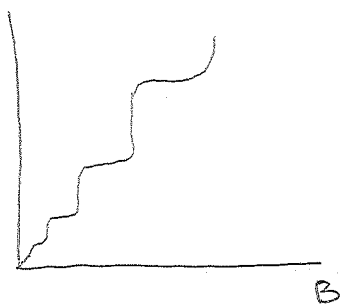
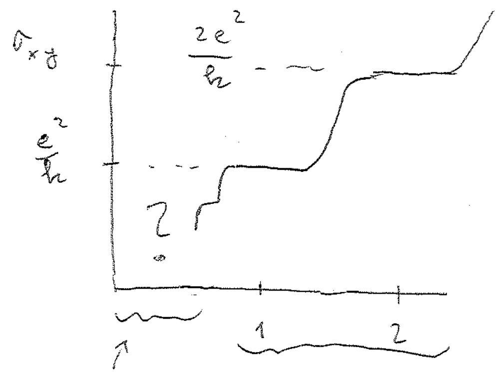
Q.H.E. and localization

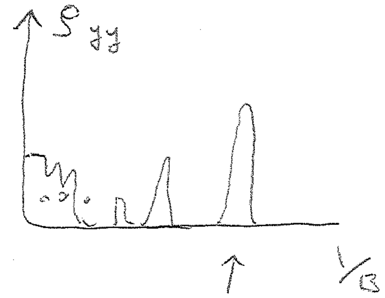
QHE : Quantum Hall effect



2D system (heterostructure, graphene ...)



$\Rightarrow$



$\nu \equiv n \frac{h}{eB} \sim 1/B$

at transitions

fractional
Chern - Simons
field theories ...

disorder crucial !
non - linear σ model
topological field theories

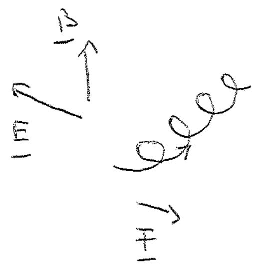
Classical motion :

$$\boxed{m \dot{\underline{v}} = -e\, \underline{v} \times \underline{B} - e\underline{E}} \qquad (e>0)$$

$z \equiv x + iy \Rightarrow \boxed{\ddot{z} = i\omega_c \dot{z} - \frac{e}{m}\mathcal{E}}$ $\qquad \mathcal{E} = E_x + iE_y$, $\omega_c = \frac{eB}{m}$

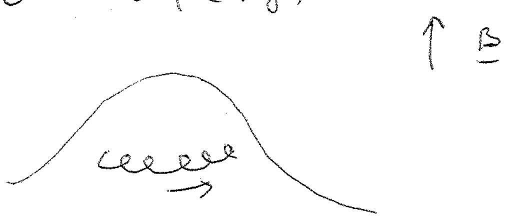
solution:

$$z(t) = \underbrace{d \cdot e^{i\omega_c t}}_{\text{cyclotron motion}} + z_0 - \underbrace{i\frac{1}{B}\mathcal{E}t}_{\text{slow drift!}} \qquad \underline{v}_D = \frac{1}{B}\,\underline{E} \times \hat{\underline{z}}$$

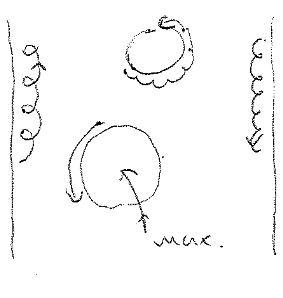


$\Rightarrow$

Landscape
$U = -e\,\phi(x,y)$



top view:



ω_c fast $\Rightarrow$ cyclotron motion quantized

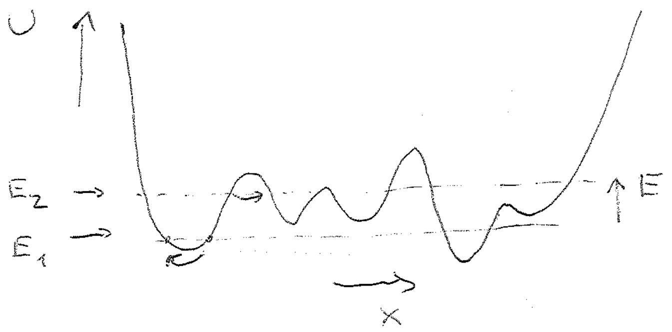
$\Rightarrow E = \hbar\omega_c\left(m+\frac{1}{2}\right) +$ potential energy

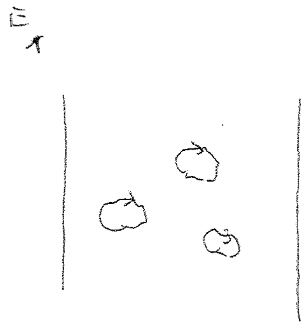
constant ! $\underline{v}_0 \sim \nabla U \times \hat{z}$!

