The Sherrington - Kirkpatrick model II

SK: "Solvable model of a Spin-Glass" PRL 1975

N sites

$$\boxed{H = - \sum_{(ij)} J_{ij} \overbrace{\sigma_i}^{\sigma_i} \sigma_j - h \sum_i \sigma_i}$$

J_{ij} Gaussian $\langle J_{ij} \rangle = \frac{1}{N} J_0$ (ferromagnetic)

$\langle \delta J_{ij}^2 \rangle = \frac{1}{N} \tilde{J}^2$ (random)

• for now: $J_0 \equiv 0$ $\Rightarrow$ $dJ\, P(J) = \sqrt{\frac{N}{2\pi}} \frac{dJ}{\tilde{J}} e^{-\frac{1}{2}\frac{N}{\tilde{J}^2} J^2}$

will set $\tilde{J} = 1$

• Random sign:

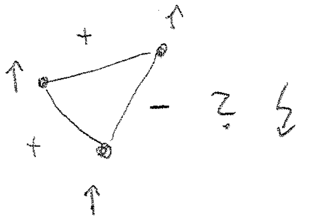
~ mimics RKKY, dipole-dipole

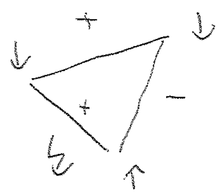
$$J(R) \sim \frac{1}{R^3} \cos(2k_F R)$$

oscillates

long-ranged

frustration:





$\Rightarrow$ degeneracy of ground states

Replica trick:

$$F = -T \ln Z \quad \Rightarrow \quad \overline{F} = -T\,\overline{\ln Z} \neq -T \ln \overline{Z}$$

disorder

this can be computed...

$$\boxed{\overline{\ln Z} = \lim_{n \to 0} \frac{1}{n}\left(\overline{Z^n} - 1\right)}$$

partition function of n copies (replicas)

$$Z_n \equiv \overline{Z^n} = \sum_{\sigma} \left\langle \exp\left\{\sum_a \sum_{i<j} \beta J_{ij}\, \sigma_i^a \sigma_j^a + \sum_{a,i} h\beta\, \sigma_i^a\right\}\right\rangle_{\{J\}}$$

$$\sigma = \{\sigma_i^a\} \qquad a = 1, \dots, n$$

Gaussian trick: x Gaussian variable with $\langle x \rangle = 0$

$$\Rightarrow \quad \langle e^{\pm x y} \rangle_x = e^{\frac{1}{2}\langle x^2 \rangle y^2}$$

$$\Rightarrow \quad Z_n = \sum_{\sigma} \exp\left\{\frac{\beta^2}{2N} \sum_{i<j}\left(\sum_a \sigma_i^a \sigma_j^a\right)^2 + \sum_i \sum_a \beta \cdot h\, \sigma_i^a\right\}$$

$$\sum_{i<j}\left(\sum_a \sigma_i^a \sigma_j^a\right)^2 = \frac{1}{2}\left\{\sum_{i,j} \sum_{a,b} \sigma_i^a \sigma_i^b \sigma_j^a \sigma_j^b - N n^2\right\}$$

diag $i = j$

no $\mathcal{O}(n)$ contr.

$$= \sum_{a<b}\left(\sum_i \sigma_i^a \sigma_i^b\right)^2 + \frac{1}{2} n N^2$$

$$\boxed{Z_n = \int \mathcal{D}Q \; \exp\{-N A(Q)\}}$$

$$\boxed{A(Q) = -\frac{\beta^2 n}{4} + \frac{1}{2}\beta^2 \sum_{a<b} Q_{ab}^2 - \ln Z(Q)}$$

$N \to \infty \quad \Rightarrow$ saddle point !

$$\frac{\partial A}{\partial Q_{ab}} = 0 = \beta^2 Q_{ab} - \frac{1}{Z(Q)} \sum_{S^a} \beta^2 S^a S^b e^{-\beta H(Q,S)}$$

mean field self consistency : $\boxed{Q_{ab} = \langle S^a S^b \rangle_{H(Q,S)} = \left\langle \frac{1}{N} \sum_i \sigma_i^a \sigma_i^b \right\rangle_{Q_{ab}}}$ $(*)$

overlap of replica "a" and "b"

program : 1) solve $(*)$ for $\tilde{Q}$

2) replace $Q \to A(\tilde{Q}) \to Z_n$

3) take $\frac{1}{N}\overline{F} = -\frac{T}{N} \lim_{n\to 0} \frac{1}{n}(Z_n - 1)$

⋮

natural assumption : overlap is same for all replicas (?) $\Rightarrow \quad Q_{ab} = q \quad \forall\, a<b$

= "replica symmetry"

- $$\sum_{a<b} Q_{ab}^2 = \frac{n(n-1)}{2} q^2 \xrightarrow[n]{0} -\frac{n}{2} q^2$$

