Problem set 4 for Quantum Field Theory course

2019.03.05.

Topics covered

- Representation of C, P, T transformations on Dirac field and bilinears
- Gordon identity and current decomposition

Recommended reading
Peskin–Schroeder: An introduction to quantum field theory

- Section 3.6

Problem 4.1 P transformation of Dirac field

Recall that the Dirac field can be expressed as

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{s=1}^{2} \left[a_{\mathbf{p}}^s u^s(\mathbf{p}) e^{-ipx} + b_{\mathbf{p}}^{s\dagger} v^s(\mathbf{p}) e^{ipx} \right] , \tag{1}$$

with two particle species a and b. Under parity, annihiliation operators transform as

$$P a_{\mathbf{p}}^s P^{-1} = \eta_a a_{-\mathbf{p}}^s \quad , \qquad P b_{\mathbf{p}}^s P^{-1} = \eta_b b_{-\mathbf{p}}^s \,. \tag{2}$$

Show that the action of parity on the spinor field $\Psi(t, \mathbf{x})$ can be written as

$$P \Psi(t, \mathbf{x}) P^{-1} = \mathcal{P} \Psi(t, -\mathbf{x}) \,. \tag{3}$$

where the constant matrix $\mathcal{P}$ is given by

$$\mathcal{P} = \eta_a \gamma^0 \,. \tag{4}$$

(Hint: you may find useful the relation $\tilde{p}\sigma = p\overline{\sigma}$, where $\tilde{p}^\mu = (p^0, -\mathbf{p})$.)

Note that locality of the parity transformed field leads to a relation between η_a and η_b! What is this relation?

From Eq. (3) derive the analogous transformation rule for $\overline{\Psi}$!

Problem 4.2 T transformation of Dirac field

Both momentum and spin must change sign under time reversal, therefore we can write

$$T a_{\mathbf{p}}^s T^{-1} = a_{-\mathbf{p}}^{-s} \,, \qquad T b_{\mathbf{p}}^s T^{-1} = b_{-\mathbf{p}}^{-s} \,, \tag{5}$$

where $-s$ refers to the flipped spin.

Remark: in principle one could also allow for a phase ζ in this definition. However, as it was shown in the previous problem set, it is physically irrelevant. A possible choice is such that product of phases in CPT combination is equal to one.

(a) We have to implement the spin flip on spinors. Prove the identity

$$\vec{\sigma}\sigma^2 = \sigma^2(-\vec{\sigma})^* \,, \tag{6}$$

where $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)^T$ is a vector of Pauli matrices.

(b) Consider a spinor ξ that has $+1$ spin projected on axis $\vec{n}$:

$$\vec{n}\vec{\sigma}\xi = \xi \,. \tag{7}$$

Utilize the identity (6) to show that operator $-i\sigma^2$ flips the spin (i.e. the eigenvalue of ξ under $\vec{n}\vec{\sigma}$), so

$$\xi^{-s} = -i\sigma^2\xi^{s*} \,. \tag{8}$$

(c) Using Eq. (6) show that

$$u^{-s}(\tilde{p}) = -\gamma^1\gamma^3 u^s(p)^* \,. \tag{9}$$

(Hint: think about the projector resolution of $\sqrt{p\sigma}$*.*

(d) After all this preparation, show using (1) that

$$T\Psi(t,\mathbf{x})T^{-1} = -\gamma^1\gamma^3\Psi(-t,\mathbf{x}) \,. \tag{10}$$

Problem 4.3 C transformation of Dirac field

Charge conjugation acts on operators as follows

$$Ca^s_{\mathbf{p}}C^{-1} = \chi_a b^s_{\mathbf{p}} \quad , \quad Cb^s_{\mathbf{p}}C^{-1} = \chi_b a^s_{\mathbf{p}} \,, \tag{11}$$

so it relates the two species, i.e. particle and anti-particle.

(a) Using again Eq. (6) prove that positive and negative energy spinors are connected via the relation

$$u^s(\mathbf{p}) = -i\gamma^2(v^s(\mathbf{p}))^* \,, \tag{12}$$

where in $u^s(p)$ we have the 2-spinor ξ^s while in $v^s(p)$ there is ξ^{-s} in accordance with what we learned about spin projections in Problem 2.5 e).

(b) Using (1), derive the action of this operator on the Dirac spinor field

$$C\Psi C^{-1} \equiv \Psi^c = \chi_a \mathcal{C}\overline{\Psi}^T \,. \tag{13}$$

Note that locality of the charge conjugate field leads to a relation between χ_a and χ_b! What is this relation?

Show that in Dirac representation

$$\mathcal{C} = -i\gamma^2\gamma^0 \tag{14}$$

i.e. identical (up to a sign) to the charge conjugation matrix obtained in Problem 2.6.

Perform a similar calculation (or use Eq. (13)) and derive the expression

$$C\overline{\Psi}C^{-1} \equiv \overline{\Psi}^c = -\chi_a^*\Psi^T\mathcal{C} \,. \tag{15}$$

Problem 4.4 C, P, T transformation of Dirac field bilinears

(a) The transformation properties of Dirac spinors under discrete symmetry transformations were derived in Problem 4.1. Combine the results to obtain the action of CPT on a Dirac spinor.

