Localization: theory and experiment

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Abstract

The transport properties of disordered solids have been the subject of much work since at least the 1950s, but with a new burst of activity during the 1980s which has survived up to the present day. There have been numerous reviews of a more or less specialized nature. The present review aims to fill the niche for a non-specialized review of this very active area of research.

The basic concepts behind the theory are introduced with more detailed sections covering experimental results, one-dimensional localization, scaling theory, weak localization, magnetic field effects and fluctuations.

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Contents

1. Introduction

The theory of the electronic properties of disordered solids has matured considerably during the past decade. On the one hand, this is due to the availability of new theoretical methods, and of bigger and faster computers. On the other hand, new experimental techniques which emerged from the fabrication techniques of modern microelectronics made explicit tests of the theoretical results possible, thus providing the possibility of verifying or falsifying the underlying microscopic models. The problem of localization of the quantum states in a random medium, the subject of this review, is one of the examples which may be considered as being representative of this development.

In addition to a number of textbooks on disordered systems which may be used for introductory reading (Mott and Davis 1979, Zallen 1983, Ziman 1979, Bonch-Bruevich *et al* 1984, Shklovskii and Efros 1984, Lifshitz *et al* 1988, Cusack 1987) there is a (still increasing) number of review articles on various aspects (Thouless 1974, Elliott *et al* 1974, Kramer and Weaire 1979). The problem of localization in one-dimensional random systems has been treated in reviews by Ishii (1973), Abrikosov and Ryzhkin (1978), Erdös and Herndon (1982), and Gogolin (1982). The localization problem was considered in the review by Lee and Ramakrishnan (1985) with special emphasis on the metallic limit. Efetov (1983) treated elaborately the field theoretical aspects using the supersymmetric formulation. Experiments in the regime of *weak localization* have been reviewed by Bergmann (1984). Chakravarty and Schmid (1986) treated the limit of weak localization from the wave mechanical point of view. Vollhardt and Wölfle (1992) described the perturbational aspects. There are also a number of introductory summaries (Vollhardt 1987, MacKinnon 1988, Altshuler and Lee 1988, Kramer 1988, Bonch-Bruevich 1983, Toulouse and Balian 1979).

Conference volumes devoted to the topic of localization are also available (Friedman and Tunstall 1978, Nagaoka and Fukuyama 1982, Nagaoka 1985, Kramer *et al* 1985, Kramer and Schweitzer 1984, Weller and Ziesche 1984, Finlayson 1986, Garrido 1985, Ando and Fukuyama 1987, Kramer and Schön 1990, Benedict and Chalker 1991, Kramer 1991). One might ask, whether or not, and why, another review article on localization is necessary. The answer to this question lies in the rather specialized nature of most of the above mentioned works. One might argue that the *most recent* general review article on the problem of the quantum states in disordered media which covers most of the aspects in a more or less equally balanced way is that by Thouless (1974). The above mentioned theoretical, computational and experimental work, however, seems to make a general review article necessary again, especially since the most recent developments indicate that, in spite of all the progress made, the Anderson transition is not yet understood and the one-parameter scaling approach possibly incomplete.

The phenomenon of localization, being, in the first instance, a property of the states in random quantum mechanical systems, has its most striking experimental manifestation in the transport properties of condensed matter systems.

The common belief is that the one-particle wavefunctions in macroscopic, disordered quantum systems at the absolute zero of temperature can be exponentially localized. More precisely, this means that, on the average, their amplitudes are exponentially decaying in space at infinity. This is expected if the *disorder* is sufficiently strong or in energy regions where the density of states is sufficiently small. Energy regions with small densities of states are typically associated with the tails of quantum mechanically allowed energy bands. For weak disorder or in energy regions with sufficient density of states, the wavefunctions will extend throughout the whole system with their phases and amplitudes varying randomly in space. Physically the disorder can be imagined to be connected with the presence

of impurities, vacancies and dislocations in an otherwise ideal crystal lattice. Another possibility is to distribute atoms or molecules at more or less random positions. Strong disorder can then be achieved by using a large concentration of impurities, for instance, independently of the strength of the individual impurities. A completely disordered assembly of atoms will be one in which the atoms are sitting on sites that are chosen completely independently and randomly.

Particles that occupy exponentially localized states are restricted to finite regions of space. They cannot contribute to transport at the absolute zero of temperature, $T = 0$ K, when the coupling to other degrees of freedom, such as phonons, and particle–particle interactions has become negligible. On the other hand, particles in extended states can escape to infinity and contribute to transport. As a consequence, if there are only localized states near the Fermi energy the system will be an insulator, in the sense that, at $T = 0$ K, the DC conductivity, σ_{dc} (the zero-frequency limit of the linear conductivity) vanishes. On the other hand, when the Fermi level lies in a region of extended wavefunctions σ_{dc} ($T = 0$ K) will be finite and the system will be metallic. Localization of the quantum mechanical wavefunctions as a consequence of the presence of disorder is one of the fundamental ingredients for the understanding of the existence of insulators and metals, and, in particular, the transition between the insulating and the metallic states of matter. The latter is one of the main issues to be discussed in this review.

There are also profound effects of the presence of the disorder deep in the metallic regime, usually associated with very weak randomness. Here, coherent quantum mechanical backscattering, which may be viewed as the precursor of the exponential localization, gives rise to a rich variety of quantum transport phenomena, such as, for instance, the logarithmic increase of the resistance of thin metallic films with decreasing temperature when approaching absolute zero (Gorkov *et al* 1979). Coherent backscattering can be understood quite generally in terms of the interference of different, quantum mechanically allowed, paths of the particles that contribute to the transport process, and can be treated quantitatively by diagrammatic perturbational techniques. These so-called *weak localization* effects have been one of the important discoveries of the past decade. They not only initiated a whole new area of research concerning the phase-breaking mechanisms in metals, such as electron–phonon, spin–orbit and electron–electron scattering, but were also of crucial importance for the development of one of the most fruitful, though recently also most heavily attacked, approaches to the disorder-induced metal–insulator transition, namely the one-parameter scaling theory (Abrahams *et al* 1979).

In the insulating regime, when the temperature is above absolute zero, transport is possible via thermal activation of charge carriers from localized into extended states. In contrast to the metallic regime, the conductivity in this case depends exponentially on the inverse temperature. At temperatures close to absolute zero the activation processes die out. Transport is then only possible via hopping of the charge carriers between localized states associated with the absorption or emission of phonon-like excitations (i.e. phonons, magnons, plasmons, etc). Hopping conduction is characterized by a most peculiar temperature behaviour of the transport coefficients, which is related to the statistics of the phonon-like excitations, and the energetic and spatial distribution of the localized states. Mott's '$T^{-1/4}$ law' (Mott and Davis 1979), in the case of the DC conductivity of amorphous semiconductors, is only one, though very characteristic, example, which provides a most convincing case for the existence of localized states in disordered solids.

