Multifractality of the critical wave function

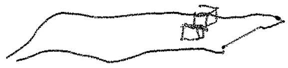
fractals: objects of non-integer dimension

Hausdorf dim.:

take boxes of side ϵ

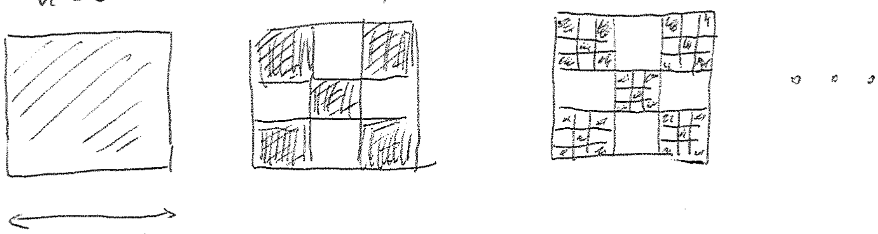
# of boxes needed to cover object $N_\epsilon \sim \left(\frac{L}{\epsilon}\right)^D$ $\epsilon \to 0$

line:  $D = 1$

surface  $D = 2$

self-sim. objects

$n = 0$ $n = 1$ $n = 2$



$1 = L$

$$\epsilon = \frac{1}{3} \longrightarrow N_\epsilon = 5$$

$$\epsilon = \frac{1}{9} \longrightarrow N_\epsilon = 5^2$$

$$\vdots$$

$$\epsilon = \frac{1}{3^n} \longrightarrow N_\epsilon = 5^n$$

$D = ?$ $$\ln \epsilon = -n \ln 3 \Rightarrow n = -\frac{\ln \epsilon}{\ln 3} \Rightarrow$$

$$\Rightarrow N_\epsilon = (5)^n = e^{n \ln 5} = e^{-\frac{\ln \epsilon \ln 5}{\ln 3}}$$

$$= \left(\frac{1}{\epsilon}\right)^{\ln 5 / \ln 3}$$

$$\Rightarrow D = \frac{\ln 5}{\ln 3} \cong 1,46\ldots$$

$E \to E_c$: $\xi \to \infty$ ~ critical behavior

- there is a divergent scale on both sides
- $\xi(E) \sim |E - E_c|^{-\nu}$

$\xi = \infty$ self-similarity ~ fractal (Ising model at $T = T_c$)

more tricky here ... <u>multifractal</u>

we measured the size of $\psi_E(\mathbf{r})$ by the IPR

$$\mathrm{IPR} = \sum_{\mathbf{r}} |\psi(\mathbf{r})|^4$$

define the q-the moment:

$$\boxed{P_q \equiv \sum_{r} |\psi(\mathbf{r})|^{2q}} \longrightarrow \boxed{\langle P_q(E)\rangle_{\text{disord}} = L^d \underset{\substack{\uparrow \\ \text{at a given pos.}}}{\langle |\psi(\mathbf{r})|^{2q}\rangle_{\text{diso}}}}$$

- in a metal, we expect

$$\langle P_q\rangle \propto L^d L^{-dq} = L^{-d(q-1)} \quad \begin{array}{l} \nearrow 0 \text{ for } q > 1 \\ \searrow \infty \text{ for } q < 1 \end{array}$$

- in the insulator ($E \leftrightarrow$ loc.)

$$\langle P_q\rangle \propto \xi^{-d(q-1)}$$

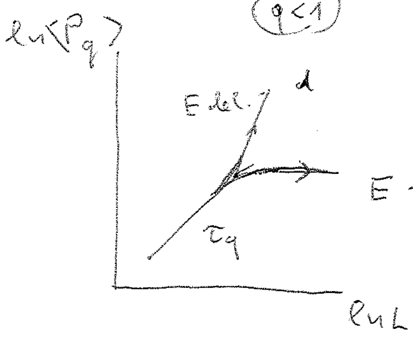
this is not quite true, at criticality

$$\boxed{\langle P_q\rangle \propto L^{-\tau_q}} \qquad E = E_c$$

$$\boxed{\tau_q = d(q-1) + \Delta_q}$$

off criticality:

(q<1)



$$\langle P_q\rangle \propto \begin{cases} L^{-\tau_q} & L < \xi(E) \\ L^{-d(q-1)} \text{ or } \xi^{-d(q-1)} & L > \xi(E) \end{cases}$$

scaling of prefactor can be worked out

$\langle P_q \rangle \quad \longleftrightarrow \quad$ moments of $\quad P(|\psi(r)|^2) = P(|\psi|^2)$

multifractal spectrum:

$$\#\,\text{sites}\,(|\psi^2(r)| \sim L^{-\alpha}) \sim L^{f(\alpha)}$$

$$\Rightarrow \underbrace{L^d \langle |\psi|^{2q} \rangle}_{= \sum_r |\psi(r)|^{2q}} \sim \int d\alpha \; L^{-\alpha q} \cdot L^{f(\alpha)}$$

$$= \int d\alpha \; e^{(f(\alpha) - \alpha q) \ln L}$$

saddle point $\Rightarrow \; \approx L^{-\underbrace{(\alpha(q) \cdot q - f(\alpha(q)))}_{\tau_q}}$

with
$$\boxed{\begin{aligned} \tau_q &= \alpha(q) \cdot q - f(\alpha(q)) \\ q &= \frac{df}{d\alpha} \end{aligned}}$$

$f(\alpha)$ is the Legendre transform of τ_q

backwards transformation
$$\boxed{\frac{d\tau_q}{dq} = \frac{d\alpha}{dq} \cdot q + \alpha - \frac{df}{d\alpha}\frac{d\alpha}{dq} = \alpha}$$

