Autocorrelation function

$$\langle v \rangle = \int_0^\infty v\, P_0(v)\, dv = \frac{1}{\Gamma(c)} \int_0^\infty v^c \exp(-v)\, dv = c$$

$$R(\Delta u) = \int_0^\infty dv \int_0^\infty dv' (v-c)(v'-c) P(v', \Delta u | v) P_0(v)$$

$$R(\Delta u) = c \exp(-|\Delta u|)$$

$$S_v(\omega) = \frac{2c}{1+\omega^2} \qquad -\infty < \omega < \infty$$

c = autocorrelation function

$$c = \langle (v-c)^2 \rangle = \langle v^2 \rangle - c^2$$

$$\langle v^2 \rangle = c + c^2 = \langle v \rangle + \langle v \rangle^2$$

physical units

$$\Gamma G \left\langle \left(\frac{d\theta}{dt}\right)^2 \right\rangle = \left| A S_p \frac{d\langle\theta\rangle}{dt} \right| + \Gamma G \left(\frac{d\langle\theta\rangle}{dt}\right)^2$$

Sherrington-Kirkpatric model

$$H = \sum_{ij} J_{ij} \sigma_i \sigma_j \qquad J_{ij} \text{ quenched, time independent}$$

- SK model : J are random Gaussian (or bimodal) variance $\frac{1}{\sqrt{N}}$, coordination number $z = N-1$
- long range Edwards-Anderson model : dim D, $J_{ij} \neq 0$ in radius R of i ij
  $J \sim 1/R^{D/2}$ $\qquad R \to \infty \to SK$
- EA model : $R = 1$ J only nearest neighbors
  $D \to \infty \to SK$

SK model

$$\langle J_{ij}\rangle = 0 \qquad \langle J_{ij}^2\rangle = \frac{J^2}{z} \qquad (J_{ij} = J_{ji})$$

$$\mathcal{H} = -\sum_{\langle ij\rangle} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i = -\frac{1}{2}\sum_{ij} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i$$

$m_i = \langle\sigma_i\rangle$

$z = N-1$

$$\langle J_{ij}^2\rangle = \frac{J^2}{N-1} \approx \frac{J^2}{N} \qquad J_{ij} \sim \frac{1}{\sqrt{N}}$$

Mean field ($h_i = 0$)

$$\mathcal{H} = -\underbrace{\left(\sum_{ij} J_{ij}\sigma_j - h_i\right)}\sigma_i \qquad h_i = 0$$

$$H_i = \sum_j J_{ij} m_j + h_i \qquad H_i = \sum_j J_{ij}\sigma_j$$

$$\langle J_{ij}\rangle = 0 \Rightarrow \langle m_i\rangle = 0 \qquad m_i = \tanh\Big(\beta \overbrace{\sum_j J_{ij} m_j}^{H_i}\Big)_{MF}$$

local variations $q = \langle m_i^2\rangle$

Self-consistent calculation of q

H_i has zero mean with Gaussian variance

$$\left\langle\left(\sum_j J_{ij} m_j\right)^2\right\rangle = \left\langle\sum_{jk} J_{ij}J_{ik} m_j m_k\right\rangle \approx \sum_j\langle J_{ij}^2\rangle\langle m_j^2\rangle =$$

$$= \sum_j \langle J_{ij}^2\rangle q = J^2 q$$

$$q = \int \frac{dH}{\sqrt{2\pi J^2 q}} \tanh^2(\beta H)\exp\left(-\frac{H^2}{2J^2 q}\right)$$

. linear expansion eqs RR6.

$$q = \int \frac{dH}{\sqrt{2\pi J^2 q}} \underbrace{\left(\beta H - \frac{1}{3}(\beta H)^3 + \dots\right)^2}_{(\beta H)^2 - \frac{2}{3}(\beta H)^4 \dots} \exp\left(-\frac{H^2}{2J^2 q}\right)$$

$$q = \beta^2 J^2 q - \frac{2}{3} 3 \beta^4 J^4 q^2 \dots$$

$\underline{T_c = J}$ $\quad (q = q - 2q^2)$

Below T_c $(\beta^2 J^2 - 1) q = 2 q^2 \Rightarrow q = T_c - T$

wrong!

