Problem set 8 for Quantum Field Theory course

2019.04.09.

Topics covered

- Particle creation by external field
- Quasiclassical fields, coherent states
- Path integral: harmonic oscillator
- Stationary phase approximation
- Generating functionals

Recommended reading
Peskin–Schroeder: An introduction to quantum field theory

- Sections 2.4, 9.1, 9.2

Problem 8.1 Particle creation by external source

In this exercise we consider a real scalar field in presence of an external source $j(x)$. This corresponds to the following Lagrangian for the field:

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + j\phi. \tag{1}$$

(a) Write down the Euler–Lagrange equations to obtain

$$\left(\partial^\mu \partial_\mu + m^2\right)\phi(x) = j(x). \tag{2}$$

We are going to solve this equation using Green's function technique.

(b) Show by explicit calculation that the retarded Green's function $D_R(x-y) = \theta(x^0-y^0)\langle 0[\phi(x),\phi(y)]|0\rangle$ satisfies

$$\left(\partial^\mu \partial_\mu + m^2\right) D_R(x-y) = -i\delta^{(4)}(x-y), \tag{3}$$

that is, $iD_R(x-y)$ is a Green's function of the Klein–Gordon equation.
(Hint: use that $\delta'(x)f(x) = -\delta(x)f'(x)$ and exploit the canonical commutation relations.)

(c) If there were no source, the solution would be the usual

$$\phi_0(x) = \int d^3k \frac{1}{\sqrt{2E_\mathbf{k}}}\left(a_\mathbf{k} e^{-ikx} + a^\dagger_\mathbf{k} e^{ikx}\right). \tag{4}$$

With $j(x) \neq 0$ we have

$$\phi(x) = \phi_0(x) + i\int d^4y D_R(x-y)j(y), \tag{5}$$

where explicitly

$$D_R(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\theta(x^0-y^0)\left(e^{-ip(x-y)} - e^{ip(x-y)}\right)\Big|_{p^0=E_\mathbf{p}} \tag{6}$$

Assume $j(x)$ has a compact support in time (i.e. it was "turned on" for a finite time interval) and write $\phi(x)$ for large times, that is when $j(x) \equiv 0$.
Hint: write the result in terms of $\tilde{j}(\mathbf{p}) = \int d^4x e^{ipx} j(x)|_{p^0=E_\mathbf{p}}$, the Fourier transform of j.

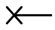 $= \int d^4x\, j(x)\cdot$

Figure 1: Feynman rule for a classical source.

(d) Write down the Hamiltonian of the system. Show that its expectation value in free vacuum is

$$\langle 0|\, H\, |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} E_{\mathbf{p}} |\tilde{j}(\mathbf{p})|^2 \,. \tag{7}$$

That is, the probability (density) for our source $j(x)$ to create a particle mode with momentum $\mathbf{p}$ is

$$P(\mathbf{p}) = |\tilde{j}(\mathbf{p})|^2 \,. \tag{8}$$

Integration over all momenta yields the expectation value of particle number after interaction with the source:

$$\langle N\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} |\tilde{j}(\mathbf{p})|^2 \equiv \lambda \,. \tag{9}$$

(e) There is another approach to this problem using Wick's theorem. The Hamiltonian of this system can be written as

$$H = H_0 + \int d^3x (-j(t,\mathbf{x})\phi(t,\mathbf{x})) \,. \tag{10}$$

Then the probability for no particle creation is just

$$P(0 \text{ part.}) = |\, \langle 0|\, T e^{i\int d^4x\, j(x)\phi_I(x)}\, |0\rangle\, |^2 \,. \tag{11}$$

We are going to do a perturbative expansion in source strength j. Show that up to fourth order in j the result is

$$P(0 \text{ part.}) = 1 - \lambda + O(j^4) \,, \tag{12}$$

where λ is defined in Eq. (9).

Hint: $j(x)$ has a compact time support so for $|p^0| \to \infty$ in any direction on the complex p^0 plane it vanishes. Utilize this fact to perform the p^0 integration coming from the propagator.

(f) One can represent terms of this perturbative expansion as Feynman diagrams. The propagator can be dealt with in the usual way and we have an additional rule, shown in Fig. 1.

