Problem set 2 for Quantum Field Theory course

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Topics covered

- Angular momentum tensor
- Dirac equation in Weyl representation: spinors, spin sums and spin projectors
- Charge conjugation in Dirac equation

Recommended reading
G. Takács: Lecture notes for Particle Physics MSc course, the following chapters:

- Chapter 5.4, 5.6-5.9
- Chapter 8

Problem 2.1 Generalized angular momentum

The effect of an infinitesimal Lorentz transformation on the coordinates is described by the antisymmetric $\omega_{\mu\nu}$ tensor

$$x'^{\mu} = x^{\mu} + \varepsilon \omega^{\mu}{}_{\nu} x^{\nu}. \tag{1}$$

It is then possible to define the generalized angular momentum tensor with three indices:

$$j^{\lambda} = -\frac{1}{2}\omega_{\mu\nu} J^{\lambda\mu\nu}. \tag{2}$$

(a) Calculate variation under (1) of a scalar field $\phi(x)$ up to first order in ε.

(b) Perform the same calculation with a spinor $\phi_\alpha(x)$ that transforms according to

$$\phi'_\alpha(x) = S(\Lambda)_\alpha{}^\beta \phi_\beta(\Lambda^{-1}x). \tag{3}$$

using the general form of Lorentz group generators:

$$S(\Lambda)_\alpha{}^\beta = \delta_\alpha^\beta - \frac{i}{2}\varepsilon\omega_{\mu\nu} \left(\mathcal{J}^{\mu\nu}\right)_\alpha{}^\beta + O(\varepsilon^2). \tag{4}$$

(c) Show that the angular momentum of scalar fields can be expressed using the canonical energy momentum tensor:

$$J^{\lambda\mu\nu}_{scalar} = T^{\lambda\mu}x^{\nu} - T^{\lambda\nu}x^{\mu}. \tag{5}$$

(Hint: Make use of the fact that Lagrangian density is also a Lorentz scalar and write its variation as a four-divergence.)

(d) Complete the calculation by showing that spinor transformation gives an extra term to angular momentum that can be interpreted as an internal angular momentum (spin):

$$J^{\lambda\mu\nu}_{spinor} = J^{\lambda\mu\nu}_{scalar} + S^{\lambda\mu\nu}, \tag{6}$$

where

$$S^{\lambda\mu\nu} = i\frac{\partial\mathcal{L}}{\partial\partial_\lambda\phi_\alpha} \left(\mathcal{J}^{\mu\nu}\right)_\alpha{}^\beta \phi_\beta(x). \tag{7}$$

Problem 2.2 Angular momentum of Dirac field

The symmetrized Lagrangian density of the Dirac field is

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \Psi - m\overline{\Psi}\Psi\,, \tag{8}$$

and the generators of the Lorentz group are

$$\mathcal{J}^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu}. \tag{9}$$

(a) Calculate $S^{\lambda\mu\nu}$ for a Dirac field. *(Note: Conjugate spinors are transformed by S^{-1}.)*

(b) Let us take $\lambda = 0$ and consider rotations only ($\mu, \nu = j, k = 1, 2, 3$). Derive a formula for S^{0jk}. *(Hint: You can get a nice expression by computing and using the value of the $[\gamma^0, \sigma^{jk}]$ commutator.)*

(c) In Weyl representation (see below for details) we have

$$\sigma^{jk} = \varepsilon^{jkl}\begin{pmatrix} \sigma^l & \\ & \sigma^l \end{pmatrix} \tag{10}$$

where ε^{jkl} is the usual Levi–Civita symbol and the σ^l ($l = 1, 2, 3$) are the Pauli matrices. Use this fact to show that Dirac spinors describe spin-$\frac{1}{2}$ particles.

(d) Derive the general commutation relation of Lorentz generators using $\sigma^{\mu\nu}$:

$$[\sigma^{\mu\nu}, \sigma^{\rho\lambda}] = 2i(\eta^{\lambda\mu}\sigma^{\nu\rho} - \eta^{\lambda\nu}\sigma^{\mu\rho} - \eta^{\rho\mu}\sigma^{\nu\lambda} + \eta^{\rho\nu}\sigma^{\mu\lambda})\,. \tag{11}$$

Problem 2.3 Weyl representation and plane wave solutions

The Dirac γ matrices take the following form in Weyl representation:

$$\gamma^0 = \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}. \tag{12}$$

(a) Calculate $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and the Lorentz generators $\sigma^{\mu\nu}$.

(b) Write the Dirac equation $(i\gamma^\mu\partial_\mu - m)\Psi = 0$ in a block-matrix form by introducing Ψ_L for left-handed and Ψ_R for right-handed two-spinors:

$$\begin{pmatrix} -m\mathbf{1} & i\sigma^\mu\partial_\mu \\ i\overline{\sigma}^\mu\partial_\mu & -m\mathbf{1} \end{pmatrix}\begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = 0\,, \tag{13}$$

where $\sigma^\mu = (\mathbf{1}, \vec{\sigma})$ and $\overline{\sigma}^\mu = (\mathbf{1}, -\vec{\sigma})$.