In addition to changing the temperature, the application of a magnetic field constitutes one of the most powerful probes, especially for the experimental investigation of localization effects. There are basically three discoveries of the past two decades which have initiated

considerable enhancement of the experimental efforts. First, application of a magnetic field induces characteristic changes in the phase of a wavefunction, thus producing pronounced changes in the interference of the quantum mechanical transport paths. These can be detected and systematically analysed to investigate the quantum mechanical coherence properties of the system. This was most strikingly demonstrated a decade ago when the experimental discovery of the Aharonov–Bohm-like oscillations of the magnetoresistance of thin normally metallic cylinders provided direct evidence for the existence of quantum interference in the presence of disorder (Sharvin and Sharvin 1981. In the second important discovery, the Hall conductivity of thin layers of charge carriers when subject to a strong magnetic field proved to be quantized in units of e^2/h at low temperatures (von Klitzing *et al* 1980). Although the theoretical understanding of this effect is still far from complete there is now good evidence that it can be used to investigate quantitatively the localization properties of quasi-two-dimensional electron systems subject to a strong magnetic field. The third discovery came when the metal–insulator transition mentioned above was associated with a second-order phase transition in the course of the development of the one-parameter scaling theory. As a consequence, the critical behaviour of the DC conductivity, i.e. how it approaches zero when approaching the critical point by changing the disorder and/or the position of the Fermi energy, should be the same within certain *universality classes* of Hamiltonians. The universality properties of the system are dictated by its fundamental symmetries; for instance, whether or not there is time-reversal invariance. A magnetic field destroys time-reversal invariance. Thus, it constitutes a most simple tool for changing the universality class of a system experimentally, and, as a consequence, the critical behaviour at the metal–insulator transition. Experimental and theoretical investigation of the effects of a magnetic field on the transport properties, especially in the critical regime close to the metal–insulator transition, can therefore be expected to be of crucial importance in examining the validity of the one-parameter scaling theory.

As a result of the statistical nature of the disorder—impurities are distributed more or less at random, atoms in a glassy structure sit at random positions—the quantum states are random objects. Their amplitudes and phase vary randomly in space. As physical quantities are represented by quantum mechanical expectation values of the operators representing the observables, they are also randomly distributed.

Conventional statistical physics associates experimentally meaningful quantities with averages over a statistical ensemble of macroscopically different systems. For example, for a given macroscopic concentration of impurities there is an essentially uncountable number of possible arrangements of their positions in the host crystal. Different members of the statistical ensemble characterized by the macroscopic concentration have different spatial configurations of the impurity atoms. Although the results of measurements of a physical quantity, when performed on specific members of the statistical ensemble, will be different and dependent on the specific configuration of the impurities, or, more generally, on the specific realization of the disorder, the statistical fluctuations of the results will become vanishingly small when compared with, say, the ensemble average, provided the system is sufficiently large, i.e. macroscopic.

Physical quantities are usually assumed to fulfil this criterion, namely that they effectively do not fluctuate within the statistical ensemble of macroscopically equivalent systems in the thermodynamic limit. They are often called *self-averaging*. Formally, the ensemble average of a self-averaging quantity and the most probable value within the ensemble practically coincide when the size of the system is assumed to be infinite. Self-averaging implies that, in practice, measurements done on specific samples—specific realizations of the disorder—can be described in terms of ensemble averages.

A most striking consequence of quantum mechanical localization is that the transport properties of the disordered systems are *not* self-averaging at very low temperatures: there are experimentally accessible, and reproducible, stochastic fluctuations of the conductance (Fowler *et al* 1982), even in the asymptotic regime of weak localization, when an external system parameter, such as the strength of an applied magnetic field, is varied. The fluctuations are much larger than expected from the application of simple considerations borrowed from classical statistical physics, like *random walk*. In the insulating regime they even diverge in the thermodynamic limit.

Besides providing insight into the quantum mechanical nature of macroscopic condensed matter systems at very low temperatures, such as in the limit of weak localization, the study of localization by theory and experiment offers, at least in principle, the unique possibility of investigating systematically the statistical properties of non-self-averaging quantities, and the related more general concepts like the ergodic hypothesis.

In the following sections we will introduce the field of localization in more detail. We describe briefly the historical development in section 2 and review the key experimental results in section 3. The basic theoretical models and methods used in the description of random systems are described in section 4, and the variety of definitions of localization used in the past are summarized in section 5. The fundamental features of the strongly localized regime are discussed in section 6 which contains a summary of work done for one-dimensional disordered electronic systems where many aspects can be treated mathematically exactly. The physics of the weakly disordered regime is touched on in section 7, and the scaling approach to the disorder-induced metal–insulator transition is described to some extent in section 8. Section 9 contains as a specific example the treatment of localization in the presence of a magnetic field with special emphasis on two-dimensional disordered systems, which are of outstanding practical importance in connection with the quantized Hall effect. The problem of the reproducible stochastic fluctuations of the conductance and the resistance is described in section 10.

Although, as we shall see, a complete understanding of the experimentally observed metal–insulator transition, especially in highly doped semiconductors, is not possible without taking interaction effects, as well as disorder, into account, these will not be considered in this review†. We refer the reader to the volume edited by Efros and Pollak (1985) which contains a number of carefully written reviews on the interplay between Anderson localization and interactions.

2. The milestones

In this section we shall provide a brief review of the history of the field of localization, together with the references that we think are relevant, although we cannot claim to achieve completeness.