- $$Z(Q) = \sum_{S^a} \exp\Big\{\beta^2 q \underbrace{\sum_{a<b} S^a S^b}_{\frac{1}{2}(\sum S^a)^2 - \frac{n}{2}}\Big\} =$$

$h=0$

$$= e^{-\frac{n}{2} q\beta^2} \int \frac{dx}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \sum_{S^a} e^{\overbrace{\beta\sqrt{q}\cdot x}^{\text{random field}} \sum_a S^a}$$

$$Z_n = \sum_{\sigma} \exp\left\{ \frac{\beta^2}{4} n \cdot N + \frac{\beta^2}{2N} \sum_{a<b} \left(\sum_i \sigma_i^a \sigma_i^b \right)^2 + \beta \cdot h \sum_{i,a} \sigma_i^a \right\}$$

$\mathcal{O}(n^2)$

trick

$$e^{\frac{\beta^2}{2N} \left(\sum_i \sigma_i^a \sigma_i^b \right)^2} = \int \frac{d y_{ab}}{\sqrt{2\pi}} \, e^{-\frac{1}{2} y_{ab}^2} \cdot e^{\frac{1}{\sqrt{N}} \beta \, y_{ab} \left(\sum_i \sigma_i^a \sigma_i^b \right)}$$

$$Q_{ab} \equiv \frac{y_{ab}}{\beta \sqrt{N}}$$

$$= \int \left(\frac{\beta^2 N}{2\pi} \right)^{1/2} d Q_{ab} \; e^{-\frac{1}{2} N \beta^2 Q_{ab}^2} \; e^{Q_{ab} \beta^2 \left(\sum_i \sigma_i^a \sigma_i^b \right)}$$

$$\int DQ \ldots \equiv \int \prod_{a<b} \left(\frac{\beta^2 N}{2\pi} \right)^{1/2} d Q_{ab}$$

$$Z_n = \int DQ \sum_{\sigma} \exp\left\{ \frac{\beta^2}{4} n \cdot N - \frac{1}{2} N \beta^2 \sum_{a<b} Q_{ab}^2 + \sum_{a<b} Q_{ab} \beta^2 \sum_i \sigma_i^a \sigma_i^b + \beta \cdot h \sum_{i,a} \sigma_i^a \right\}$$

can carry out site by site

$$\underline{Z_n = \int DQ \, \exp\left\{ \frac{\beta^2}{4} n \cdot N - \frac{1}{2} N \beta^2 \sum_{a<b} Q_{ab}^2 \right\} \cdot Z(Q)^N}$$

$$Z(Q) = \sum_{s^a} \exp\left\{ \beta^2 \sum_{a<b} Q_{ab} s^a s^b + h\beta \sum_a s^a \right\}$$

$$\boxed{Z(Q) = \sum_{s^a} \exp\left[-\beta \, H(Q, s) \right]}$$

single site

$$\boxed{H(Q, s) = -\beta \sum_{a<b} Q_{ab} s^a s^b - h \sum_a s^a}$$

couples replicas ! mean field !

$$Z(q) = e^{-\frac{n}{2} q \beta^2} \cdot \underbrace{\int \frac{dy}{\sqrt{2\pi q}}\, e^{-\frac{1}{2}\frac{y^2}{q}} \left[2\,\mathrm{ch}\,\beta(y+h)\right]^n}_{\sim\ n \text{ indep. spins in random field } y \text{ of var } \langle y^2\rangle = q}$$

$h=0$ case

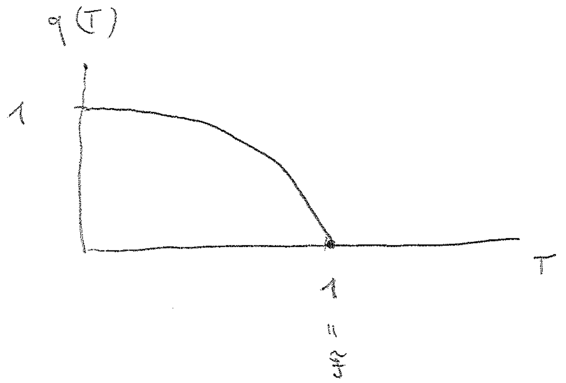
$$A(q) = -\frac{\beta^2}{4}\cdot n - \frac{\beta^2}{4} q^2 \cdot n + \frac{n}{2} q \cdot \beta^2$$
$$- \ln\left\{ \int \frac{dy}{\sqrt{2\pi q}}\, e^{-\frac{1}{2}\frac{y^2}{q}} \left(2\,\mathrm{ch}\,\beta(y+h)\right)^n \right\}$$

saddle point ($h=0$)

$$\frac{\partial A}{\partial q} = 0 \quad \Rightarrow \quad \boxed{q = \langle \mathrm{th}^2(y\cdot\beta)\rangle_y} \quad \Rightarrow \quad q(T)$$

$$\langle \ldots \rangle_y = \int \frac{dy}{\sqrt{2\pi q}}\, e^{-\frac{1}{2}\frac{y^2}{q}} \ldots$$

solution:



