Percolation

- site p
- bond p

} occupation probability

clusters connected components

s: size of a cluster

n_s: number of clusters with size s

p_c: for $p > p_c$ there is an infinite cluster

for $L \to \infty$ $\left(\frac{s_{max}}{L^d} > 0 \right)$

finite system:

$\Pi(p,L)$ probability that a system of size L percolates at p (has a size of L

$$\lim_{L \to \infty} \Pi(p,L) = \begin{cases} 0, & p < p_c \\ 1, & p > p_c \end{cases}$$

Example: 1d

$$\lim_{L \to \infty} \Pi(p,L) = \lim_{L \to \infty} p^L = \begin{cases} 0, & p < 1 \\ 1, & p = 1 \end{cases}$$

prob RHS of a spin a clusters of size s

$$n_s = (1-p)\, p^s (1-p) = (1-p)^2 p^s$$

$$n_s(p) = (1-p)^2 p^s = (1-p)^2 \exp(\ln(p^s)) = (p_c - p)^2 e^{-s/s_\xi}$$

s_ξ: characteristic cluster size

$$s_\xi = \frac{-1}{\ln p} = \frac{-1}{\ln(p_c - (p_c - p))} \underset{(p \to p_c)}{\Longrightarrow} \frac{1}{p - p_c} = (p - p_c)^{-1}$$

using $p_c = 1$

$\ln(1-x) \approx -x$

$s_\xi \sim |p_c - p|^{-1/\sigma}$ $\quad p \to p_c$

for 1 d $\quad \sigma = 1$

- prob that a site belongs to any cluster (finite)

$$\sum_{s=1}^{\infty} s \cdot n_s(p) = p \qquad (p < p_c)$$

$$\sum_{s=1}^{\infty} s \cdot n_s(p) = \sum_{s=1}^{\infty} s(1-p)^2 p^s = (1-p)^2 \sum_{s=1}^{\infty} p \frac{d p^s}{dp} =$$

$$= (1-p)^2 p \frac{d}{dp} \sum_s p^s = (1-p)^2 p \frac{d}{dp} \frac{p}{1-p} = p$$

Average cluster size → how large is a cluster on average to which an occupied site belongs?

$w_s = \frac{s n_s}{p}$ prob. that the cluster of the site contains s sites

$\sum w_s = 1$

Mean cluster size

$$S(p) = \sum_{s=1}^{\infty} s w_s = \sum_s \frac{s^2 n_s}{p} = \frac{1}{p}(1-p^2) \sum_s s^2 p^s =$$

$$S(p) = \frac{1+p}{1-p} = \frac{p_c + p}{p_c - p} \sim |p_c - p|^{-\gamma} \Rightarrow \gamma = 1$$

$$\approx \frac{2p_c}{p_c - p}$$

- pair correlation function $g(r)$ prob that a site at distance r from occupied site belongs to the same cluster:

$$g(r) = p^r$$

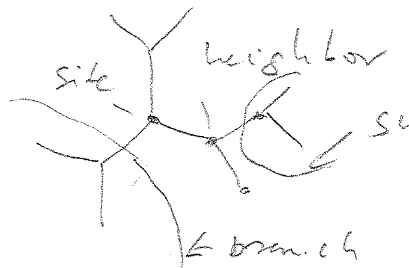
$$g(r) = p^r = \exp(\ln p^r) = \exp(r \ln(p)) = \exp\left(-\frac{r}{\xi}\right)$$

$$\xi = -\frac{1}{\ln p} \approx (p - p_c)^{-1} \Rightarrow \xi \sim |p_c - p|^{-\nu} \qquad p \to p_c$$

$$\sum_r g(r) = S(p)$$

Other analytical solution : $d = \infty$

Bethe lattice (no loops) sites have z neighbours



$z = 3$

g generation $(\sim L)$

number of sites :

$$N(g) = 1 + 3 \cdot (1 + 2 + 2^2 + \dots + 2^{g-1}) = 3 \cdot 2^g - 2$$

number of surface sites

$$N_S = N(g) - N(g-1)$$

$$\frac{N_S(g)}{N(g)} = \frac{3 \cdot 2^{g-1}}{3 \cdot 2^g - 2} \to \frac{1}{2} \qquad g \to \infty$$

general : $$\frac{N_S(z,g)}{N(z,g)} = \frac{z-2}{z-1} \qquad g \to \infty$$

— p_c

$$p_c(z-1) = 1 \Rightarrow p_c = \frac{1}{z-1} \qquad (z = 2 \Leftrightarrow 1d)$$