Draw representation of higher order terms and calculate their symmetry factor (we can permute "crosses", choose connection of crosses via propagators and there is a symmetry factor from the fact that propagators do not have a direction). Show that the series exponentialize to yield the final result

$$P(0 \text{ part.}) = e^{-\lambda} \,. \tag{13}$$

(g)* Similarly, one can show that the probability of creating 1 particle is $P(1) = \lambda + O(j^4)$, summing up all the diagrams gives $P(1) = \lambda e^{-\lambda}$. Finally, the probability of creating n particles is $P(n) = \frac{\lambda^n}{n!} e^{-\lambda}$, i.e. it is a Poisson process. It follows that the mean particle number is $\overline{n} = \sum_{n=0}^{\infty} P(n) n = \lambda$, in accordance with Eq. (9).

Problem 8.2 Field with external source: coherent states

Consider the free boson coupled to a space-dependent external field $j(\mathbf{x})$,

$$H = H_0 - \int d^3x\, j(\mathbf{x})\phi(t,\mathbf{x}) \,. \tag{14}$$

Using the mode expansion of the field show that the Hamiltonian is

$$H = \int \frac{d^3p}{(2\pi)^3}\left[E_{\mathbf{p}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} - \frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}} J^\star(\mathbf{p}) + a^\dagger_{\mathbf{p}} J(\mathbf{p})\right)\right]. \tag{15}$$

where $J(\mathbf{p}) = \int d^3x e^{i\mathbf{px}} j(\mathbf{x})$ is the 3-dimensional Fourier transform of $j(\mathbf{x})$.

(a) Introduce a new operator $b_{\mathbf{p}}$ to bring this expression to the form

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}}\left(b^\dagger_{\mathbf{p}} b_{\mathbf{p}} - \eta^\star_{\mathbf{p}}\eta_{\mathbf{p}}\right). \tag{16}$$

What is $b_{\mathbf{p}}$ and $\eta_{\mathbf{p}}$?

(b) Show that if we define a new vacuum as

$$b_{\mathbf{p}}\left|\tilde{0}\right\rangle = 0, \tag{17}$$

then we get for the "free" annihilation operator the following result:

$$a_{\mathbf{p}}\left|\tilde{0}\right\rangle = \eta_{\mathbf{p}}\left|\tilde{0}\right\rangle. \tag{18}$$

Consequently, the new vacuum is the eigenstate of the annihilation operator of each mode. Such states are called coherent states.

(c) Let us explore some general properties of coherent states focusing on a single mode. They cannot have a definite particle number (by definition), so we can write a coherent state $\left|\eta\right\rangle$ as

$$\left|\eta\right\rangle = \sum_n c_n \left|n\right\rangle, \tag{19}$$

with

$$\left|n\right\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}\left|0\right\rangle. \tag{20}$$

Calculate $\langle m| a \left|\eta\right\rangle$ in two different ways to express c_n in terms of η and n. Note that c_0 does not get determined but will be fixed by the normalization.

(d) Normalising the coherent state as $\left\langle\eta|\eta\right\rangle = 1$, show that

$$\left|\eta\right\rangle = e^{-|\eta|^2/2}\sum_{n=0}^{\infty}\frac{\eta^n}{\sqrt{n!}}\frac{a^{\dagger n}}{\sqrt{n!}}\left|0\right\rangle = e^{-|\eta|^2/2}e^{\eta a^\dagger}\left|0\right\rangle. \tag{21}$$

Note that using the Baker–Campbell–Hausdorff formula ($e^X e^Y = e^{X+Y+1/2[X,Y]}$) this can also be written as $\left|\eta\right\rangle = e^{\eta a^\dagger - \eta^* a}\left|0\right\rangle$.

(e) Calculate particle number expectation value $\langle n\rangle = \left\langle a^\dagger a\right\rangle$ in a coherent state and also $\left\langle (n-\langle n\rangle)^2\right\rangle$ to recognize the first two momenta of a Poisson distribution.

(f) In the quantum mechanical problem of a particle in a harmonic oscillator the creation and annihilation operators are related to the position and momentum as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \qquad \hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger). \tag{22}$$

Compute the expectation value of position and momentum operators in a coherent state for which the eigenvalue η depends on time as $\eta(t) = \eta(0)e^{-i\omega t}$ to show that they follow the classical equations of motion.
Show that the fluctuations of these operators (e.g. $\Delta x^2 = \left\langle \hat{x}^2\right\rangle - \langle\hat{x}\rangle^2$) are independent of time and compute $\Delta x \cdot \Delta p$.
(Bonus: Solving the differential equation following from $a\left|\eta\right\rangle = \eta\left|\eta\right\rangle$ show that at any time the real space wave function of $\eta(t)$ is a Gaussian.)