Free energy:

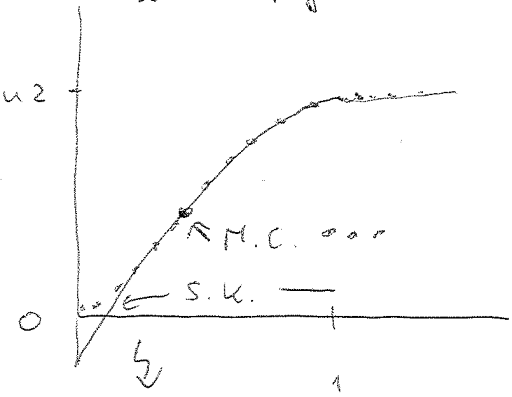
$$+\frac{F}{T} = \lim_{n\to 0} \frac{1}{n} N A(q) \quad \Rightarrow \quad f = \frac{1}{N} F = T \cdot \lim_{n\to 0} \frac{1}{n} A(q)$$

$$\boxed{f(T) = -\frac{\beta}{4}(1-\underset{q(T)}{q})^2 - \frac{1}{\beta} \int \frac{dy}{\sqrt{2\pi q}}\, e^{-\frac{1}{2} y^2/q} \cdot \ln\left(2\,\mathrm{ch}\,\beta(y+h)\right)}$$

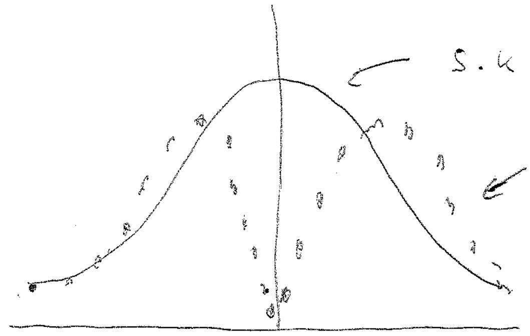
Problems :

- $s(T)$

entropy



- local field distribution $T \to 0$



"pseudogap" ! $\Leftrightarrow$ stability against two spin-flip

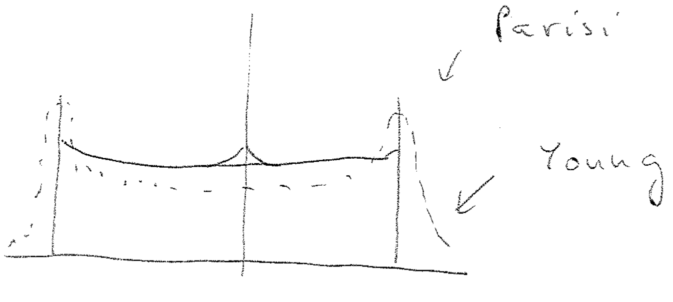
- $P(Q_{ab})$ for $T \to \phi$ (simulated annealing)

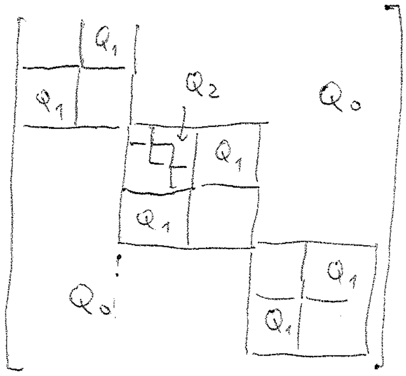
Peter Young '83

$$W(q) \equiv \Big\langle \sum_{s,s'} P(s)\, P(s')\, \delta(Q^{ss'} - q) \Big\rangle_J$$

indep. mean field solutions

SK $\to$ $W(q) = \delta(q - q_{SK})$



$Q_{ab} = Q =$ 

ultrametric structure

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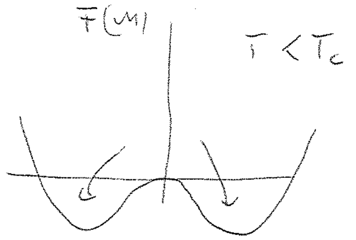
Parisi matrices

"ultrametric structure"

ferromagnet

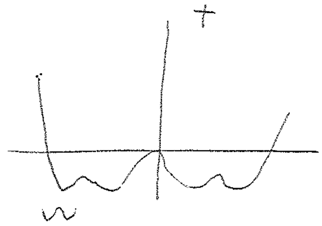


two possible states

overlap between states $\hat{q} \xrightarrow[T]{} \pm 1$

spin glass

$T < T_c$



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exponentially many possible ground stat configurations

tree - structure