# 1 The scattering cross section

## 1.1 Derivation of the differential cross section for a general $2 \to n$ particle scattering process

Let us write down the oncoming wave packets as

$$|i\rangle = \int \frac{d^3k_A}{(2\pi)^3\sqrt{2\omega_A}} \int \frac{d^3k_B}{(2\pi)^3\sqrt{2\omega_B}} \phi_A(\vec{k}_A)\phi_B(\vec{k}_B)|\vec{k}_A, \vec{k}_B\rangle$$

and the outgoing wave packets as

$$|f\rangle = \prod_{i=1}^{n} \left\{ \int \frac{d^3p_i}{(2\pi)^3\sqrt{2\omega_i}} \phi_i(\vec{p}_i) \right\} |\vec{p}_1, \ldots, \vec{p}_n\rangle$$

The S matrix is written as

$$\mathbb{S} = 1 + i\mathbb{T}$$

and the transition amplitude can be computed easily

$$T(\vec{b}) = \langle f|\mathbb{T}|i\rangle = \prod_{i=1}^{n} \left\{ \int \frac{d^3p_i}{(2\pi)^3\sqrt{2\omega_i}} \phi_i(\vec{p}_i)^* \right\} \int \frac{d^3k_A}{(2\pi)^3\sqrt{2\omega_A}} \int \frac{d^3k_B}{(2\pi)^3\sqrt{2\omega_B}} \phi_A(\vec{k}_A)\phi_B(\vec{k}_B)e^{-i\vec{b}\cdot\vec{k}_B}$$
$$\times \langle \vec{p}_1, \ldots, \vec{p}_n|\mathbb{T}|\vec{k}_A, \vec{k}_B\rangle$$

where

$$\langle \vec{p}_1, \ldots, \vec{p}_n|\mathbb{T}|\vec{k}_A, \vec{k}_B\rangle = -\mathcal{M}\left(\{\vec{k}_A, \vec{k}_B\} \to \{\vec{p}_1, \ldots, \vec{p}_n\}\right)(2\pi)^4\delta^{(4)}\left(k_A + k_B - \sum_{i=1}^{n} p_i\right)$$

and the factor $e^{-i\vec{b}\cdot\vec{k}_B}$ corresponds to an incoming projectile $B$ with an impact parameter $\vec{b}$ relative to the target particle $A$. The transition probability for this process is

$$P(\vec{b}) = |T(\vec{b})|^2$$
$$= \prod_{i=1}^{n} \left\{ \int \frac{d^3p_i}{(2\pi)^3\sqrt{2\omega_i}} \phi_i(\vec{p}_i)^* \right\} \int \frac{d^3k_A}{(2\pi)^3\sqrt{2\omega_A}} \int \frac{d^3k_B}{(2\pi)^3\sqrt{2\omega_B}} \phi_A(\vec{k}_A)\phi_B(\vec{k}_B)e^{-i\vec{b}\cdot\vec{k}_B}\langle \vec{p}_1, \ldots, \vec{p}_n|\mathbb{T}|\vec{k}_A, \vec{k}_B\rangle$$
$$\times \prod_{i=1}^{n} \left\{ \int \frac{d^3p'_i}{(2\pi)^3\sqrt{2\omega'_i}} \phi_i(\vec{p}'_i) \right\} \int \frac{d^3k'_A}{(2\pi)^3\sqrt{2\omega'_A}} \int \frac{d^3k'_B}{(2\pi)^3\sqrt{2\omega_B}'} \phi_A(\vec{k}'_A)^*\phi_B(\vec{k}'_B)^*e^{i\vec{b}\cdot\vec{k}'_B}\langle \vec{p}'_1, \ldots, \vec{p}'_n|\mathbb{T}|\vec{k}'_A, \vec{k}'_B\rangle^*$$
$$= \prod_{i=1}^{n} \left\{ \int \frac{d^3p_i}{(2\pi)^3\sqrt{2\omega_i}} \phi_i(\vec{p}_i)^* \right\} \int \frac{d^3k_A}{(2\pi)^3\sqrt{2\omega_A}} \int \frac{d^3k_B}{(2\pi)^3\sqrt{2\omega_B}}$$
$$\times \prod_{i=1}^{n} \left\{ \int \frac{d^3p'_i}{(2\pi)^3\sqrt{2\omega'_i}} \phi_i(\vec{p}'_i) \right\} \int \frac{d^3k'_A}{(2\pi)^3\sqrt{2\omega_A}'} \int \frac{d^3k'_B}{(2\pi)^3\sqrt{2\omega'_B}}$$
$$\times \phi_A(\vec{k}_A)\phi_B(\vec{k}_B)\phi_A(\vec{k}'_A)^*\phi_B(\vec{k}'_B)^*e^{i\vec{b}\cdot(\vec{k}'_B-\vec{k}_B)}(2\pi)^4\delta^{(4)}\left(k_A + k_B - \sum_{i=1}^{n} p_i\right)(2\pi)^4\delta^{(4)}\left(k'_A + k'_B - \sum_{i=1}^{n} p'_i\right)$$
$$\times \mathcal{M}\left(\{\vec{k}_A, \vec{k}_B\} \to \{\vec{p}_1, \ldots, \vec{p}_n\}\right)\mathcal{M}\left(\{\vec{k}'_A, \vec{k}'_B\} \to \{\vec{p}'_1, \ldots, \vec{p}'_n\}\right)^*$$