2.1. The pre-scaling era

Perhaps the first paper in which the problem of localization was discussed in connection with quantum mechanical diffusion is that of P W Anderson (1958) (figure 1). He formulated the problem and gave a first quantitative estimate of the strength of the random potential which is necessary for *the absence of diffusion in certain random lattices*. The relevance

† In fact, there are examples of MITs that are induced by interactions instead of disorder, the Peierls transition which is due to electron–lattice coupling, and the Mott–Hubbard transition induced by electron–electron interaction.
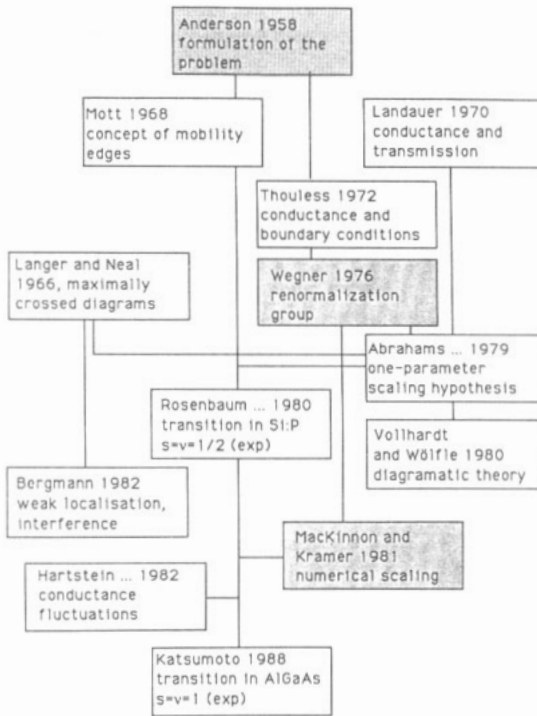
Figure 1. The history of the field of localization.
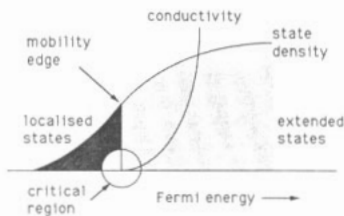
Figure 2. The concept of the mobility edge. Electronic states below and above the mobility edge are localized and extended, respectively. If the Fermi energy lies in the region of the localized states, the system is insulating at $T = 0$. In the extended-states region it is metallic.

of localization with regard to the transport properties of amorphous semiconductors was discussed by Mott (1968). He proposed the concept of the mobility edge which separates the localized states from the extended states energetically (figure 2).

Here the conductivity drops to zero for $T = 0$ and $\omega = 0$ such that the mobility edge represents the critical energy for the transition from a metallic to an insulating state (metal–insulator transition, MIT). In the 1970s, Thouless and many others tried to clarify the quantitative aspect of the problem (Thouless 1974). It was in particular the idea of a connection between the DC conductivity and the sensitivity of the eigenvalues of the Hamiltonian of a finite (but very large) system to changes of the boundary conditions

(Edwards and Thouless 1972, Licciardello and Thouless 1975a, b) which formed the basis for the later reformulation of the problem in terms of the renormalization group and the corresponding scaling theory by Wegner (1976) that eventually culminated in the formulation in terms of the non-linear σ model (Wegner 1979a, b, Hikami 1981, Efetov 1983). Thus, the link between the disorder-induced MIT and second-order phase transitions was established.

2.2. *The scaling theory*

The essential hypothesis of the scaling theory is that close to the transition between localized and extended states there is only one relevant scaling variable which is sufficient to describe the critical behaviour of the DC conductivity (on the metallic side) and the localization length (on the insulating side). Physically, this is equivalent to the statement that close to the transition it does not make any sense to distinguish between the various mechanisms for localization discussed below.

In 1970 Landauer pointed out that since the DC conductivity is vanishing in the localized regime, and at $T = 0$, it is no longer a useful quantity for the description of transport through *finite* systems. Instead, the conductance must be considered. He proposed an alternative description of the conductance of 1D disordered systems in terms of their transmission properties. The Landauer relation (Landauer 1970) gives explicitly the scaling properties of the conductance as a function of the length of the system. There have been several generalizations to quasi-1D systems (with many transmission channels) (Anderson *et al* 1980, Langreth and Abrahams 1981, Fisher and Lee 1981, Economou and Soukoulis 1981a, b, Thouless 1981, Anderson 1981, Büttiker *et al* 1985). The Landauer approach can also be considered as one of the precursors of the one-parameter scaling theory, especially with respect to the numerical implementations of the latter.

Based on Wegner's work in addition to the ideas of Thouless and Landauer, it was possible to formulate the one-parameter scaling theory of localization (Abrahams *et al* 1979, Gorkov *et al* 1979), in which the conductance itself was taken as the scaling variable. In this work, although intuitively, and in an *ad hoc* manner, for the first time an explicit description of the dependence of the conductance on the size of a disordered system was given, and an elementary description of role of the dimensionality was provided.

In order to describe the conductance of a hypercube of the volume L^d, $g(L)$, its logarithmic derivative β was introduced:

$$\beta = \frac{\mathrm{d}\ln g}{\mathrm{d}\ln L}. \tag{1}$$

It was assumed that it depends only on the conductance itself, and not on energy, disorder, or L separately. The qualitative behaviour of $\beta(g)$ was obtained by interpolating from the asymptotic behaviour at large and small conductance assuming that β is a continuous, monotonically increasing function (figure 3).

This behaviour was subsequently corroborated by a quantitative extrapolation from the weak disorder limit (Vollhardt and Wölfle 1982) using standard diagrammatic perturbation techniques for the two-particle propagator. The role of interactions was studied by Hikami *et al* (1980), Finkelstein (1983a, b, 1984a, b), Fukuyama (1985), Altshuler and Aronov (1979a, b, 1985), Castellani *et al* (1984), and Raimondi *et al* (1990).

If $\beta > 0$ the conductance increases with the size of the sample, reflecting metallic behaviour. The metallic region may be characterized by the classical behaviour, namely $\beta(g) = d - 2$ which may be obtained from the classical relation between the conductance and the conductivity. On the other hand, if $\beta < 0$, $g(L)$ decreases with L, eventually terminating
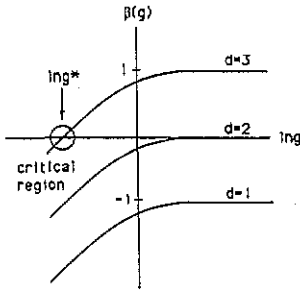
Figure 3. The β function for the zero-temperature conductance of a disordered system for dimensionality $d = 1, \ldots, 3$. $g(L)$ increases with increasing L if $\beta > 0$, but decreases for $\beta < 0$. $\beta = 0$ defines the critical point corresponding to the Anderson transition. It is only achievable in $d = 3$.

in the localized regime where $\beta(g) = \ln g$. A fixed point is defined by $\beta(g_c) = 0$. It corresponds to a disorder-induced MIT. One of the essential results of the one-parameter scaling theory is that such a MIT can only exist in three dimensions since this is the only dimensionality where β can have positive and negative values. In one and two dimensions $g(L)$ is always decreasing with L. Thus, the insulating regime is always eventually reached in the thermodynamic limit at the absolute zero of the temperature, for non-interacting electrons, and without magnetic scattering effects.