(b) Show that the product $\overline{\Psi}\Psi$ transforms as a scalar under each discrete symmetry.

(c) Derive the transformation properties of $\overline{\Psi}\gamma^\mu\Psi$, $\overline{\Psi}\sigma^{\mu\nu}\Psi$, $\overline{\Psi}\gamma^\mu\gamma^5\Psi$ and $\overline{\Psi}\gamma^5\Psi$ under C, P and T. Note that the P and T transformation show a particular dependence on Lorentz indices.

(d) Obtain the CPT transformation of the Dirac bilinears.

(e) How does ∂_μ transform under C, P and T?

(f) Now we are ready to compute the effect of CPT on the free Dirac Lagrangian $\mathcal{L}_0$

$$\mathcal{L}_0(x) = \overline{\Psi}(x)(i\gamma^\mu\partial_\mu - m)\Psi(x)\,. \tag{16}$$

Is the action invariant under CPT?

Problem 4.5 Gordon identity

In this exercise we prove a useful identity and then consider some of its applications.

(a) Prove the Gordon identity

$$\overline{u}(p')\gamma^\mu u(p) = \overline{u}(p')\frac{(p+p')^\mu + i\sigma^{\mu\nu}(p'-p)_\nu}{2m}u(p)\,, \tag{17}$$

where $u(p)$ are positive energy Dirac spinors.

Hint: starting from the right hand side rewrite $\sigma^{\mu\nu}$ using the Clifford algebra and exploit the Dirac equation in momentum space.

(b) Recall that the $U(1)$ Noether current of Dirac field is

$$j^\mu = \overline{\Psi}\gamma^\mu\Psi\,. \tag{18}$$

Now consider a positive energy wave packet

$$\Psi^+(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{s=1}^{2} a^s_\mathbf{p} u^s(\mathbf{p})e^{-ipx}\,, \tag{19}$$

where the coefficients $a^s_\mathbf{p}$ are complex numbers, and calculate the current $j^{\mu(+)}$ associated to it. Show that if the total charge is normalised to unity

$$\int d^3x j^{0(+)}(t,\mathbf{x}) = 1 \tag{20}$$

one can express the total current as

$$\mathbf{J}^+ = \int d^3x \mathbf{j}^{(+)}(t,\mathbf{x}) = \sum_s \int d^3p |a^s_\mathbf{p}|^2 \frac{\mathbf{p}}{E_\mathbf{p}} = \left\langle \frac{\mathbf{p}}{E_\mathbf{p}} \right\rangle, \tag{21}$$

i.e. it equals the group velocity.

Hint: use the Gordon-identity and orthogonality of spinors (cf. Problem 2.3(f)).

(c) Gordon decomposition of Dirac current

Decompose the current (18) into two parts as

$$j^\mu = \frac{1}{2}(\overline{\Psi}\gamma^\mu\Psi + \overline{\Psi}\gamma^\mu\Psi)\,, \tag{22}$$

and use free Dirac equation in the first term and the conjugate equation in the second. Writing the $\gamma^\mu\gamma^\nu$ terms in the form

$$\gamma^\mu\gamma^\nu = \eta^{\mu\nu} + \frac{1}{i}\sigma^{\mu\nu} \tag{23}$$

derive the Gordon decomposition:

$$j^\mu = \frac{i}{2m}\overline{\Psi}\overleftrightarrow{\partial^\mu}\Psi + \frac{1}{2m}\partial_\nu(\overline{\Psi}\sigma^{\mu\nu}\Psi)\,. \tag{24}$$

Note that the first term is similar to scalar $U(1)$ current and so the second can be attributed to the spin. The coupling to an external electromagnetic field corresponds to an interaction term

$$H_{int} = -\int d^3x e j^\mu A_\mu \,, \tag{25}$$

where e is the electric charge, so ej^μ is the electromagnetic current. Focus on the second term from Gordon decomposition and perform a partial integration to derive the expression

$$H^{(2)}_{int} = -\frac{e}{2m}\int d^3x \frac{1}{2} F_{\mu\nu}(\overline{\Psi}\sigma^{\mu\nu}\Psi) \tag{26}$$

Take positive energy solutions in the Dirac representation

$$\Psi = \begin{pmatrix} \Psi_+ \\ \frac{\vec{\sigma}\vec{p}}{p_0+mc}\Psi_+ \end{pmatrix} \tag{27}$$

in the nonrelativistic approximation $p = |\vec{p}| \ll mc$, where the lower component is negligible compared to the upper. Show that in this limit the above interaction can be approximated at leading order in p/c as

$$H^{(2)}_{int} \simeq -\frac{e}{m}\int d^3x \vec{B} \cdot \left(\Psi^\dagger_+ \frac{\vec{\sigma}}{2}\Psi_+\right) \,, \tag{28}$$

which is a Pauli interaction term between the spin and the external magnetic field $\vec{B}$. Given the non-relativistic spin operator

$$\vec{S} = \frac{\vec{\sigma}}{2} \,, \tag{29}$$

what is the gyromagnetic ratio?