What is the number of events per second of the incident beam scattering on the target particle? If the beam has a particle flux $j_B$ (particles per second per area) then the number of events per second (event frequency) is

$$f = \int d^2b j_B P(\vec{b}) = j_B \int d^2b P(\vec{b})$$

where I supposed that the incoming beam flux is homogeneous across it's transverse area so it can be brought out of the integration. Then the cross section is just

$$\sigma = \int d^2b P(\vec{b})$$

Now our first step is to suppose that the outgoing states are narrow wave packets so that $\phi_i(\vec{p}_i)^*\phi_i(\vec{p}_i')$ is only nonzero when $\vec{p}_i \approx \vec{p}_i'$ (what this really means that our detectors measuring the final state have a good momentum resolution). Using the wave packet normalisation

$$\int \frac{d^3p}{(2\pi)^3}|\phi(\vec{p})|^2 = 1$$

we can perform the integral over the $\vec{p}_i'$ leaving us with

$$P(\vec{b}) = \prod_{i=1}^{n}\left\{\int \frac{d^3p_i}{(2\pi)^3 2\omega_i}\right\} \int \frac{d^3k_A}{(2\pi)^3\sqrt{2\omega_A}} \int \frac{d^3k_B}{(2\pi)^3\sqrt{2\omega_B}} \int \frac{d^3k_A'}{(2\pi)^3\sqrt{2\omega_A}'} \int \frac{d^3k_B'}{(2\pi)^3\sqrt{2\omega_B'}}$$
$$\times \phi_A(\vec{k}_A)\phi_B(\vec{k}_B)\phi_A(\vec{k}_A')^*\phi_B(\vec{k}_B')^* e^{i\vec{b}\cdot(\vec{k}_B'-\vec{k}_B)}$$
$$\times (2\pi)^4\delta^{(4)}\left(k_A + k_B - \sum_{i=1}^{n} p_i\right)(2\pi)^4\delta^{(4)}\left(k_A' + k_B' - \sum_{i=1}^{n} p_i\right)$$
$$\times \mathcal{M}\left(\{\vec{k}_A, \vec{k}_B\} \to \{\vec{p}_1, \ldots, \vec{p}_n\}\right)\mathcal{M}\left(\{\vec{k}_A', \vec{k}_B'\} \to \{\vec{p}_1, \ldots, \vec{p}_n\}\right)^*$$