For ferromagnet:

$$H_i = \sum_j J_{ij} \sigma_j \approx \sum_j J_{ij} m_j$$

$\langle m_i \rangle = 0$? next term

$$\mathcal{H}_i = \sum_j J_{ij} \sigma_j = \sum_j J_{ij} m_j - \beta m_i \sum_j J_{ij}^2 (1 - m_j^2) + \dots$$

for FM $\mathcal{O}(\frac{1}{N})$

then first term is zero

Thouless - Anderson - Palmer (TAP) equations:

$$F_{TAP}(\{m\}) = \sum_{ij} \frac{1}{2} J_{ij} m_i m_j - \frac{1}{4} \beta J_{ij}^2 (1 - m_i^2)(1 - m_j^2) - T \sum_i S(m_i)$$

$$m_i = \tanh\left(\beta \sum_j J_{ij} m_j - \beta^2 m_i \sum_j J_{ij}^2 (1 - m_j^2) + \beta h_i\right)$$

Monte-Carlo Glauber dynamics

$$\tau_0 \frac{dP(\{\sigma\},t)}{dt} = \frac{1}{2}\sum_i \left(1 + \sigma_i \tanh(\beta h_i(t))\right) P(\sigma_1 \ldots -\sigma_i \ldots \sigma_N) -$$
$$- \frac{1}{2}\sum_i \left(1 - \sigma_i \tanh(\beta h_i(t))\right) P(\sigma_1 \ldots +\sigma_i \ldots \sigma_N)$$

$$\tau_0 \frac{d\langle \sigma_i \rangle}{dt} = -\langle \sigma_i(t) \rangle + \left\langle \tanh\left(\beta \sum_j J_{ij} \sigma_j(t)\right) \right\rangle$$

Mean field

$$\tau_0 \frac{dm_i}{dt} = -m_i + \tanh(\beta H_i)$$

above T_c $\quad -m_i + \tanh(\beta H_i) \approx -m_i + \beta H_i$

$$\tau_0 \frac{dm_i}{dt} \simeq -m_i + \beta \sum_j J_{ij} m_j - \beta^2 m_i \sum_j J_{ij}^2 (1-q) + \beta h_i$$
$$= -m_i + \beta \sum_j J_{ij} m_j - m_i \beta^2 J^2 + \beta h_i \qquad (q = 0 \text{ above } T_c)$$

In the basis where J is diagonal

$$\tau_0 \frac{dm_\lambda}{dt} = -m_\lambda (1 + \beta^2 J^2 - \beta J_\lambda) + \beta h_\lambda$$

Susceptibility in Fourier space

$$\chi_\lambda = \frac{\partial m_\lambda}{\partial h_\lambda} = \frac{\beta}{1 - i\omega\tau_0 + \beta^2 J^2 - \beta J_\lambda}$$

instability when max eigenvalue

$$(J_\lambda)_{max} = \frac{1 + \beta^2 J^2}{\beta}$$

RR 6.

Dense random matrix with mean square element $\frac{J^2}{N}$
the eigenvalue density is semicircular

$$\rho(J_\lambda) = \frac{1}{2\pi J}\sqrt{4J^2 - J_\lambda^2}$$

$$(J_\lambda)_{max} = 2 \Rightarrow \beta_c = 1$$

$J=1$ $$\chi(\omega) = \frac{1}{N}\sum_\lambda \chi_\lambda(\omega) = \int d\lambda\, \rho(\lambda)\chi_\lambda(\omega)$$