During the time when the scaling theory of localization was developed, experimental techniques became available which made explicit tests of the theory possible. In the metallic regime the asymptotic (perturbation) theory for weak disorder (weak elastic scattering) predicted a disorder-induced logarithmic correction to the temperature dependence of the conductivity at very low temperature for 2D systems. The corresponding quantitative theory was formulated by Hikami *et al* (1980) and Altshuler *et al* (1980), and verified experimentally in a series of beautiful experiments done on very thin Mg films (Bergmann 1982a, b, c, 1984).

2.3. The critical behaviour

A further important result of the one-parameter scaling theory concerns the critical behaviour at the MIT (Anderson transition). If, in analogy with second-order phase transition theory, the DC conductivity σ_{dc} and the localization length λ near the mobility edge are assumed to behave according to

$$\sigma_{dc} \propto (E - E_c)^s$$
$$\lambda \propto (E_c - E)^{-\nu} \tag{2}$$

respectively, then one obtains $s = \nu$ from the scaling relations. The numerical value was evaluated by using the ϵ expansion (Wegner 1985) or diagrammatic techniques (Vollhardt and Wölfle 1982), $s = \nu = 1$.

The critical behaviour of the conductivity near the MIT in doped semiconductors has been carefully investigated experimentally in recent years (Thomas and Paalanen 1985, Thomas 1986, Katsumoto 1991). These experiments yielded a variety of exponents which group around $s = \nu = \frac{1}{2}$ (standard example Si:P), and $s = \nu = 1$ (standard example Al$_x$Ga$_{1-x}$As). Whereas it was generally believed that exponent 1 is due to an Anderson
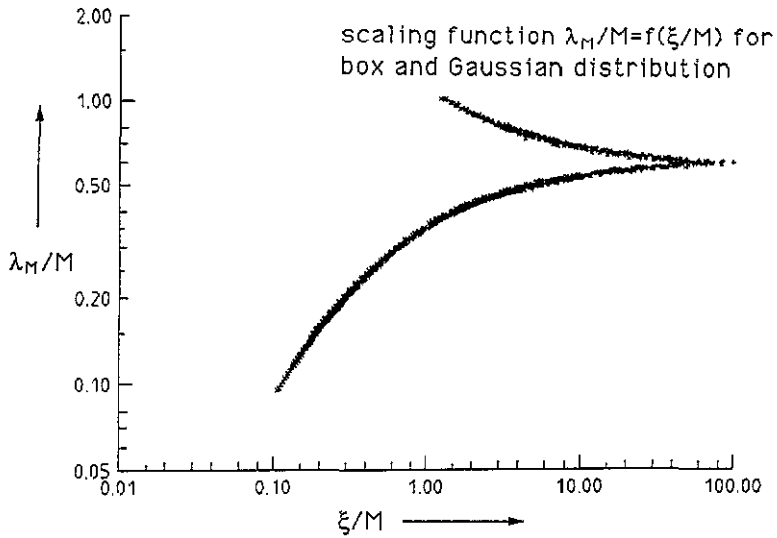
Figure 4. The numerically determined scaling function for the Anderson model in three dimensions with the potential energy distributed randomly according to box and Gaussian distributions. Although the (about 1000) data points for the exponential decay lengths λ_M in a system of the size M are taken for many different sizes and widths of the distribution functions, they can be scaled onto one and the same curve by changing the scale of M with a disorder (and energy) dependent scaling parameter ξ.

transition, i.e. described by the current theory of localization, the exponent $\frac{1}{2}$ was ascribed to the presence of local magnetic moments induced by Coulomb interaction. Although there have been several efforts to treat this effect theoretically (Belitz and Kirkpatrick 1989a, b, 1991, Kirkpatrick and Belitz 1989, 1990a, b, Milovanovic *et al* 1989, Bhatt *et al* 1988) a generally accepted theory for this type of transition is not available at present.

A different approach has been developed by Götze (1981) by starting from the mode-coupling theory, and calculating explicitly experimentally observable quantities like the frequency-dependent conductivity, and the dielectric susceptibility, as a function of the various system parameters like the disorder and the Fermi energy (Götze 1985). In this way a quantitative description of the Anderson transition was claimed.

2.4. Numerical scaling

The one-parameter scaling theory contains a number of assumptions, the most important of these being the one-parameter scaling hypothesis. In order to see whether or not this is really valid an explicit numerical test has been performed by the authors of the present review (MacKinnon and Kramer 1981, 1983a, MacKinnon 1985b) using a recursive technique which was developed in connection with the calculation of the conductivity of disordered systems by one of the authors (MacKinnon 1980, 1985a). A similar technique was developed simultaneously by Pichard and Sarma (1981a, b). Using a real space renormalization technique which was connected with the ideas of Landauer and Wegner, it was possible to establish numerically the existence of a scaling function for the exponential decay length of the transmission probability through the random medium (figure 4). However, the obtained critical exponents were only partially in agreement with the results of the other theories and the above mentioned experimental data. Also, the validity of the scaling concept was demonstrated only for the centre of the band (of the Anderson model).

The numerical values for the localization length near the critical disorder in three dimensions and for small disorder in two dimensions which were obtained in these calculations turned out to be macroscopically large. The question arises how the wavefunctions behave before the asymptotic exponential decay sets in. The idea of an inverse power law decrease as a function of the distance from some localization centre was introduced by Kaveh and Mott (1981). Some numerical results were found which seemed to be in favour of such a behaviour (Pichard and Sarma 1981a, b, Schreiber 1985). Power law localization has also been found recently in 1D disordered systems subject to an electric field (Delyon *et al* 1984, Cota *et al* 1985, Leo and Movaghar 1988). However, such a behaviour would be in severe disagreement with the one-parameter scaling theory. Presumably, further theoretical and numerical studies are necessary in order to clarify the issue.

3. Experiments on localization

A large number of phenomena exist which can be explained in terms of localization of quantum states. A *classical* example is Mott's celebrated $T^{-1/4}$ law for the low-temperature behaviour of the conductivity of amorphous semiconductors (Mott and Davis 1979). The existence of quantum interference has been demonstrated in thin metallic films (Bergmann 1984). The metal–insulator transition has been investigated in a variety of systems including doped semiconductors and amorphous metal–non-metal mixtures (Thomas 1986, Katsumoto 1990), and magnetic-field-induced transitions (Biskupski and Briggs 1988, Chen *et al* 1989). Conductance fluctuations as a result of disorder have been observed in small systems at low temperature (Washburn and Webb 1986, Lee *et al* 1987). The existence of the quantized Hall resistance in inversion layers was interpreted as a direct manifestation of the existence of localized states in the presence of a magnetic field (Aoki and Ando 1981). Classical waves, such as electromagnetic waves, and water waves, for instance, exhibit many of the localization phenomena predicted and investigated in quantum systems quite directly on a macroscopic scale (Etemad *et al* 1986, Akkermans and Maynard 1985, Lindehof *et al* 1986). In this section we want to discuss the essential features of some of these phenomena in greater detail.