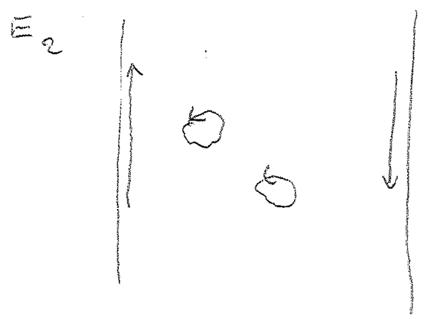
length scale $\hbar\omega_c = \frac{\hbar^2}{m}\cdot\frac{1}{\ell_B^2} \Rightarrow \ell_B^2 = \frac{h}{Be}\,\frac{1}{2\pi} \Rightarrow \nu = 2\pi\,\ell_B^2\, n$

phase transition

$m = 0$

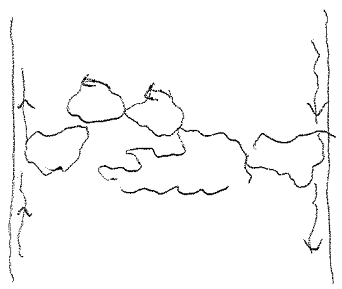


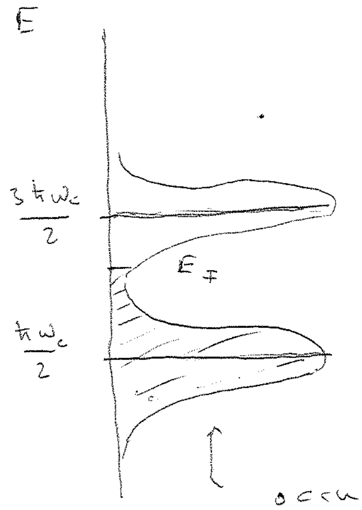




transition $\sim$ percolation !

$E = E_c$





$\leftarrow$ second Landau band $(m=1)$

$\leftarrow$ single percolating state !

$\Rightarrow \boxed{\xi \sim |E-E_c|^{-\nu}}$

occupied levels

Hall conductance quantization ??

qualitative answer ... - $V(x,y) \to V(y)$ + Landau gauge

- edge channel picture

... $\Rightarrow$ topological phase transition

$$\left[\quad \sigma_{xy} = -\frac{e^2}{\hbar}\,\mathcal{F}_{xy} = -\frac{e^2}{\hbar}\left(\frac{\partial \mathcal{A}_x}{\partial \vartheta_y} - \frac{\partial \mathcal{A}_y}{\partial \vartheta_x}\right)\quad\right]$$

Topology and $\sigma_{yx} = G_{yx}$

Kubo formula:

use $\underline{A}(\underline{r},t) \equiv \underline{A}_0(\underline{r}) + \delta\underline{A}(t)$ $\qquad \phi(\underline{r})$

$$\underline{E}(t) = -\delta\dot{\underline{A}}(t) \quad \Rightarrow \quad \underline{\delta A_x \equiv \frac{E_x}{i\omega} e^{-i\omega t}}$$

$$\hat{H} \Rightarrow \hat{H} + \underbrace{e\,\delta A_x \sum_i \frac{1}{m}\left(p_i^x + e A_0^x\right)}_{-\delta A_x \, J_x} + \dots$$

Kubo: $\langle J_y\rangle_\omega = \chi_{J_y J_x}(\omega)\,\delta A_{x;\omega}$

$$\Downarrow$$

$$\langle j_y\rangle_\omega = \underbrace{\frac{1}{L_x L_y}\frac{1}{i\omega}\chi_{J_y J_x}(\omega)}_{\sigma_{yx}(\omega)} \cdot E_{x;\omega}$$

$$\chi_{J_y J_x}(\omega) = -\sum_n \left\{ \frac{1}{\hbar\omega_+ - (E_n - E_0)} \langle 0|J_y|n\rangle\langle n|J_x|0\rangle - \frac{1}{\hbar\omega_+ + (E_n - E_0)} \langle 0|J_x|n\rangle\langle n|J_y|0\rangle \right\}$$

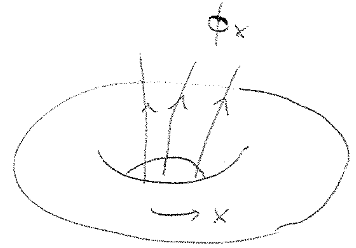
expand in ω

$\omega \to 0$

$$\boxed{\sigma_{yx} = \frac{\hbar}{L_x L_y}\frac{1}{i}\sum_n \frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle - \langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{(E_n - E_0)^2}}$$

Claim: This is a Chern # ...