$$\chi(\omega) = \frac{1}{2}\beta\left(T^2(1-i\omega\tau_0) + 1 - \sqrt{(T^2(1-i\omega\tau_0)+1)^2 - 4T^2}\right)$$

$$\chi(\omega) \approx \frac{1}{T(1-i\omega\tau)} \Rightarrow \tau \sim \frac{1}{T-T_c}$$ critical slowing down

softest mode

$$\chi_\lambda = \frac{\beta}{1-i\omega\tau_0+\beta^2J^2-2\beta J} = \frac{\beta}{(1-\beta J)^2 - i\omega\tau_0}$$

$$\tau \sim \frac{1}{(T-T_c)^2}$$

Cavity method

Back to $F_{TAP} = \sum_{ij}\left(\frac{1}{2}J_{ij}m_im_j - \frac{1}{4}\beta J_{ij}^2(1-m_i^2)(1-m_j^2)\right)$

$$-T\sum_i S(m_i)$$

When is the correction correct

it is defined if $\beta^2\langle(1-m_i^2)^2\rangle \leq 1$

– System with N spins $i = 1 \ldots N$
we add a new spin σ_0, J_{0i}

Assumption! there is only one nontrivial sol. of TAP $m_0^2 = \langle\langle m_i\rangle\rangle$
(apart from $m_i \to -m_i$)

Hamiltonian of the new spin:

$$\mathcal{H}_0 = \sigma_0 \sum_{i=1..N} J_{0i}\,\sigma_i$$

- mean field (vanishing correlations)

$$m_0 = \langle \sigma_0 \rangle = \tanh(\beta h^{eff})$$

$$h^{eff} = \sum_{i=1..N} J_{0i}\, m_i^c$$

cavity magnetization of system without $\varnothing$

$$\overline{h^2} = q = \frac{\sum_i (m_i^c)^2}{N}$$

→ average squared magnetization of new spin is the same as the average of the old points

$$(1) \qquad q = \overline{m_0^2} = \int dp_q(h) \tanh^2(\beta h)$$

normalized Gauss dist, zero mean, var. q

$$\beta\,\Delta F(h) = -\ln(\cosh(\beta h))$$

h dependent increase of the free energy from $N \to N+1$

→ other method: perturbative approach of magnetization from $N \to N+1$

$$m_i' \approx m_i + J_{0i}\, m_0' \frac{\partial m_i}{\partial h_i} =$$

N+1 sys   MF approx

$$= m_i + J_{0i}\, m_0' \underbrace{\beta(1-(m_i')^2)}$$

We can recover the TAP equations:

$$m_0 = \tanh(\beta h)$$

$$h = \sum_{i=1,N} J_{0i} m_i \simeq \sum_{i=1,N} J_{0i} m_i^1 - m_0 \sum_i J_{0i}^2 \beta (1-(m_i^1)^2) \simeq$$

$$\simeq \sum_{i=1,N} J_{0i} m_i^1 - m_0 \beta (1-q)$$

where

$$(N+1) q = \sum_{i=0,N} m_i^2 \qquad \left(J_{0i}^2 = N^{-1} \right)$$

internal energy density

$$e = \frac{1}{2} \sum_i J_{0i} \langle \sigma_0 \sigma_i \rangle = \frac{1}{2} \beta (1-q^2)$$

TAP free energy as function of q

$$\beta f(q) = -\frac{1}{4} \beta^2 (1-q^2) - \int dP_q(h) \ln(\cosh(\beta h))$$

Stationary point $\frac{\partial f}{\partial q} = 0$

nontrivial solution

$q(T) = 0$ for $T > 1$

$q(T) \sim 1-T$ for $T \lesssim 1$

$q(T) \sim 1 - AT$ $T \to 0$

$$\underbrace{S = \langle\langle S(m_i) \rangle\rangle}_{\sim T} - \frac{(1-q)^2}{4T^2} \Rightarrow s < 0$$

$T \to 0$ ↯

↯ single nontrivial solution of TAP eq