(g) Finally, return to our initial problem and write the ground state in the presence of the source, in finite volume as a product of coherent states for the quantized momentum modes

$$|\tilde{0}\rangle = \prod_{\mathbf{p}} |\eta_{\mathbf{p}}\rangle = e^{\sum_{\mathbf{p}}\left(-1/2|\eta_{\mathbf{p}}|^2+\eta_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}\right)}|0\rangle \,. \tag{23}$$

which in infinite volume can be written as $e^{\int \frac{d^3p}{(2\pi)^3}\left(-1/2|\eta_{\mathbf{p}}|^2+\eta_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}\right)}|0\rangle$. Calculate the expectation value of the free Hamiltonian on this vacuum to show that

$$\langle\tilde{0}| H_{\text{free}} |\tilde{0}\rangle = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} |\eta_{\mathbf{p}}|^2 \,. \tag{24}$$

Problem 8.3 Path integral of quantum harmonic oscillator

Feynman's path integral formalism states that the propagator can be expressed as it follows:

$$\langle q_f | e^{-iHt} | q_i \rangle = \int Dq e^{iS[q]} \,, \tag{25}$$

so it is a functional integral, where $S[q] = \int_0^t dt' L(q,\dot{q})$ is the action.

(a) Write down the action of a harmonic oscillator which has the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \,. \tag{26}$$

(b) Perform the functional integral on paths that are slightly fluctuating around the classical trajectory. First, solve the equation of motion for the classical trajectory $q_{cl}(t')$ using the boundary conditions $q_{cl}(0) = q_i$ and $q_{cl}(t) = q_f$.

(c) Write the path on which we integrate as

$$q(t') = q_{cl}(t') + \delta q(t') \,. \tag{27}$$

We can then separate the action as

$$S[q] = S[q_{cl}] + S[\delta q] \,. \tag{28}$$

Calculate $S[q_{cl}]$ explicitly.

(d) For integration over δq quantum fluctuations we perform a Fourier series expansion satisfying the boundary conditions:

$$\delta q(t') = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{t} t'\right) , \tag{29}$$

so the functional integration is changed to integration over a_n coefficients:

$$\int Dq \to c \int \prod_{n=1}^{\infty} da_n \,, \tag{30}$$

where c is a normalization factor that can be obtained from path integral of a free particle. Utilize the orthogonality condition

$$\int_0^t dt' \sin\left(\frac{n\pi}{t} t'\right) \sin\left(\frac{k\pi}{t} t'\right) = \delta_{kn} t/2 \tag{31}$$

to obtain the following result

$$S[\delta q] = \sum_{n=1}^{\infty} \frac{m}{2} \frac{a_n^2}{2} \left(\frac{(n\pi)^2}{t} - \omega^2 t \right) . \tag{32}$$

Hint: terms linear in a_n are excluded by the extremum principle so it is sufficient to restrict the calculation to second order terms.

(e) Use the normalization of the integration measure obtained from the free particle case:

$$Dq = \sqrt{\frac{m}{2\pi i t}} \prod_{n=1}^{\infty} \sqrt{\frac{m}{2\pi i t}} \frac{n\pi}{\sqrt{2}} da_n , \tag{33}$$

and perform the a_n integrals. Utilize the identity

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) = \frac{\sin \pi x}{\pi x} \tag{34}$$

to obtain the final result:

$$\langle q_f | e^{-iHt} | q_i \rangle = \sqrt{\frac{m\omega}{2\pi i \sin \omega t}} \exp\left[\frac{i m\omega}{2}\left((q_i^2 + q_f^2)\cot(\omega t) - \frac{2q_i q_f}{\sin(\omega t)}\right)\right]. \tag{35}$$

Problem 8.4 Stationary phase approximation

The previous exercise showed a solution of the path integral of a quantum harmonic oscillator that consisted of integration of quantum oscillations superposed on a classical path. We can explore this line of thought further in the context of stationary phase approximation.

To do so, we reintroduce units such that $\hbar \neq 1$ can play the role of parameterizing strength of the quantum behaviour. This way the path integral is

$$\langle q_f | e^{-\frac{i}{\hbar}Ht} | q_i \rangle = \int Dq e^{\frac{i}{\hbar}S[q]} . \tag{36}$$

(a) Show that by taking the limit $\hbar \to 0$ we recover the classical extremum principle for the path.