3.1. Hopping transport in amorphous semiconductors

Figure 5 shows the DC conductivity of amorphous silicon as a function of the temperature. It can be seen directly from the plot that at low enough temperatures (Beyer 1974)

$$\sigma_{dc}(T) = \sigma_0 \exp(-(T_0/T)^{1/4}) \tag{3}$$

over two orders of magnitude, with suitable constants σ_0 and T_0. At higher temperatures this behaviour changes into an activated one.

The behaviour of the conductivity can be understood when assuming that the transport is mediated by *phonon-assisted hopping processes between localized* states which are energetically close to each other but with localization lengths that are small compared with the spatial distance between the centres of localization (figure 6).

At low temperatures the hopping probability p between the states will be proportional to the overlap integral of the two wavefunctions which depends exponentially on their spatial distance R, and a Boltzmann factor containing their mean energetical distance Δ,

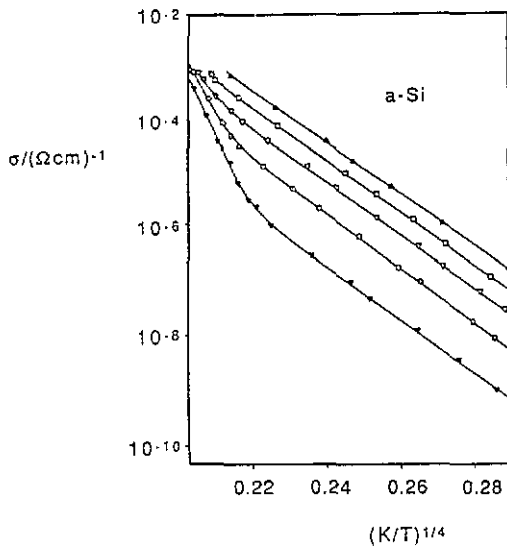$$p \propto \exp(-\alpha R - \beta \Delta). \tag{4}$$

Figure 5. Hopping transport in amorphous silicon. The logarithm of the DC conductivity depends on the temperature according to Mott's $T^{-1/4}$ law. The different curves are for different deposition temperatures (after Beyer (1974)).
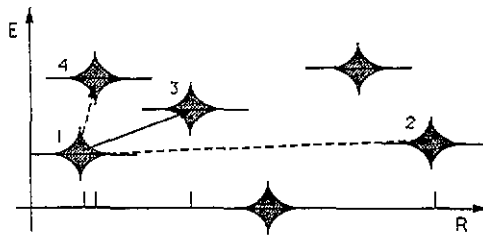
Figure 6. Hopping between states localized at different sites R. States that are close in normal space (1 and 4) must be well separated in energy E. On the other hand, states at almost the same energies are far apart in R-space (1 and 2).

Here α is proportional to the inverse of the exponential decay length of the states, and β is the inverse temperature. For small hopping distances R, necessary in order to obtain a large hopping probability, the number of states which are available for a hopping process from a given state will be small on the average. Correspondingly, the mean energy separation of the states will be large. The Boltzmann factor will decrease the total hopping probability. On the other hand, when R is large, there are many states to which an electron can hop from a given site. Δ is small. The Boltzmann factor will enhance the total hopping probability. In order to maximize p one has to know how Δ depends on R.

Assuming that the localization centres are distributed homogeneously in space we estimate (for a d-dimensional system)

$$\Delta \propto [R^d n(E_F)]^{-1}. \tag{5}$$

The distance R_{max} which maximizes p is then obtained by minimizing the exponent in (4),

$$R_{max} = \left(\frac{d\beta}{dn(E_F)} \right)^{1/(d+1)}. \tag{6}$$

Inserting this result into (4), and noting that $p \propto \sigma$, yields the d-dimensional version of the $T^{-1/4}$ law, which should be valid as long as the density of states is approximately constant.

More rigorous treatments of this remarkable idea to explain the low-temperature transport properties of many amorphous materials which is, as many others, due to Mott, can be found in the literature (Böttger and Bryskin 1985, Efros and Shklovskii 1975, Gogolin 1982).

3.2. Weak localization

At low temperatures thin, weakly disordered metal films exhibit anomalies in the behaviour of the electrical resistance which can only be understood when taking into account the quantum mechanical nature of the electrons. These anomalies are found in the temperature dependence as well as in the magnetic field dependence of the resistance. In terms of the above mentioned scaling picture, they concern the regime of *almost classical* transport (large conductance, see figure 7) in a 2D system. Since the quantum corrections to the conductivity can be interpreted as being due to an interference of the electron wavefunctions which favours backscattering, and since the localization length considerably exceeds all of the other relevant lengths in the system, this is called the regime of *weak localization.*

One of the most striking effects is seen in the temperature dependence. Many thin metallic films show a logarithmic increase of the resistance when the temperature is decreased. Some of the first results obtained are shown in figure 7. These and many other examples can be found in the review article by Bergmann (1984).

Results of this kind were first taken as confirming the scaling theory of localization in the asymptotic regime of weak localization where the β function appeared to be given by

$$\beta(g) = -\frac{\text{constant}}{g} \tag{7}$$

in two dimensions. Integrating with respect to the system size L one obtains

$$g - g_0 = -\text{constant} \times \ln(L/L_0) \tag{8}$$

where g_0 and L_0 are constants of integration. For $L < L_0$, g ($> g_0$) can be treated classically. For $L > L_0$ the conductance decreases logarithmically with the length L. In order to obtain the temperature dependence the geometrical system size has to be replaced by an effective system size L_i, the mean distance between successive inelastic scattering events suffered by the particle (Anderson *et al* 1979, Thouless 1980, Altshuler and Aronov 1985). With the (temperature-dependent) inelastic scattering time τ_i (phase coherence time) we have

$$L_i^2 = D\tau_i(T). \tag{9}$$

Taking $\tau_i \propto T^{-p}$ ($p = \mathrm{O}(1)$) at low temperatures one obtains the desired $\log T$ behaviour.

However, some time later doubts were shed on this interpretation since, based on earlier work on the density of states (Altshuler and Aronov 1979a, b, Altshuler *et al* 1980), it was discovered that the Coulomb interaction between the electrons leads to the same temperature dependence in the presence of disorder, except that the prefactor is different (Fukuyama 1980, 1981). Thus, a tool was needed that could distinguish between localization and interaction effects.