Now we omit the final state integral and integrate over the impact parameter $\vec{b}$ to write the differential cross section as

$$d\sigma = \prod_{i=1}^{n}\left\{\frac{d^3p_i}{(2\pi)^3 2\omega_i}\right\} \int d^2b \int \frac{d^3k_A}{(2\pi)^3\sqrt{2\omega_A}} \int \frac{d^3k_B}{(2\pi)^3\sqrt{2\omega_B}} \int \frac{d^3k_A'}{(2\pi)^3\sqrt{2\omega_A}'} \int \frac{d^3k_B'}{(2\pi)^3\sqrt{2\omega_B'}}$$
$$\times \phi_A(\vec{k}_A)\phi_B(\vec{k}_B)\phi_A(\vec{k}_A')^*\phi_B(\vec{k}_B')^* e^{i\vec{b}\cdot(\vec{k}_B'-\vec{k}_B)}$$
$$\times (2\pi)^4\delta^{(4)}\left(k_A + k_B - \sum_{i=1}^{n} p_i\right)(2\pi)^4\delta^{(4)}\left(k_A' + k_B' - \sum_{i=1}^{n} p_i\right)$$
$$\times \mathcal{M}\left(\{\vec{k}_A, \vec{k}_B\} \to \{\vec{p}_1, \ldots, \vec{p}_n\}\right)\mathcal{M}\left(\{\vec{k}_A', \vec{k}_B'\} \to \{\vec{p}_1, \ldots, \vec{p}_n\}\right)^*$$

We can now perform the integral over the impact parameter to get

$$d\sigma = \prod_{i=1}^{n}\left\{\frac{d^3p_i}{(2\pi)^3 2\omega_i}\right\} \int \frac{d^3k_A}{(2\pi)^3\sqrt{2\omega_A}} \int \frac{d^3k_B}{(2\pi)^3\sqrt{2\omega_B}} \int \frac{d^3k_A'}{(2\pi)^3\sqrt{2\omega_A'}} \int \frac{d^3k_B'}{(2\pi)^3\sqrt{2\omega_B'}}$$
$$\times \phi_A(\vec{k}_A)\phi_B(\vec{k}_B)\phi_A(\vec{k}_A')^*\phi_B(\vec{k}_B')^*(2\pi)^2\delta^{(2)}\left(\vec{k}_{B\perp}' - \vec{k}_{B\perp}\right)$$
$$\times (2\pi)^4\delta^{(4)}\left(k_A + k_B - \sum_{i=1}^{n} p_i\right)(2\pi)^4\delta^{(4)}\left(k_A' + k_B' - \sum_{i=1}^{n} p_i\right)$$
$$\times \mathcal{M}\left(\{\vec{k}_A, \vec{k}_B\} \to \{\vec{p}_1, \ldots, \vec{p}_n\}\right)\mathcal{M}\left(\{\vec{k}_A', \vec{k}_B'\} \to \{\vec{p}_1, \ldots, \vec{p}_n\}\right)^*$$

where $\perp$ denotes the components $x, y$ perpendicular to the direction $z$ of the collision. Now the $\delta$-functions enforce the following identities between the $A, B$ four vectors:

$$\vec{k}_A + \vec{k}_B = \vec{k}'_A + \vec{k}'_B$$
$$\vec{k}_{B\perp} = \vec{k}'_{B\perp}$$
$$\sqrt{\vec{k}_A^2 + m_A^2} + \sqrt{\vec{k}_B^2 + m_B^2} = \sqrt{\vec{k}_A'^2 + m_A^2} + \sqrt{\vec{k}_B'^2 + m_B^2}$$

