\beta d\tau \frac{1}{2}\left(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\right). \tag{10}$$

(a) Perform two partial integrations and use arguments from Problem 8.4 to show that the partition function is proportional to

$$Z \sim \left[\det\left(-\partial_\tau^2 - \nabla^2 + m^2\right)\right]^{-1/2}. \tag{11}$$

(b) This determinant can be calculated explicitly with the help of mode expansion. Write $\phi(\tau, \mathbf{x})$ as

$$\phi(\tau, \mathbf{x}) = \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n\tau/\beta}}{\sqrt{\beta}} \int \frac{d^d k}{(2\pi)^d} e^{i\mathbf{k}\mathbf{x}} \phi_{n\mathbf{k}}. \tag{12}$$

Write this back into (10) and use the tricks of Problem 9.1(d) to show that the result is

$$Z = C \prod_{\mathbf{k}} \frac{e^{-\beta\omega_{\mathbf{k}}/2}}{1 - e^{-\beta\omega_{\mathbf{k}}}}, \tag{13}$$

where C is a β-dependent constant and

$$\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}. \tag{14}$$

(c) Do a similar calculation for a fermionic oscillator. The Euclidean action is

$$S_E = \int_0^\beta d\tau \left(\overline{\Psi}\dot{\Psi} + \omega\overline{\Psi}\Psi\right). \tag{15}$$

If we impose *anti-periodic* boundary conditions, the mode expansion looks like

$$\Psi = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{\infty} \Psi_n e^{2\pi i(n+1/2)\tau/\beta}, \tag{16}$$

and similarly for $\overline{\Psi}$. Utilize these expressions to write

$$S_E = \sum_{n=-\infty}^{\infty} \overline{\Psi}_n \left[\frac{2\pi i}{\beta}(n+1/2) + \omega\right] \Psi_n. \tag{17}$$

(d) Calculate the partition function

$$Z = \int D\Psi D\overline{\Psi} e^{-S_E}. \tag{18}$$

Bear in mind that these are fermionic fields so they behave as Grassmann numbers which simplifies the exponential term. A useful identity here is

$$\cosh x = \prod_{n=0}^{\infty} \left(1 + \frac{x^2}{\pi^2(n+1/2)^2}\right). \tag{19}$$

Show that in the end the partition function is

$$Z = C(\beta)\cosh\big(\beta\omega/2\big), \tag{20}$$

that is the partition function for a two-state system, the correct result for the case of Fermi–Dirac statistics with a single degree of freedom.

(e) Bonus question: why did we have to impose antiperiodic boundary condition?
Consider the Euclidean two-point function:

$$G_F(\tau_1, \tau_2) = \frac{1}{Z} Tr\left(e^{-\beta H} T(\Psi(\tau_1)\bar{\Psi}(\tau_2))\right) \tag{21}$$

Using the relations

$$T(\Psi(\tau_1)\bar{\Psi}(\tau_2)) = \theta(\tau_1 - \tau_2)\Psi(\tau_1)\bar{\Psi}(\tau_2) - \theta(\tau_2 - \tau_1)\bar{\Psi}(\tau_2)\Psi(\tau_1), \tag{22}$$

$$\Psi(\tau) = e^{\tau H}\Psi(0)e^{-\tau H} \tag{23}$$

show that

$$G_F(0, \tau_2) = -G_F(\beta, \tau_2) \tag{24}$$

implying

$$\Psi(\beta) = -\Psi(0). \tag{25}$$

Show that the same argument leads to periodicity for bosonic fields!
Food for thought: why is the antiperiodic boundary condition of the fermion field consistent with the periodic nature of the trace operation?

Problem 9.3 Euclidean path integral: Feynman rules

In this exercise we consider the Wick-rotated (Euclidean) path integral formulation of the ϕ^4 theory. In Minkowski space, this theory is described by the Lagrangian density

$$\mathscr{L}_M = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{26}$$

The action is given by

$$S_M = \int d^4x \mathscr{L}_M. \tag{27}$$

(a) Perform a Wick rotation in time

$$x^4 = i x^0 \tag{28}$$

and calculate the Euclidean action S_E and Lagrangian $\mathscr{L}_E$

$$S_E = \int d^4 \bar{x} \mathscr{L}_E , \tag{29}$$

where $\bar{x} = (x^1, x^2, x^3, x^4)$.

(b) Working with the free Lagrangian

$$\mathscr{L}_{E0} = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \tag{30}$$

derive the Feynman rule for the propagator.