The latter was provided by applying an external magnetic field (Uren *et al* 1980). While localization leads to a negative magnetoresistance usually associated with an increase
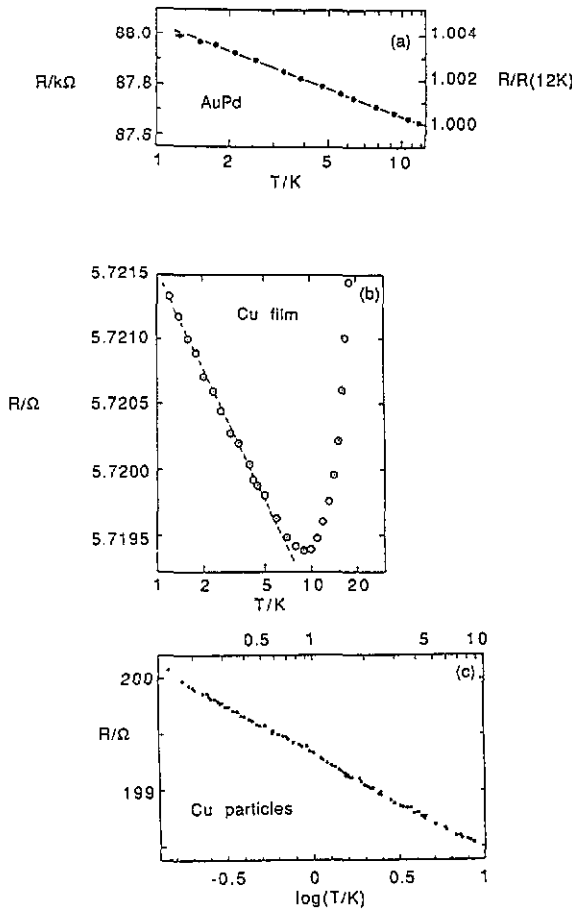
Figure 7. (*a*) Logarithmic temperature dependence of the resistance of thin Au–Pd films (Dolan and Osheroff 1979), (*b*) thin Cu films (van den Dries *et al* 1981) and (*c*) fine Cu particles (Kobayashi *et al* 1980).

of the localization length when a magnetic field is applied, Coulomb interaction gives a positive magnetoresistance. The interesting point that was made by Bergmann (1983a) is that localization effects may be *switched off* systematically by introducing spin–orbit scattering into the system (figure 8). Using the asymptotic theory of Hikami *et al* (1980) the experimental results may be used to determine the inelastic scattering time, the spin–orbit scattering time, and the magnetic scattering time (Bergmann 1982a, b, c, 1983a, b) (see also section 7).

The fact that the regime of weak localization in the transport properties of disordered metals can be described within the picture of the interference between quantum mechanical probability waves has been demonstrated explicitly by Sharvin and Sharvin (1981). When measuring the magnetoresistance of a Mg cylinder which was about 1.5 μm in diameter and 1 cm long, they observed oscillations on varying the magnitude of an externally applied axial magnetic field. They were periodic with a period $\Delta\Phi = h/2e$, half the normal Aharonov–Bohm period (figure 9). The results were in quantitative agreement with the theoretical predictions made a few months earlier by Altshuler *et al* (1981) who applied
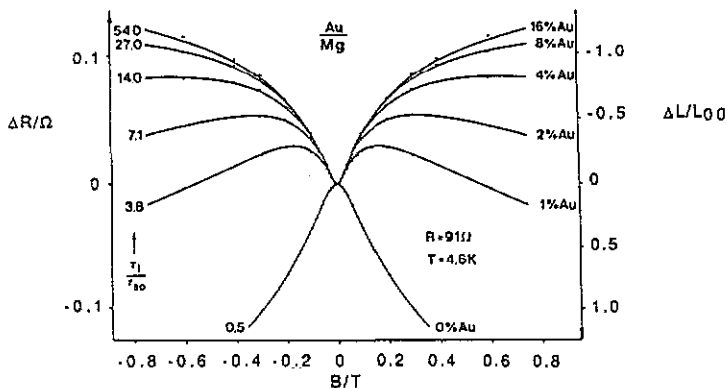
Figure 8. The magnetoresistance ΔR of thin Mg films. The clean film shows a negative magnetoresistance indicating localization. When the film is covered with a small amount of gold atoms the magnetoresistance becomes positive due to increasing spin–orbit scattering. The right scale shows the magnetoconductance ΔL. On the left, the ratio of the inelastic scattering time and the spin–orbit scattering time is indicated (after Bergmann (1984)).

the same technique which was used for the treatment of *weak localization* (see below) to describe the low-temperature magneto-transport properties of loops and cylinders.
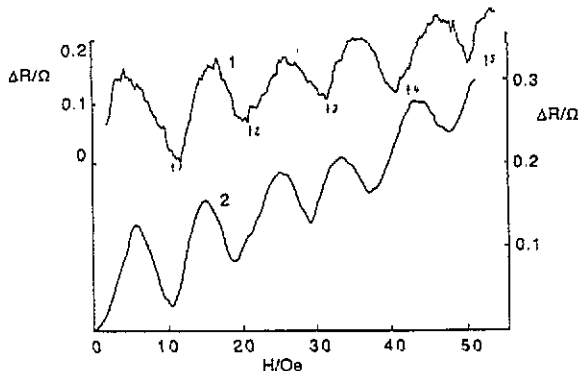
Figure 9. Aharonov–Bohm-like magnetoconductance oscillations observed in normally conducting Mg cylinders of diameter 1.5 μm by Sharvin and Sharvin (1981). Left and right resistance scales correspond to samples 1 and 2, respectively. The periodicity of the oscillations corresponds to $\Delta \Phi = h/2e$.

Magnetoresistance oscillations were observed, and successfully treated theoretically, by several other groups in cylindrical systems as well as in large 2D arrays of loops (Gijs *et al* 1984a, b, Pannetier *et al* 1984, Washburn and Webb 1986, Aronov and Sharvin 1987). Thus, the asymptotic regime of weak scattering is physically understood and experimentally well confirmed.