The first two relations together enforce

$$\vec{k}_{A\perp} = \vec{k}'_{A\perp} \qquad \vec{k}_{B\perp} = \vec{k}'_{B\perp}$$

while the equations for the $z$ components are

$$k_A^z + k_B^z = k_A'^z + k_B'^z$$
$$\omega_A + \omega_B = \sqrt{\left(k_A'^z\right) 2 + \vec{k}_{A\perp}^2 + m_A^2} + \sqrt{\left(k_B'^z\right) 2 + \vec{k}_{B\perp}^2 + m_B^2}$$

which enforces

$$k_A'^z = k_A^z \qquad k_B'^z = k_B^z$$

So the result of the $k'$ integration is

$$\int \frac{d^3k'_A}{(2\pi)^3\sqrt{2\omega'_A}} \int \frac{d^3k'_B}{(2\pi)^3\sqrt{2\omega'_B}} (2\pi)^2\delta^{(2)}\left(\vec{k}'_{B\perp} - \vec{k}_{B\perp}\right)(2\pi)^4\delta^{(4)}\left(k'_A + k'_B - k_A - k_B\right)(\dots)$$
$$= \frac{1}{\sqrt{2\omega_A}}\frac{1}{\sqrt{2\omega_B}} \int dk_A'^z\, \delta\left(\sqrt{\left(k_A'^z\right)^2 + \vec{k}_{A\perp}^2 + m_A^2} + \sqrt{\left(k_B'^z\right)^2 + \vec{k}_{B\perp}^2 + m_B^2} - \omega_A - \omega_B\right)\Bigg|_{k_B'^z = k_A^z + k_B^z - k_A'^z}$$
$$(\dots)|_{\vec{k}_{A,B} = \vec{k}'_{A,B}}$$
$$= \frac{1}{\sqrt{2\omega_A}}\frac{1}{\sqrt{2\omega_B}}\frac{1}{\left|\frac{k_A^z}{\omega_A} - \frac{k_B^z}{\omega_B}\right|}(\dots)|_{\vec{k}_{A,B} = \vec{k}'_{A,B}}$$

where we used the rule

$$\delta(f(x)) = \frac{1}{|f'(a)|}\delta(x - a)$$

when the function $f(x)$ has a single zero at $x = a$, and the result

$$\frac{\partial}{\partial k_A'^z}\left(\sqrt{\left(k_A'^z\right)^2 + \vec{k}_{A\perp}^2 + m_A^2} + \sqrt{\left(k_B'^z\right)^2 + \vec{k}_{B\perp}^2 + m_B^2} - \omega_A - \omega_B\right)\Bigg|_{k_B'^z = k_A^z + k_B^z - k_A'^z}$$
$$= \frac{k_A'^z}{\sqrt{\left(k_A'^z\right)^2 + \vec{k}_{A\perp}^2 + m_A^2}} - \frac{k_B'^z}{\sqrt{\left(k_B'^z\right)^2 + \vec{k}_{B\perp}^2 + m_B^2}}$$

which is just equal to

$$\frac{k_A^z}{\omega_A} - \frac{k_B^z}{\omega_B}$$

once we use that $\vec{k}_A = \vec{k}'_A$ and $\vec{k}_B = \vec{k}'_B$. Now note that

$$\frac{k_A^z}{\omega_A} - \frac{k_B^z}{\omega_B} = v_A - v_B$$

where $v_A$ and $v_B$ are the velocities of the particles $A$ and $B$ in the direction of the scattering axis $z$.

So the result is

$$d\sigma = \prod_{i=1}^{n} \left\{ \frac{d^3 p_i}{(2\pi)^3 2\omega_i} \right\} \int \frac{d^3 k_A}{(2\pi)^3 2\omega_A} \int \frac{d^3 k_B}{(2\pi)^3 2\omega_B} \frac{1}{|v_A - v_B|} \left| \phi_A(\vec{k}_A) \right|^2 \left| \phi_B(\vec{k}_B) \right|^2$$
$$(2\pi)^4 \delta^{(4)} \left( k_A + k_B - \sum_{i=1}^{n} p_i \right) \left| \mathcal{M} \left( \{\vec{k}_A, \vec{k}_B\} \to \{\vec{p}_1, \ldots, \vec{p}_n\} \right) \right|^2$$

Now we assume that the wave packets $\phi_{A,B}$ are very narrow (i.e. the incoming particles have a sharply defined value of momentum) and that the amplitude $\mathcal{M}$ varies slowly enough so that it can be taken constant on their support. Then we can write

$$d\sigma = \prod_{i=1}^{n} \left\{ \frac{d^3 p_i}{(2\pi)^3 2\omega_i} \right\} \frac{1}{2\omega_A} \frac{1}{2\omega_B} \frac{1}{|v_A - v_B|} (2\pi)^4 \delta^{(4)} \left( k_A + k_B - \sum_{i=1}^{n} p_i \right) \left| \mathcal{M} \left( \{\vec{k}_A, \vec{k}_B\} \to \{\vec{p}_1, \ldots, \vec{p}_n\} \right) \right|^2$$

which is our main result.