(c) Consider plane-wave solutions of Dirac equation with positive and negative energy, respectively:

$$\Psi^{(+)}_{L/R}(x) = u_{L/R}(p)e^{-ip_\mu x^\mu}, \qquad \Psi^{(-)}_{L/R}(x) = v_{L/R}(p)e^{ip_\mu x^\mu}. \tag{14}$$

Solve the Dirac equation for $u = \begin{pmatrix} u_L \\ u_R \end{pmatrix}$ in the rest frame ($p^i = 0$).

(d) Solve the equation if $p^i \neq 0$. Look for the solution in the form

$$u(p) = \begin{pmatrix} \sqrt{p\sigma}\xi \\ \sqrt{p\overline{\sigma}}\xi \end{pmatrix}. \tag{15}$$

where ξ is a two-component spinor parametrizing the degenerate subspace.

(e) Calculate $u^\dagger u$ and $\overline{u}u$. Which of them is Lorentz invariant?

(f) Derive orthogonality relations of these spinors:

$$u_r^\dagger(p) u_s(p) = 2E_p\, \delta_{rs} \,, \tag{16}$$

$$\overline{u}_r(p) u_s(p) = 2m\, \delta_{rs} \,, \tag{17}$$

where $r, s = 1, 2$ labels the basis in the degenerate subspace corresponding to

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \,. \tag{18}$$

Problem 2.4 Negative energy plane-wave and spin sums

(a) Perform the above (2.3 (c)-(f)) calculation for negative energy spinor v. Look for the solution in the form

$$v(p) = \begin{pmatrix} \sqrt{p\sigma}\eta \\ -\sqrt{p\overline{\sigma}}\eta \end{pmatrix} , \tag{19}$$

and now the degenerate subspace is parameterized by η. Notice the sign difference in $\overline{v}v$ product compared to $\overline{u}u$.

(b) Calculate $u_r^\dagger(p) v_s(p)$ and $\overline{u}_r(p) v_s(p)$. Which is the suitable choice to set orthogonality conditions?

(c) Derive the expression for spin sums:

$$\sum_s u^s(p)\overline{u}^s(p) = \not{p} + m\mathbf{1} \,, \tag{20}$$

$$\sum_s v^s(p)\overline{v}^s(p) = \not{p} - m\mathbf{1} \tag{21}$$

utilizing completeness of ξ: $\sum_s \xi^s \xi^{s\dagger} = \mathbf{1}$ (similar relation applies for $\eta_{1,2}$).

Problem 2.5 Projectors to energy and spin subspaces

(a) Calculate square of the spin sums (20) and (21). Construct projectors from them (denoted by $\Lambda_\pm$). *(Hint: show that $\not{p}^2 = m^2$ by symmetrizing the product and utilizing the dispersion relation.)*

(b) Show that these projectors project to orthogonal subspaces (i.e. $\Lambda_+\Lambda_- = 0$) and they cover the whole space together (their sum is equal to 1). What are their respective eigenspaces?

(c) One can construct a spin projector with respect to some axis n ($n^\mu n_\mu = -1$ and $n^\mu p_\mu = 0$) as follows:

$$P_\uparrow(n) = \frac{1 + \gamma_5 \not{n}}{2}. \tag{22}$$

Show that this operator commutes with $\Lambda_\pm$.

(d) Show that the above matrix is really a projector, i.e. $P_\uparrow(n)^2 = P_\uparrow(n)$.

(e) Show that $P_\uparrow(n)$ is indeed a spin projector. Choose a special reference frame in which the particle is at rest and the space-like unit vector n is aligned with the z axis (so $n_\mu = (0, 0, 0, 1)^T$). What is the action of $P_\uparrow(n)$ in the positive energy spinor basis $u_1 = \sqrt{m}(1, 0, 1, 0)^T$, and on $u_2 = \sqrt{m}(0, 1, 0, 1)^T$? Repeat this for the similarly defined v_1 and v_2. How can you interpret the spin of a negative energy mode?

Problem 2.6 Charge conjugation

The Dirac equation in the presence of an external electromagnetic field takes the form

$$(i\not{\partial} - q\not{A} - m)\Psi = 0. \tag{23}$$

We intend to find a charge-conjugate spinor Ψ^c that satisfies

$$(i\partial\!\!\!/ + qA\!\!\!/ - m)\Psi^c = 0, \tag{24}$$

where $\Psi^c = \xi^c \Psi^*$ ($\xi^c \in \mathbb{C}$ denotes the charge conjugation parity satisfying $\xi^c \xi^{c*} = 1$) as charge conjugation must involve to complex conjugation (check the canonical momentum operator!).