Place <u>particles</u> on <u>torous</u>



<u>add flux</u> $\phi_x \Rightarrow$

$$A_x \to A_x + \frac{\phi_x}{L_x}$$

$$\boxed{H \to H - \frac{\phi_x}{L_x} J_x - \frac{\phi_y}{L_y} J_y} \quad + \text{P.B.C.} \, !$$

<u>Remark:</u> $\phi_{x,y}$ are defined up to multiples of $\phi_0 = \frac{h}{e}$

$$\psi(x,y) \to e^{i 2\pi x / L_x} \psi(x,y) \quad \text{shifts} \quad A_x \to A_x + \frac{h}{e} \frac{1}{L_x})$$

i.e. $\phi_x \to \phi_x + \frac{h}{e}$

introduce angle $\boxed{\vartheta_x = 2\pi \frac{\phi_x}{\phi_0}} = \frac{e}{\hbar} \phi_x$

perturbation theory:

$$\boxed{|\phi_0\rangle \Rightarrow |\phi_0\rangle - \frac{\phi_x}{L_x} \sum_{n \neq 0} \frac{|n\rangle \langle n | J_x | 0\rangle}{E_n - E_0}} - \frac{\phi_y}{L_y} \ldots$$

$$\left| \frac{\partial \phi_0}{\partial \vartheta_x} \right\rangle = |\partial_x \phi_0\rangle = -\frac{\hbar}{e} \frac{1}{L_x} \sum_n \frac{|n\rangle\langle n | J_x | 0\rangle}{E_n - E_0}$$

$$\langle \partial_y \phi_0 | \partial_x \phi_0 \rangle = \frac{\hbar^2}{e^2} \frac{1}{L_x L_y} \sum_n \frac{\langle 0 | J_y | n\rangle \langle n | J_x | 0\rangle}{(E_n - E_0)^2}$$

$$\Rightarrow \boxed{\sigma_{yx} = \frac{e^2}{\hbar} \frac{1}{i} \underbrace{\left(\langle \partial_y \phi_0 | \partial_x \phi_0 \rangle - \langle \partial_x \phi_0 | \partial_y \phi_0 \rangle \right)}_{\text{Berry curvature}}}$$

$$\boxed{-i \langle \phi_0 | \partial_\alpha \phi_0 \rangle \equiv A_\alpha} \leftarrow \underline{\text{Berry - connection}}$$

curvature

$$\boxed{\sigma_{yx} = \frac{e^2}{h} \cdot 2\pi \left(\partial_y A_x - \partial_y A_x \right) = 2\pi \frac{e^2}{h} \mathcal{F}_{yx}}$$

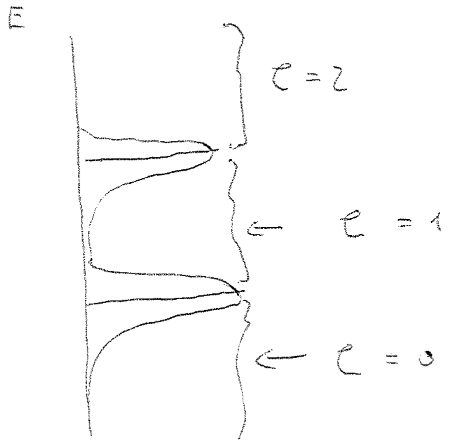
average $\Rightarrow$ $$\langle \sigma_{yx} \rangle_{\underline{\vartheta}} = \frac{e^2}{h} \underbrace{\frac{1}{2\pi} \int d^2\vartheta \, \mathcal{F}_{yx}(\underline{\vartheta})}_{\text{Chern number!}}$$

$$\boxed{\sigma_{yx} = G_H = \frac{e^2}{h}\cdot \mathcal{C}}$$

Remarks: • For non-interacting electrons

$$\mathcal{C} = \sum_{E<E_F} C(E)$$

↑ single particle Chern #



• Thouless - pump:

$\phi_x = 0 \to \phi_0 \qquad \phi_x(t) \qquad \Rightarrow V_x = \dot{\phi}_x$

$\Rightarrow$ current $\quad I_y = \frac{e^2}{h}\cdot V_x = \frac{e^2}{h}\dot{\phi}_x$

$\Rightarrow \quad \Delta Q_y = \int dt\, I_y = \frac{e^2}{h}\phi_0 = e$!