## 1.2 Lorentz properties of the cross section

All the ingredients of $d\sigma$ are Lorentz invariant with the exception of the factor

$$\frac{1}{\omega_A \omega_B |v_A - v_B|} = \frac{1}{|k_A^z \omega_B - k_B^z \omega_A|}$$

This should transform as area element in the $xy$ plane. Let us check this explicitly. Lorentz transformations consist of rotations and boosts in three principal directions $x$, $y$ and $z$. Let us go through each of these one by one.

1. Rotations around $z$ axis do not change neither the $z$ component of momentum, nor the energy, so this is invariant. This is right since rotation leaves an area element perpendicular to the axis invariant.

2. Boosts in direction $z$ with velocity $u$ (in units of $c = 1$) result in

   $$\omega' = \gamma(\omega - u k^z)$$
   $$k'^z = \gamma(k^z - u\omega) \qquad \gamma = (1 - u^2)^{-1/2}$$

   So we get

   $$|k_A^z \omega_B - k_B^z \omega_A| \to \gamma^2 \left| (k_A^z - u\omega_A)(\omega_B - u k_B^z) - (k_B^z - u\omega_B)(\omega_A - u k_A^z) \right| = |k_A^z \omega_B - k_B^z \omega_A|$$

   which is correct as area elements perpendicular to the boost direction do not undergo Lorentz contraction.

3. Boosts in direction $x$ with velocity $u$ (in units of $c = 1$) result in

   $$\omega' = \gamma(\omega - u k^x)$$
   $$k'^x = \gamma(k^x - u\omega) \qquad \gamma = (1 - u^2)^{-1/2}$$

Now since the particles move in direction $z$, $k^x$ is zero, so we get

$$|k_A^z \omega_B - k_B^z \omega_A| \rightarrow \gamma \, |k_A^z \omega_B - k_B^z \omega_A|$$

so

$$d\sigma \rightarrow \frac{1}{\gamma} d\sigma = \sqrt{1-u^2} d\sigma$$

This is just the result of Lorentz contraction in direction $x$.
The calculation for a boost in direction $y$ is similar, and leads to the same result.

4. Rotation around $x$or $y$ axis does not change the energy, but changes the direction of the momentum. Since the original $x$ and $y$ components are zero, the $z$ components of the momentum picks up a factor $\cos\alpha$ where $\alpha$ is the rotation angle, so we get

$$d\sigma \rightarrow \frac{d\sigma}{\cos\alpha}$$

This is exactly the correct way for the transformation of a cross sectional area element under rotation.

## 1.3 $2 \rightarrow 2$ processes

For the special case of a two-particle scattering process of two particles of masses $m_A$ and $m_B$ to two other particles with masses $m_1$ and $m_2$ we have the conservation equations

$$\vec{k}_A + \vec{k}_B = \vec{p}_1 + \vec{p}_2$$
$$E_A + E_B = E_1 + E_2$$

and the differential cross section is

$$d\sigma = \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} \frac{1}{2E_A} \frac{1}{2E_B} \frac{1}{|v_A - v_B|} (2\pi)^4 \delta^{(4)} \left(k_A + k_B - p_1 - p_2\right) \left| \mathcal{M} \left( \{\vec{k}_A, \vec{k}_B\} \rightarrow \{\vec{p}_1, \vec{p}_2\} \right) \right|^2$$