3.3. The Anderson transition

There is a large body of experimental work concerning the metal–insulator transition (Thomas 1986). The most extensively studied examples are doped silicon, Si:P

(figures 10(*a*) and (*b*) (Paalanen and Thomas 1983)), as well as the persistent photoconductor $Al_xGa_{1-x}As$ (figure 10(*c*), (Katsumoto *et al* 1987)). In both cases both the insulating and the metallic sides of the transition were studied by measuring the dielectric susceptibility and the conductivity, respectively. In Si:P the tuning parameter was the magnitude of uniaxial stress applied to the sample (at fixed doping level), or the concentration of the doping atoms. In the persistent photoconductor the concentration of the charge carriers was tuned via optical excitation.
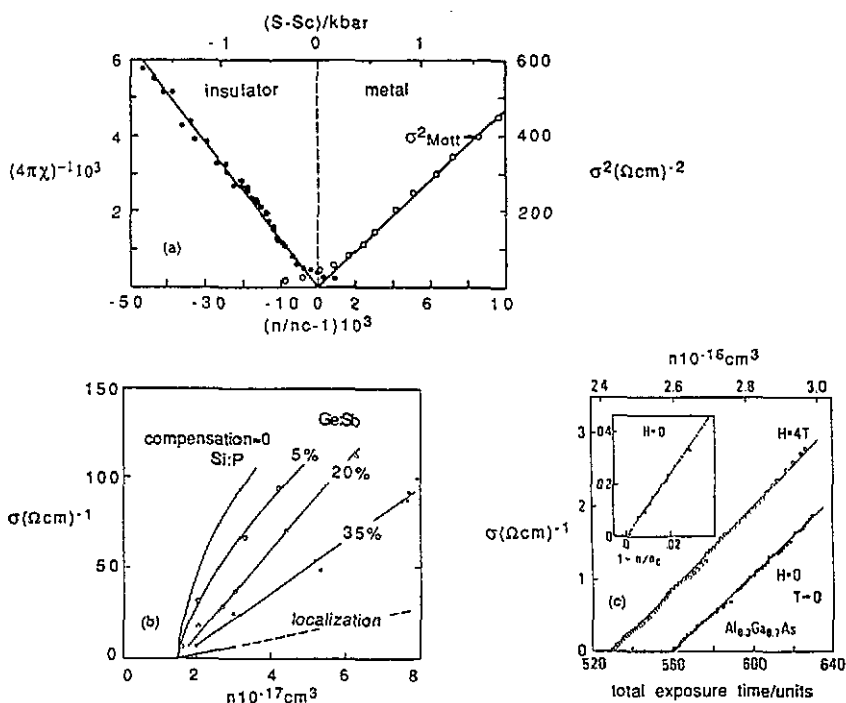
Figure 10. The metal–insulator transition. (*a*) Uncompensated Si:P. The square of the DC conductivity σ (right scale) and the inverse of the dielectric susceptibility (left scale) are plotted against the renormalized concentration $n/n_c - 1$ of the dopant (P) atoms. The upper scale shows the uniaxial stress S applied to the system in order to achieve the transition (after Paalanen and Thomas (1983)). (*b*) Compensation dependence of the transition in Ge:Sb. The DC conductivity is plotted as a function of the concentration of the dopant (Sb) atoms for different degrees of compensation (in %). The uppermost curve is the result obtained for Si:P. The critical exponent changes between about 0.5 and 1 depending on compensation (after Thomas *et al* (1982)). (*c*) Persistent photoconductor $Al_{0.3}Ga_{0.7}As$:Si. The DC conductivity σ extrapolated to zero temperature for zero magnetic field and $H = 4$ T is plotted against the light exposure time. The latter is a measure of the density n of the charge carriers. The exponent is 1 (after Katsumoto *et al* (1987)).

The critical exponents turned out to be equal on both sides. An absolute value close to $\frac{1}{2}$ was obtained for Si:P (and other uncompensated doped materials), whereas exponents of the order of 1 were obtained for $Al_xGa_{1-x}As$, and other amorphous materials, as well as compensated semiconductors.

The three examples shown in figure 10 characterize completely the experimental situation concerning the critical exponent. One can distinguish two large groups of

transitions. On the one hand, there are uncompensated doped semiconductors. Here the transition is characterized by an exponent close to $\frac{1}{2}$. On the other hand, there is the large group of systems including amorphous metals, amorphous semiconductors, and compensated doped semiconductors where the exponent is close to 1. To our knowledge, there is up to now no experimental indication of a disorder-induced transition with different exponents on the two sides, as far as they have been determined.

3.4. Scattering of light

If quantum interference is the dominant mechanism for the localization of states in a random medium, localization effects should be of importance in other wave phenomena, too (Sheng 1990). That this is indeed the case has been demonstrated in recent years theoretically (by considering classical wave equations) (John and Stephen 1983, John 1984, Azbel 1983, Kirkpatrick 1984, Flesia *et al* 1987, Akkermans and Maynard 1985, 1986, Guazzelli *et al* 1983) as well as experimentally in light scattering experiments (Wolf and Maret 1985, Kuga nad Ishimaru 1984, van Albada and Lagendijk 1985, van Albada *et al* 1991). An example is shown in figure 11 (Etemad *et al* 1986).
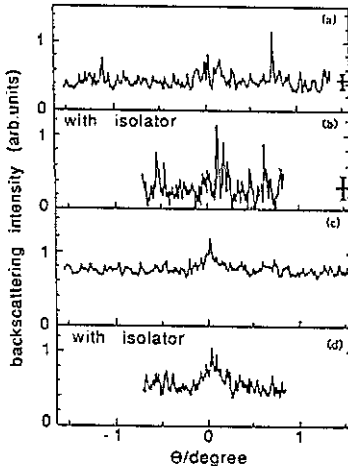
Figure 11. (*a*) Backscattering of light from a solid assembly of submicron SiO$_2$ particles in air, (*b*) with single-scattering component removed, (*c*) ensemble averages of nine and (*d*) sixteen scans (after Etemad *et al* (1986)).

The light of a 5 mW low-divergence He–Ne laser has been scattered from a solid sample made of colloidal silica particles which were produced by flame hydrolysis of silicon tetrachloride vapour, and deposited on various substrates as a solid layer of uniform thickness d ranging from a few microns to a few millimetres. Thus the full range of elastic scattering length $\ell \ll d$ to $\ell > d$ could be covered. The size of the particles was between 0.1 and 0.2 μm, the solid fraction of the as grown samples varied between 0.05 and 0.12. Enhanced intensity in the backscattering direction was only observed after removing the single-scattering contribution and averaging over several noise-like spectra (speckle pattern). In experiments using a liquid sample instead of a solid sample enhanced backscattering is more readily observed because the Brownian motion of the particles carries out the averaging process very effectively (Wolf and Maret 1985).

There are indications that light waves may even be strongly localized if scattering is strong enough (Economou 1990, Sheng 1990).