Hint: a similar calculation was performed at the lecture in Minkowski space-time. It can be done by calculating the functional integral

$$\langle 0| T\phi(\bar{x}_1)\phi(\bar{x}_2) |0\rangle = \frac{\int D\phi\, \phi(\bar{x}_1)\phi(\bar{x}_2) e^{-S_E}}{\int D\phi e^{-S_E}} . \tag{31}$$

(c) Now treat the interaction term perturbatively with λ small enough:

$$\mathscr{L}_E = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 . \tag{32}$$

Expand the exponential with the interaction term up to first order in λ and read off the Feynman rule for the vertex.

(d) For an arbitrary connected graph the number of loops equals the number of vertices minus internal propagators plus 1:

$$L = P - V + 1 . \tag{33}$$

Combine this knowledge with your previous result to show that for any **connected** n-point function W the Minkowski and Euclidean results are simply related:

$$W_M(p_1, p_2, ..., p_n) = i(-i)^n W_E(\bar{p}_1, \bar{p}_2, ..., \bar{p}_n) , \tag{34}$$

where $\bar{p}$ are Euclidean momenta.

Problem 9.4 Casimir effect

This exercise aims to demonstrate the richness of quantum field theory vacuum. The attractive force in vacuum experienced by two metallic plates is called the Casimir effect. Let us consider two metallic plates that are squares of size L, their distance (say, along the z axis) is a.

(a) The ground state energy of a Hamiltonian describing an infinite set of harmonic oscillators is

$$\langle H \rangle = \sum_{\mathbf{k}} \frac{1}{2}\hbar\omega_{\mathbf{k}} . \tag{35}$$

Show that using the dispersion relation $\omega_{\mathbf{k}} = c\mathbf{k}$ the energy E stored between the plates can be written as

$$E = \frac{\hbar c}{2} L^2 \int \frac{d^2 k_\parallel}{(2\pi)^2} \left[|\mathbf{k}_\parallel| + 2 \sum_{n=1}^{\infty} \sqrt{\mathbf{k}_\parallel{}^2 + \left(\frac{n\pi}{a}\right)^2} \right] , \tag{36}$$

where $\mathbf{k}_\parallel$ is the wavenumber component parallel to the plates.

Hint: recall k_z is quantized and also that for $k_z = 0$ there is only one polarization instead of two (why?).

(b) Write the same energy in the $V = aL^2$ volume without the presence of these plates.

$$E_0 = \frac{\hbar c}{2} L^2 \int \frac{d^2 k_\parallel}{(2\pi)^2} a \int \frac{dk_z}{2\pi} 2\sqrt{{\mathbf{k}_\parallel}^2 + k_z^2}. \tag{37}$$

Perform a variable transform

$$k_z = \frac{n\pi}{a} \tag{38}$$

in the k_z integral to get an expression involving an integral over n (similarly to sum over n in the previous expression).

(c) Let us write the energy difference due to the plates over unit area as

$$\frac{E - E_0}{L^2}. \tag{39}$$

Write it as an integral over $k = |\mathbf{k}_\parallel|$ (the angular integral gives a factor of 2π). This integral is divergent. Let us introduce a cutoff-function $\Lambda(k)$ such that $\Lambda(0) = 1$, $\Lambda(\infty) = 0$, and $\Lambda^{(n)}(0) = \Lambda^{(n)}(\infty) = 0$ for any $n > 0$th derivative of $\Lambda(k)$.

By multiplying all divergent contributions with Λ bring the integral to the form

$$\frac{\Delta E}{L^2} = \frac{\hbar c}{2\pi} \int_0^\infty k dk \left[\frac{k}{2}\Lambda(k) + \sum_{n=1}^{\infty} \sqrt{k^2 + \left(\frac{n\pi}{a}\right)^2} \Lambda(\surd) - \int_0^\infty dn \sqrt{k^2 + \left(\frac{n\pi}{a}\right)^2} \Lambda(\surd) \right], \tag{40}$$

where we used the brief notation

$$\Lambda(\surd) = \Lambda\left(\sqrt{k^2 + \left(\frac{n\pi}{a}\right)^2} \right). \tag{41}$$

(d) Introduce a new variable such that

$$u = \frac{a^2 k^2}{\pi^2}, \tag{42}$$

so the energy difference per unit area is

$$\frac{\hbar c \pi^2}{4a^3} \left[\frac{1}{2} F(0) + \sum_{n=1}^{\infty} F(n) - \int_0^\infty dn F(n) \right]. \tag{43}$$

What is $F(n)$?