Note that we have 6 differentials, but also 4 Dirac deltas, which means that we can fix four of the differential variables (i.e. integrate them out). It is simplest to do this in the center-of-mass (or zero-momentum frame, in which $\vec{k}_A + \vec{k}_B = 0$ which means that the spatial part of $\delta^{(4)}$ enforces $\vec{p}_2 = -\vec{p}_1$. This allows us to integrate over $\vec{p}_2$ to obtain

$$d\sigma = \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{1}{2E_2} \frac{1}{2E_A} \frac{1}{2E_B} \frac{1}{|v_A - v_B|} 2\pi \delta \left(E_{CM} - E_1 - E_2\right) |\mathcal{M}|^2$$

where

$$E_{CM} = E_A + E_B$$

is the center-of-mass total energy. Now the last $\delta$ fixes $p_1 = |\vec{p}_1|$ since it means that

$$E_{CM} - \sqrt{p_1^2 + m_1^2} - \sqrt{p_1^2 + m_2^2} = 0$$

To use this we write $d^3p_1 = p_1^2 dp_1 d\Omega$ where $\Omega$ is the solid angle parameterising of the direction of the outgoing particle $A$. The derivative of the Dirac delta argument with respect to $p_1$ is

$$-\left(\frac{p_1}{E_1} + \frac{p_1}{E_2}\right)$$

so after integration we are left by

$$
\begin{aligned}
d\sigma &= \frac{p_1^2 d\Omega}{4\pi^2} \frac{1}{2E_1} \frac{1}{2E_2} \frac{1}{2E_A} \frac{1}{2E_B} \frac{1}{|v_A - v_B|} \frac{1}{\frac{p_1}{E_1} + \frac{p_1}{E_2}} |\mathcal{M}|^2 \\
&= d\Omega \frac{1}{2E_A} \frac{1}{2E_B} \frac{1}{|v_A - v_B|} \frac{|\vec{p}_1|}{16\pi^2 (E_1 + E_2)}
\end{aligned}
$$

What is all the masses are equal $m_A = m_B = m_1 = m_2 = m$. This leads to $\left|\vec{k}_A\right| = \left|\vec{k}_B\right| = |\vec{p}_1| = |\vec{p}_2|$ and so $E_A = E_B = E_1 = E_2 = E_{CM}/2$. In addition

$$
v_A - v_B = \frac{k_A^z}{E_A} - \frac{k_B^z}{E_B} = 2\frac{k_A^z}{E_A} = 2\frac{|\vec{p}_1|}{E_{CM}/2} = 4\frac{|\vec{p}_1|}{E_{CM}}
$$

so we get

$$
\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{CM}^2}
$$

In our scalar example we had $\mathcal{M} = \lambda + O(\lambda^2)$, so the differential cross section to first order is

$$
\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\lambda^2}{64\pi^2 E_{CM}^2}
$$

Note that

$$
E_{CM}^2 = (p_1 + p_2)^2
$$

where the square denotes the Lorentz "norm". The Mandelstam variable

$$
s = (p_1 + p_2)^2
$$

provides a Lorentz-invariant expression for the center-of-mass energy (squared).

The total cross section can be obtained by

$$
\sigma = \frac{1}{2} \int d\Omega \left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\lambda^2}{32\pi s}
$$

where the factor 1/2 takes into account that there are two identical particles in the end state and we cannot distinguish which one was emitted in the given angular direction over which we integrate.

# 2 The decay rate

If there is only a single particle in the initial state, the process describes the decay. As discussed in the class, the proper way of obtaining the decay rate $\Gamma$ is via the construction of the relativistic Breit-Wigner distribution of the unstable (resonant) excitation, but the end result can be easily guessed modifying the formula for $d\sigma$ by eliminating one of the incoming particles. The result is

$$
d\Gamma = \prod_{i=1}^{n} \left\{ \frac{d^3 p_i}{(2\pi)^3 2\omega_i} \right\} \frac{1}{2\omega_A} (2\pi)^4 \delta^{(4)} \left( k_A - \sum_{i=1}^{n} p_i \right) \left| \mathcal{M} \left( \{\vec{k}_A\} \to \{\vec{p}_1, \ldots, \vec{p}_n\} \right) \right|^2
$$