<u>metallic phase</u>:

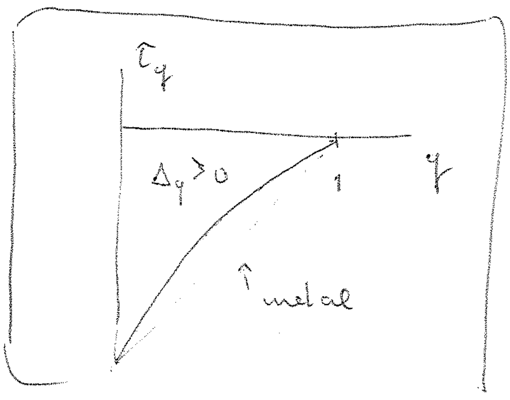
$\tau_q = d(q-1) \quad \Rightarrow \quad \alpha = \frac{d\tau_q}{dq} = d \quad !$

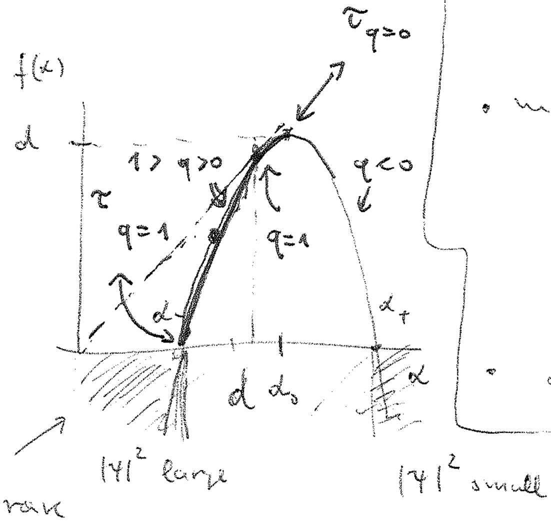
the wave function has a single dim. in the spectrum $f(\alpha)$!

<u>general props.</u>

τ_0 : $P_0 = L^d \Rightarrow \tau_0 = -d$

τ_1 : $P_1 = 1 \Rightarrow \tau_1 = 0$





rare events ~ with exp small ampl.

• maximum at α_0 $\iff$ $\left.\frac{df}{d\alpha}\right|_{\alpha_0} = q = 0$

$$\Rightarrow \quad \tau_{q=0} = -d = \alpha\cdot q - f(\alpha)$$

$$\Rightarrow \quad \boxed{f(\alpha_0) = d}$$

• $q=1$ $\rightarrow$ $\tau_{q=1} = 0 = \alpha_1 - f(\alpha_1)$

$$f'(\alpha_1) = 1 \; (= q)$$

simplest correction:

$$\tau_q = d(q-1) + \underbrace{\gamma\, q(1-q)}_{\Delta_q} + \dots$$

$$\Rightarrow \quad \alpha = \frac{\partial \tau_q}{\partial q} = d + \gamma(1-2q) \quad \Rightarrow \quad q = -\frac{1}{2\gamma}(\alpha - \underset{d+\gamma}{\alpha_0})$$

$$f(\alpha) = \alpha q - \tau_q =$$

$$= -\alpha\frac{1}{2\gamma}(\alpha-\alpha_0) + d + \frac{d}{2\gamma}(\alpha-\alpha_0)$$

$$+ \frac{1}{2}(\alpha-\alpha_0)\left(1 + \frac{1}{2\gamma}(\alpha-\alpha_0)\right)$$

$$= d + \frac{(\alpha-\alpha_0)^2}{4\gamma} + \frac{\alpha-\alpha_0}{2\gamma}\big(\underbrace{d - \alpha + \gamma}_{\alpha_0-\alpha}\big)$$

$$\boxed{f(\alpha) \cong d - \frac{1}{4\gamma}(\alpha-\alpha_0)^2}$$

weak multifractality

symmetries:

$$\Delta_q = \Delta_{1-q} \quad \Rightarrow \quad f(2d-\alpha) = f(\alpha) + d - \alpha$$

Correlations:

spatial correlations ~

$$\langle |\psi(r)|^{2q} \, |\psi(r')|^{2q'} \rangle \quad \begin{cases} \nearrow \; r = r': & L^{-(d(q+q') + \Delta_{q+q'})} \\ \searrow \; r \to \text{infty}, \; r_{12} \to L & L^{-(d(q+q') + \Delta_q + \Delta_{q'})} \end{cases}$$

$$L^{d(q+q')} \langle |\psi(r)|^{2q} \, |\psi(r')|^{2q'} \rangle \sim L^{-\beta} \left(\frac{|\mathbf{r} - \mathbf{r}'|}{L} \right)^{\alpha}$$

$$\alpha + \beta = \Delta_{q+q'} \qquad \checkmark$$
$$\beta \sim \Delta_q + \Delta_{q'}$$

in particular

$$L^{2d} \langle |\psi(r_1)|^2 \, |\psi(r_2)|^2 \rangle \sim \left(\frac{r_{12}}{L} \right)^{-\eta}$$

in the localized phase

$$\langle |\psi(r_1)|^2 \, |\psi(r_2)|^2 \rangle \sim \begin{cases} \left(\frac{r_{12}}{\xi} \right)^{-\eta} \xi^{-2d} & r_{12} < \xi \\ \emptyset & r_{12} > \xi \end{cases}$$

Problem: work out the scaling form of correlations close to E_c