(b) Take a general functional $F[q]$ and investigate the functional integral

$$\int Dq e^{-F[q]} . \tag{37}$$

Do a Taylor expansion up to second order around a stationary point of this functional. Argue that the above integral can be approximated with

$$\int Dq e^{-F[q]} \simeq e^{-F[\bar{q}]} \det\left(\frac{\hat{A}}{2\pi}\right)^{-1/2} , \tag{38}$$

where $\bar{q}$ is the stationary path and

$$\hat{A}(t,t') = \frac{\delta^2 F[q]}{\delta q(t) \delta q(t')}\bigg|_{q=\bar{q}} \tag{39}$$

is the second functional derivative at that point.
Hint: recall the result of a matrix Gaussian integral from the lecture.

(c) Do a model calculation utilizing the above result for the Γ-function

$$\Gamma(z+1) = \int_0^{\infty} dx x^z e^{-x} \tag{40}$$

to derive Stirling's formula:

$$\Gamma(s+1) \simeq \sqrt{2\pi s} e^{s(\ln s - 1)} . \tag{41}$$

(d) Perform the stationary phase approximation for a Lagrangian of the form

$$L = \frac{m\dot{q}^2}{2} - V(q) . \tag{42}$$

Write $q(t)$ as

$$q(t) = q_{cl}(t) + \delta q(t) \tag{43}$$

and show that approximating (36) we get

$$\langle q_f | e^{-\frac{i}{\hbar}Ht} | q_i \rangle \simeq e^{\frac{i}{\hbar}S[q_{cl}]} \int_{\delta q(t=t_i,t_f)=0} D(\delta q) \exp\left[-\frac{1}{2}\frac{i}{\hbar}\int dt \left(\delta q(t)(m\partial_t^2 + V''(q_{cl}))\delta q(t)\right)\right]. \tag{44}$$

(e) Expression (44) can be applied for a quantum particle sitting in a potential well. Assume $V(q)$ is minimal for $q = 0$ and we apply boundary conditions $q(t = t_i) = q(t = t_f) = 0$. Then classically $q \equiv 0$ is a solution. Use the notation $m\omega^2 = V''(0)$, so in light of (38) we have to calculate

$$\det\left(\partial_t^2 + \omega^2\right). \tag{45}$$

Calculate this determinant.

Hint: express it as an (infinite) product of eigenvalues of this operator, while its eigenfunctions are $\delta q(t)$ satisfying the boundary conditions. This infinite product can be normalized by factoring out a (formally divergent) constant factor and utilizing identity (34). Up to this constant the result coincides with that of Problem 8.3. Note that in that case it is exact since the series expansion of $V(q)$ terminates at second order.

Problem 8.5 Generating functionals

The generating functional of a scalar field theory is given by the functional integral

$$Z[J] = \mathcal{N} \int D\phi e^{i\int d^4x(\mathcal{L}+J(x)\phi(x))} \tag{46}$$

where $\mathcal{N}$ is chosen to ensure $Z[0] = 1$. The correlation functions can be obtained by taking functional derivatives with respect to the source $J(x)$:

$$\langle 0|T\phi(x_1)\ldots\phi(x_n)|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x_1)}\cdots\frac{1}{i}\frac{\delta}{\delta J(x_n)} Z[J]\bigg|_{J=0}. \tag{47}$$

For the free real scalar field ($\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$) we derived during the lecture the result

$$Z_0[J] = e^{-\frac{1}{2}\int dx\int dy J(x)D_F(x-y)J(y)}. \tag{48}$$

(a) Compute the time ordered two-point and four-point functions in the Klein–Gordon theory via functional derivatives using the above two equations.

(b) The Lagrangian density of the ϕ^4 theory is $\mathcal{L} = \mathcal{L}_0 - \frac{g}{4!}\phi^4$. Noting that the operation

$$\frac{1}{i}\frac{\delta}{\delta J(x)} \tag{49}$$

is equivalent to inserting $\phi(x)$ under the functional integral, show that the interacting generating functional can be written as

$$Z[J] = \mathcal{N}\int D\phi\, e^{i\int d^4x(\mathcal{L}_0 - \frac{g}{4!}\phi^4 + J\phi)} = \mathcal{N} e^{-i\int d^4x \frac{g}{4!}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^4}\int D\phi\, e^{i\int d^4x(\mathcal{L}_0+J\phi)} = \mathcal{N} e^{-i\int d^4x\frac{g}{4!}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^4} Z_0[J]. \tag{50}$$

Using the explicit expression (48) of $Z_0[J]$ compute $Z[J]$ up to first order in g by expanding the exponential in a Taylor series and evaluating the functional derivative. Don't forget to compute $\mathcal{N}$ to the same order!

(c) Use the resulting expression for $Z[J]$ in Eq. (47) to obtain the two-point function of the interacting theory in the first of order of perturbation theory!