The total decay rate can be obtained by integrating over all the final states. Note that the decay rate $\Gamma_0$ in the rest frame is obtained by substituting $\omega_A$ by the mass $m_A$, therefore

$$\Gamma = \frac{m_A}{\omega_A}\Gamma_0$$

Since the decay rate is related to the lifetime $\tau$by $\Gamma = 1/\tau$, we obtain

$$\tau = \frac{\omega_A}{m_A}\tau_0$$

where

$$\frac{\omega_A}{m_A} = \frac{1}{\sqrt{1-u_A^2}}$$

with $u_A$ being the velocity of particle $A$ in $c = 1$ units. This is just the correct formula for relativistic time dilation, thus supporting our guess. In fact, the formula for the decay rate can be proven rigorously using the so-called optical theorem (cf. Section 7.3 in Peskin-Schroeder).

# 3 Some dimensional analysis

The dimensions (in units of energy) of a momentum Dirac delta can be obtained from

$$\int d^3p\delta^{(3)}(\vec{p}) = 1$$

Since $[d^3p] = 3$ we get $\left[\delta^{(3)}(\vec{p})\right] = -3$. Now our states have inner products

$$\langle\vec{p}'|\vec{p}\rangle = 2\omega_{\vec{p}}\delta^{(3)}(\vec{p}-\vec{p}')$$

The left hand side has dimension -2, so a one-particle state has dimension -1. Therefore, the dimensionality of a multi-particle state is given by

$$[|\vec{p}_1,\ldots,\vec{p}_n\rangle] = -n$$

Now the $S$-matrix is dimensionless since it is a (time-ordered) exponential of the time-integrated Hamiltonian density, and so is $\mathbb{T}$. Therefore, using

$$\langle\vec{p}_1,\ldots,\vec{p}_n|\mathbb{T}|\vec{k}_1,\ldots,\vec{k}_m\rangle = -\mathcal{M}\left(\{\vec{k}_1,\ldots,\vec{k}_m\}\to\{\vec{p}_1,\ldots,\vec{p}_n\}\right)(2\pi)^4\delta^{(4)}\left(\sum_{j=1}^m \vec{k}_j - \sum_{i=1}^n p_i\right)$$

we obtain that

$$\left[\mathcal{M}\left(\{\vec{k}_1,\ldots,\vec{k}_m\}\to\{\vec{p}_1,\ldots,\vec{p}_n\}\right)\right] = -n-m+4$$

Let us check the dimensionality of the cross section

$$d\sigma = \prod_{i=1}^n\left\{\frac{d^3p_i}{(2\pi)^3 2\omega_i}\right\}\frac{1}{2\omega_A}\frac{1}{2\omega_B}\frac{1}{|v_A-v_B|}(2\pi)^4\delta^{(4)}\left(k_A+k_B-\sum_{i=1}^n p_i\right)\left|\mathcal{M}\left(\{\vec{k}_A,\vec{k}_B\}\to\{\vec{p}_1,\ldots,\vec{p}_n\}\right)\right|^2$$

We obtain that this is

$$[d\sigma] = n(3-1)-2-4+2(4-n-2) = -2$$

which is just right for an area.

For the decay rate

$$d\Gamma = \prod_{i=1}^{n} \left\{ \frac{d^3 p_i}{(2\pi)^3 2\omega_i} \right\} \frac{1}{2\omega_A} (2\pi)^4 \delta^{(4)} \left( k_A - \sum_{i=1}^{n} p_i \right) \left| \mathcal{M} \left( \{\vec{k}_A\} \to \{\vec{p}_1, \ldots, \vec{p}_n\} \right) \right|^2$$

we get

$$[d\Gamma] = n(3-1) - 1 - 4 + 2(4 - n - 1) = 1$$

which is correct since it must have the same units as energy (being inversely proportional to time).