Strength of the infinite cluster $P(p)$

$p < p_c$ no infinite cluster for $q \to \infty$

$p \geq p_c$:

Q : prob. site does not belong to the inf. cluster

$P(p) = p(1 - Q^3)$

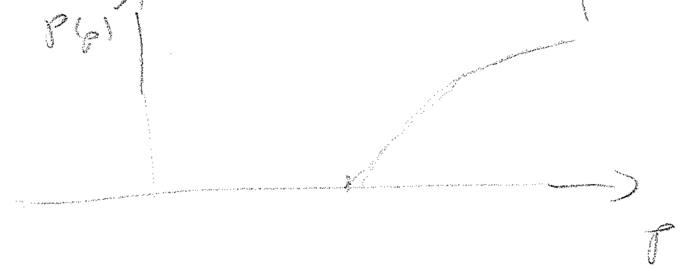
site is occ (for p); at least one goes to inf (for $1 - Q^3$)

$Q =$ (empty) + (occ.) $\cdot$ (no branch goes to inf)

$Q = (1-p) + p \cdot Q^2 \Rightarrow Q = \begin{cases} 1 \\ \frac{1-p}{p} \end{cases}$

$\Rightarrow P(p) = \begin{cases} 0 & p < p_c \\ p\left[1 - \left(\frac{1-p}{p}\right)^3\right] & p \geq p_c \end{cases}$

$P(p) \sim (p - p_c) \Rightarrow \beta = 1 \qquad (p \to p_c^+)$



- Mean cluster size $S(p)$ (of origin)

$S(p) =$ origin $+ 3 \cdot$ branches $= 1 + 3T$

$T = \underbrace{(1-p) \cdot 0}_{\text{not occ}} + \underbrace{p}_{\text{occ}} \cdot \underbrace{(1 + 2T)}_{\text{self, 2 branches}} \Rightarrow T = \frac{p}{1-2p}$

$S(p) = 1 + 3T = \frac{1+p}{(1-2p)} = \frac{1+p}{2(p_c - p)} \sim (p_c - p)^{-1} \qquad p < p_c$

$\gamma = 1$

Scaling function

$$n_s(p) = s^{-\tau} f((p-p_c)\, s^{\sigma})$$

$$f(x) = \begin{cases} \text{const} & x \ll 1 \\ \text{fast decay} & x \gg 1 \end{cases}$$

$$\langle s \rangle = \int s^{2-\tau} f((p-p_c)\, s^{\sigma})\, ds \Rightarrow \langle s \rangle \sim (p-p_c)^{-\frac{3-\tau}{\sigma}}$$

$\tau < 3$

$$\gamma = \frac{3-\tau}{\sigma}$$

$\sum_{s=1}^{\infty} P(s) \to$ prob to hit an occupied node of a finite cluster

$P_\infty \to$ prob to hit the inf. cluster

$$P_\infty + \sum_{s=1}^{\infty} P(s) = p$$

at $p = p_c$

$$P_\infty = (p-p_c) + \sum_{s=1}^{\infty} (P_c(s) - P(s))$$

$$P_\infty \sim (p-p_c)^{\frac{\tau-2}{\sigma}} \qquad \beta = \frac{\tau-2}{\sigma}$$

Radius of gyration

~~Cluster~~ center of mass $r_{cm} = \frac{1}{s} \sum_{i \in s} r_i$

$$R_s^2 = \frac{1}{s} \sum_i |\underline{r}_i - \underline{r}_{cm}|^2$$ distance square from c.m.