$$\boxed{\Delta\phi_x = \phi_0 \Rightarrow \Delta Q_y = e}$$

FQHE: degenerate ground states

$\Rightarrow \quad \Delta Q_y = \frac{p}{q}\cdot e$

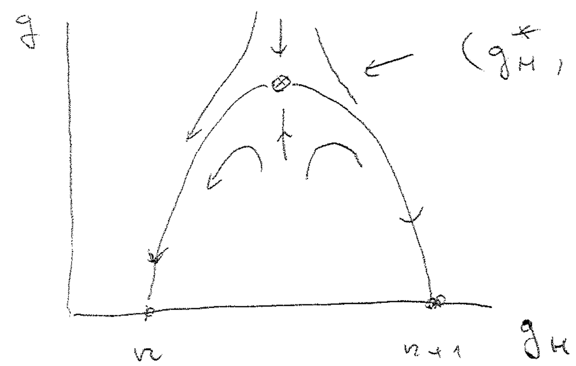
Pruisken - Khmelnitskii non-linear σ model with topological term in 2D

2 parameter scaling :

2D: $\sigma_{xx} = \frac{e^2}{h} \underset{\underset{g}{\parallel}}{g_{xx}}$, $\sigma_{xy} = \frac{e^2}{h} \underset{\underset{g_H}{\parallel}}{g_{xy}}$

$$\boxed{\frac{dg}{d\ln L} = \beta(g, g_H)} \qquad \boxed{\frac{d g_H}{d\ln L} = \beta_H(g, g_H)}$$

Flow :



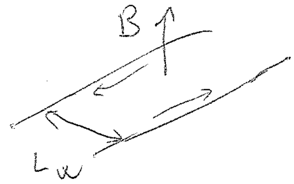
Consequence : $g = g^* + \sigma$

$g_H = g_H^* + \theta$

$$\boxed{\frac{d\sigma}{d\ln L} = -|y_\sigma|\,\sigma} \;, \quad \boxed{\frac{d\theta}{d\ln L} = y_\theta \cdot \theta}$$

$\Rightarrow$ sample of size L_0 $\Rightarrow g_{\alpha\beta}(L_0, B)$ $\Rightarrow \sigma_0(B), \theta_0(B)$

sample of width L_W , $T = 0$:



$$g_{\alpha\beta}(B) \approx F_{\alpha\beta}\left(\left(\frac{L_W}{L_0}\right)^{y_\theta} \theta_0 , \left(\frac{L_W}{L_0}\right)^{-|y_\sigma|} \sigma_0\right)$$

$\theta_0(B) \sim B - B_c$

$\Rightarrow \boxed{g_H(B) = F_H\left(\frac{L_W}{\xi(B)}\right)} \qquad \boxed{\xi(B) \sim |B - B_c|^{-\frac{1}{y_\theta}}}$

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Pruisken - Khmelnitskii
non - linear σ model with topological term in 2D

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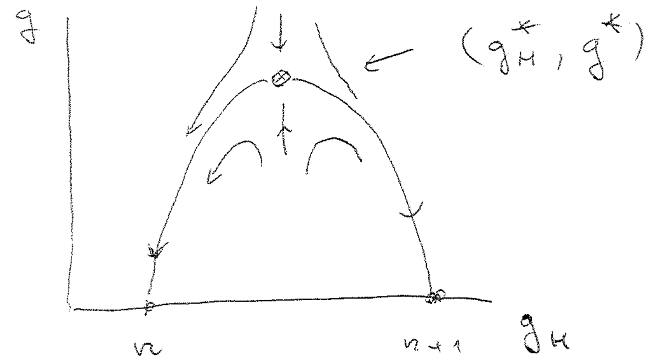

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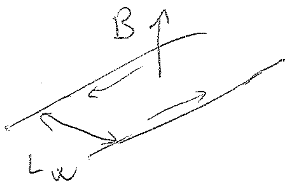

$$g_{\alpha\beta}(B) \simeq F_{\alpha\beta}\left(\left(\frac{L_W}{L_0}\right)^{y_\sigma}\theta_0,\ \left(\frac{L_W}{L_0}\right)^{-|y_\sigma|}\sigma_0\right)$$

$\theta_0(B) \sim B - B_c$ $\qquad$ $\cancel{\ }$

$$\Rightarrow \boxed{g_H(B) = F_H\left(\frac{L_W}{\xi(B)}\right)} \qquad \boxed{\xi(B) \sim |B - B_c|^{-\frac{1}{y_\sigma}}}$$

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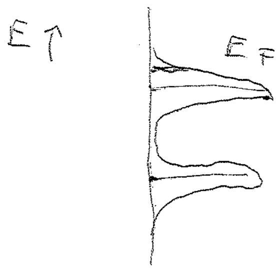
simple picture



B moves Landau level w.r.t. E_F

localization length $\sim |E_F - E_c|^{-\nu} \sim$

$E_c(B)$

$\sim |B - B_c|^{-\nu}$

$E_c(B_c) = E_F$

Backscattering possible if

$L_W < \xi(B)$

quantization spoiled