(e) For taking difference of the sum and integral use the Euler–Maclaurin formula:

$$\sum_{j=m}^{n} f(j) = \int_m^n f(x) dx + \frac{f(n) + f(m)}{2} + \sum_{k=1} \frac{B_{2k}}{(2k)!} \left[f^{(2k-1)}(n) - f^{(2k-1)}(m) \right] + R, \tag{44}$$

where B_i are Bernoulli numbers and R is the residual difference of the two sides. Keep the first non-vanishing term (with respect to the summation over k) and exploiting the properties of $\Lambda(k)$ show that

$$\frac{\Delta E}{L^2} = -\frac{\hbar c \pi^2}{720 a^3}. \tag{45}$$

Note that the result is independent of the regulator function $\Lambda(k)$.

Hint: it is useful to introduce a variable transform in the integral in $F(n)$ such that its argument appears only once in the expression.

(f) Calculate the force over unit area

$$\frac{F}{L^2} = -\frac{\partial(\Delta E / L^2)}{\partial a} \tag{46}$$

to show that the plates experience an attractive force.

Figure 1: Feynman rules in zero space-time dimensions.

Problem 9.5 Convergence of perturbative series

Perturbation theory is a very efficient method to obtain powerful predictions from Quantum Field Theory. It is interesting to ask some fundamental questions about the convergence of these perturbative series. We are going to investigate this topic with the help of a *zero*-dimensional model.

Let us define a ϕ^4-like theory with the help of the diagrammatic rules shown in Fig. 1.

The m-point function can be expanded as

$$G_m = \sum_{p=0}^{\infty} g^p G_m^{(p)}, \tag{47}$$

so $G_m^{(p)}$ is the number of diagrams with m external legs and p vertices since propagators are 1.

(a) We can write the generating functional as

$$Z[j] = \sum_{m=0}^{\infty} \frac{i^m j^m}{m!} G_m, \tag{48}$$

where i is the complex unit. Show by counting the number of graphs $G_m^{\text{free}} = G_m^{(0)}$ in the free $g = 0$ case that

$$Z_{\text{free}}[j] = \exp\left[-j^2/2\right]. \tag{49}$$

(b) By performing a Fourier transform one can write the generating functional of the interacting theory (applying formulae of Problem 8.5 to the zero-dimensional case)

$$Z[j] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} da \exp\left(-\frac{a^2}{2} - \frac{g a^4}{4!} + i a j\right). \tag{50}$$

Use this to write the $2m$-point functions as

$$G_{2m} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} da\, a^{2m} e^{-S(a)/\hbar}, \tag{51}$$

where we reintroduced $\hbar$ and

$$S(a) = \frac{1}{2} a^2 + \frac{g}{4!} a^4 \tag{52}$$

is the zero-dimensional action. Rescale the variable $a = b\hbar^{1/2}$ to show that we get a $g\hbar$ coefficient for the interaction term. This means that perturbative expansion in g is equivalent to an expansion in $\hbar$.

(c) We are going to illustrate limitations of perturbative expansion in $\hbar$ on

$$Z[0] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} da\, e^{-S(a)/\hbar}, \tag{53}$$

since this can be easily modified for the case of Green's functions. Perform a variable transform to $u(a)$ such that

$$\begin{aligned} u(a) &= \sqrt{S(a)}, \quad \text{if} \quad a \geq 0 \\ u(a) &= -\sqrt{S(a)}, \quad \text{if} \quad a < 0. \end{aligned} \tag{54}$$

Write $Z[0]$ as an integral over u.

(d) In the integral $1/u'$ appears. Expand it as a power series in u

$$(u')^{-1} = \sum_{r=0}^{\infty} c_r u^r \,. \tag{55}$$

It has a finite convergence radius r_C on the complex u plane. Perform the integration over u order by order to get

$$Z[0] = \frac{\sqrt{\hbar}}{2} \sum_{p=0}^{\infty} \hbar^p c_{2p} \Gamma(p + 1/2) \,. \tag{56}$$

Use the definition of the convergence radius

$$\lim_{p\to\infty} |c_{2p}|^{1/2p} = r_C \tag{57}$$

to show that expression (56) is divergent for any $\hbar \neq 0$.

Remark: This is a consequence of integrating beyond the convergence radius in u, thus the interchange between integral and summation leads to divergence. In an analogous way this leads to divergence of perturbative expansions in non-zero dimensional QFT as well. That, however involves some additional technicalities, the interested reader is referred to Section 5.5 of Rivers' book and to [1].

References

[1] Zinn-Justin, J. Perturbation series at large orders in quantum mechanics and field theories: Application to the problem of resummation. (1981) Phys. Rep. **70C** 109 No 2. https://doi.org/10.1016/0370-1573(81)90016-8.