Transposing the conjugate Dirac equation we find that

$$(-i(\gamma^\mu)^T \partial_\mu - q(\gamma^\mu)^T A_\mu - m)\overline{\Psi}^T = 0. \tag{25}$$

Now applying a matrix transformation satisfying $(\gamma^\mu)^T = -C^{-1}\gamma^\mu C$ show that we get the desired result and also obtain the following formula for the charge conjugate spinor:

$$\xi^c \Psi^c = C\overline{\Psi}^T. \tag{26}$$

(Remark: the existence of such a matrix C follows from the fact that both $(\gamma^\mu)^T$ and $-\gamma^\mu$ satisfy the Clifford algebra relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, and from Clifford's theorem which states that any two irreducible[1] solutions to these relations are related by a similarity transformation. In fact, given two solutions such a matrix is unique up to overall normalization.)

(a) Show by explicit calculation that in the Dirac representation

$$\gamma^0_D = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \qquad \gamma^l_D = \begin{pmatrix} 0 & \sigma^l \\ -\sigma^l & 0 \end{pmatrix} \tag{27}$$

$C = i\gamma^2\gamma^0$ is a suitable choice for C (i.e. it transforms the γ matrices as stipulated above).

(b) Perform a similar calculation for the Weyl representation

$$\gamma^0_W = \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix}, \qquad \gamma^l_W = \begin{pmatrix} & \sigma^l \\ -\sigma^l & \end{pmatrix}. \tag{28}$$

Show that the same choice for C does the trick.

(c) Calculate charge conjugate plane-wave spinors in their rest frame (see 2.5(e)). What happens to the spin under charge conjugation?

(d) Now obtain a formula for C in Majorana representation:

$$\gamma^0_M = \begin{pmatrix} & \sigma^2 \\ \sigma^2 & \end{pmatrix}, \quad \gamma^1_M = \begin{pmatrix} i\sigma^3 & \\ & i\sigma^3 \end{pmatrix}, \quad \gamma^2_M = \begin{pmatrix} & -\sigma^2 \\ \sigma^2 & \end{pmatrix}, \quad \gamma^3_M = \begin{pmatrix} -i\sigma^1 & \\ & -i\sigma^1 \end{pmatrix}. \tag{29}$$

(Hint: all γ_M-s are either symmetric or antisymmetric, which allows a very simple choice for C.)

A Majorana fermion is an eigenstate of the charge conjugation operation. What is the charge conjugation parity of a Majorana fermion?

(Remark: note that all γ_M-s are purely imaginary and so the Dirac equation is real.)

Problem 2.7 Weyl spinors, chirality and helicity

(a) Let us define

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{30}$$

Show that

$$\left(\gamma^5\right)^2 = \mathbf{1}, \qquad \gamma^{5\dagger} = \gamma^5, \\ \{\gamma^5, \gamma^\mu\} = 0, \qquad [\gamma^5, \sigma^{\mu\nu}] = 0, \tag{31}$$

and

$$\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \tag{32}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the 4-dimensional Levi-Civita with $\epsilon_{0123} = +1$.

[1] In 4-dimensional space-time, irreducible solutions of the Clifford relations have matrix size 4×4.

(b) In the Weyl representation γ^5 is diagonal. Writing the Dirac spinor as

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \tag{33}$$

show that

$$P_\pm = \frac{1 \pm \gamma^5}{2} \tag{34}$$

project on the lower/upper components. $\psi_\pm$ are called chiral right/left handed Weyl spinors. To justify this show that parity transformation can be represented by

$$S(\mathcal{P}) = \gamma^0 \tag{35}$$

and it interchanges the two chiral components, which are therefore spatial reflections of each other.

(c) Show that in the Weyl basis the Dirac equation can be written as

$$i\partial_0\psi_+ + i\sigma^k\partial_k\psi_+ - m\psi_- = 0\,, \tag{36}$$

$$i\partial_0\psi_- - i\sigma^k\partial_k\psi_- - m\psi_+ = 0\,. \tag{37}$$

Therefore for zero mass, the two components decouple into separate Weyl equations for the two chiral components:

$$i\partial_0\psi_\pm = \mp i\sigma^k\partial_k\psi_\pm\,. \tag{38}$$

(d) Helicity is defined as the projection of the spin on the direction of motion. By considering positive and negative energy plane wave solutions

$$\psi_\pm^{(+)}(x) = u_\pm(p)e^{-ip_\mu x^\mu/\hbar}\,, \tag{39}$$

$$\psi_\pm^{(-)}(x) = v_\pm(p)e^{+ip_\mu x^\mu/\hbar} \tag{40}$$

of the Weyl equations (38) show that the positive and negative chirality solutions are helicity eigenstates

$$\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}u_\pm(p) = \pm u_\pm(p)\,, \tag{41}$$

$$\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}v_\pm(p) = \pm v_\pm(p)\,. \tag{42}$$