$$R_s^2 = \frac{1}{2} \frac{1}{s^2} \sum_{ij} |\underline{r}_i - \underline{r}_j|^2$$

$$\sum_r g(r) = S(p)$$ mean cluster size

$$\xi^2 = \frac{\sum_r r^2 g(r)}{\sum_r g(r)} = \frac{\sum_s 2 R_s^2 s^2 n_s(p)}{\sum s^2 n_s(p)}$$

Fractal

$$M(L) \sim L^D$$

$$s \sim R_s^D \qquad s \gg 1 \qquad p = p_c$$

$$R_s \sim s^{1/D}$$

$$\xi^2 \sim \frac{\sum_s 2 s^{2+2/D} n_s}{\sum_s s^2 n_s} \sim$$

$$\sim \frac{|p-p_c|^{\tau - 3 - \frac{2}{D\sigma}}}{|p-p_c|^{\tau - 1 - 2/\sigma}} \sim |p-p_c|^{-\frac{2}{D\sigma}}$$

$$\xi^2 \sim |p-p_c|^{-2\nu} \Rightarrow \nu = \frac{1}{D\sigma} \qquad D = \frac{1}{\nu\sigma}$$

$$\Rightarrow \nu = \tfrac{1}{2} \quad \sigma = \tfrac{1}{2} \quad \text{Bethe} \quad D = 4$$

Percolation on graphs

Erdős - Rényi random graph:

N nodes links with probability p

percolation?

$$p(N, p; L) = \binom{L_{max}}{L} p^L (1-p)^{L_{max} - L}$$

↳ prob of having L links

$\langle L \rangle = p \cdot L_{max}$

$$p(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k} \Rightarrow \langle k \rangle = p(N-1)$$

percolation?

$u = 1 - P_\infty$

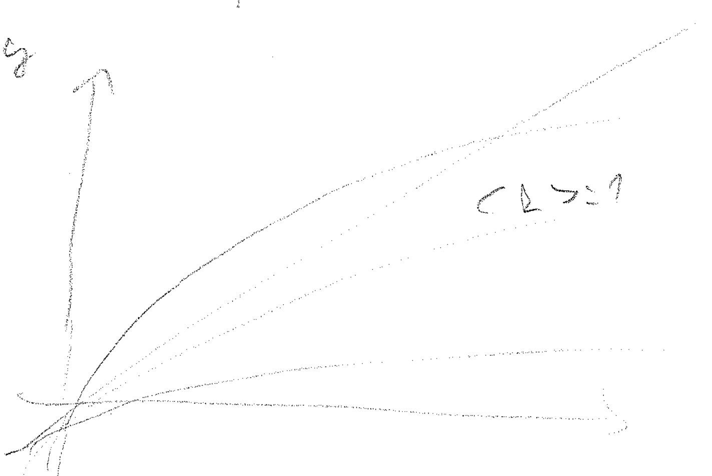
$$u = (\underbrace{1-p}_{\text{no link}} + \underbrace{pu}_{\substack{\text{link to} \\ \text{non inf}}})^{N-1} = \left[1 - \frac{\langle k \rangle}{N-1}(1-u)\right]^{N-1}$$

$$\ln u = (N-1) \ln\left(1 - \frac{\langle k \rangle}{N-1}(1-u)\right) \simeq -\langle k \rangle (1-u)$$

$u = e^{-\langle k \rangle (1-u)}$

$P_\infty = 1 - e^{-\langle k \rangle P_\infty}$

$y = 1 - e^{-\langle k \rangle x}$



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$P_\infty = 1 - u = 1 - e^{-\langle k\rangle P_\infty}$

$p = p_c$ $\qquad y = \langle k\rangle P_\infty$

$y = \langle k\rangle (1 - e^{-y})$ $\qquad y = \langle k\rangle \left(y - \frac{1}{2} y^2\right)$

$\langle k\rangle - 1 = \langle k\rangle \frac{1}{2} y \Rightarrow P_\infty \sim \varepsilon^\beta$ $\qquad \beta = 1$

$\varepsilon = \langle k\rangle - \langle k\rangle_c = \langle k\rangle - 1$