3.5. Localization of water waves

One of the *elementary* experiments showing the effect of localization of classical waves can be performed in the first year of a physics course (Lindelof *et al* 1986). It is the scattering of water waves at an assembly of scatterers (figure 12).
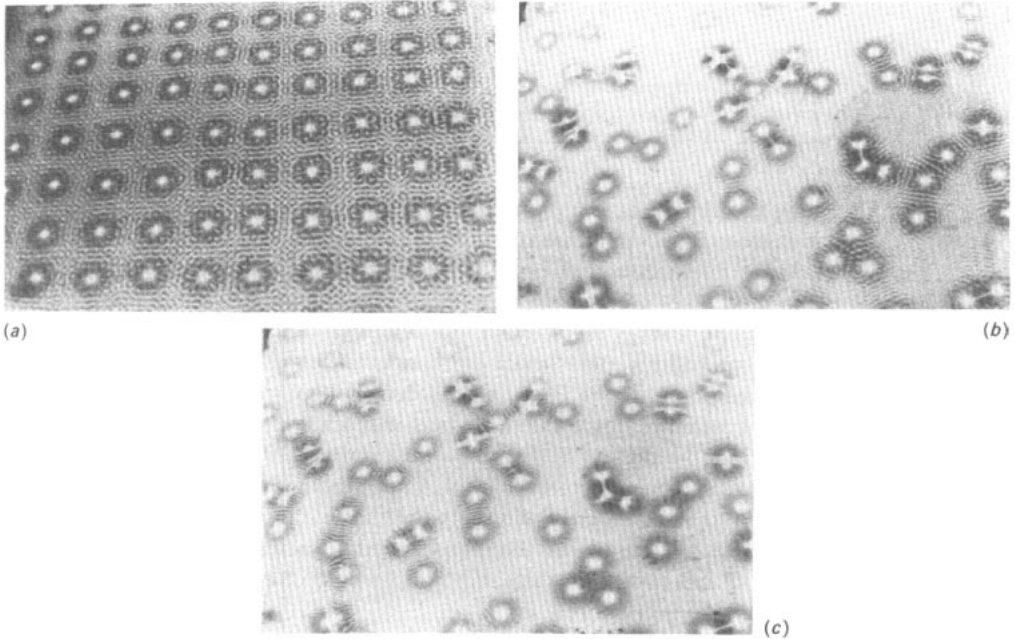
Figure 12. Scattering of water waves from a regular and a random assembly of scatterers (with the permission of the authors from Lindelof *et al* (1986)).

A number of scatterers (for instance, metal nuts) is placed regularly or at random within the area of a bath of water. The bath is filled up to a level of about 5 cm. Surface water waves are excited. In the case of a regular assembly of scatterers the waves are spreading all over the whole area of the surface. When the scatterers are random, waves are generated only within some restricted areas.

3.6. Conductance fluctuations

A phenomenon which is closely related to localization is the reproducible fluctuations of the conductance in *mesoscopic* samples. These have geometric dimensions of the order of or less than the phase coherence length (inelastic scattering length), i.e. a few hundred nm when considering temperatures of the order of 1 K. Measuring the conductance or the resistance of thin wires as a function of the magnetic field in the case of metallic samples, and of quasi-1D inversion layers in high-quality heterostructures, one observes irregular but nevertheless (for a given sample) reproducible structures (Fowler *et al* 1982, 1988, Washburn and Webb 1986, Pichard and Sanquer 1990, Mailly and Sanquer 1991, Caro *et*

Figure 13. The reproducible fluctuations of the conductance Γ as observed at low temperatures in a quasi-1D inversion layer channel when changing the voltage V_g applied to the Al gate on top of a Si-MOSFET, as shown schematically in the left-hand part (after Fowler *et al* (1982)).

al 1991). Changing the gate voltage, i.e. the charge density in quasi-1D inversion layers in MOSFETs, similar random fluctuations are detected at low temperature (figure 13).

These conductance fluctuations can be understood when assuming that at low enough temperature the transport takes place coherently wi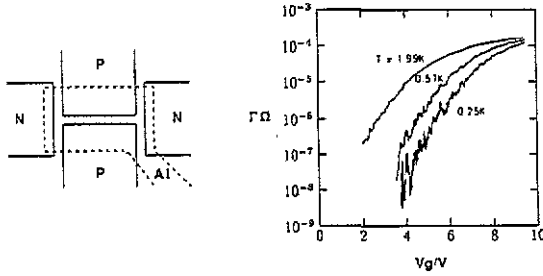thin single quantum states throughout the whole sample. If the Fermi energy is such that it accidentally hits the energy of a quantum state the transmission probability (and hence the conductance) is very high (almost 1) as compared with the situation where the Fermi energy does not coincide with a quantum state energy. In this case the variance of the fluctuations is independent of the size of the sample and other parameters like the degree of disorder (universal conductance fluctuations, UCF, see figure 14). In this regime, which corresponds to that of weak localization since the states are essentially assumed to extend throughout the whole sample, one can use perturbational techniques for the determination of the magnitude of the fluctuations. Random matrix theory is another useful tool for the mathematical treatment of the stochastic behaviour of the transport properties (see section 10).
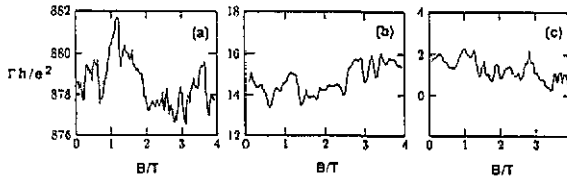
Figure 14. The universal fluctuations of the magnetoconductance Γ of thin metallic wires. (*a*) Mesoscopic Au ring of about 0.8 μm diameter, (*b*) quasi-1D inversion layer channel in a Si-MOSFET and (*c*) numerical result for the Anderson model (see below). Although the three *samples* are microscopically different, as one can judge from the fact that the zero-magnetic field conductances are about an order of magnitude different, the fluctuations are about of the same magnitude (after Lee *et al* (1987)).

When the sample size is larger than the localization length, which is the case in the insulating regime, one expects a second kind of fluctuation, the variance of which increases exponentially with the length of the sample. These are due to the non-ergodic nature of the localized phase. To our knowledge they have not yet been unambiguously observed experimentally. They could, however, be connected with the above mentioned conductance fluctuations in the quasi-1D MOSFET channels (see also Orlov *et al* 1989a, Millikan and Ovadyahu 1990).

Indications of reproducible stochastic fluctuations are also seen in the above mentioned light scattering experiments (figure 11).

4. Basic concepts

4.1. Models of disorder

The characterization of the properties of ideally ordered materials is comparatively easy. Due to the presence of the long-range order one has translational symmetry. The quantum objects (electrons, phonons, magnons, etc) are of the Bloch type. As a consequence, they are freely itinerant, and can move unrestrictedly throughout the whole system. In reality, however, there are no ideally ordered media. There are always distortions of the ideal order due to the presence of impurities, dislocations, vacancies and other defects. As long as the concentration of these is small, one may still use the concepts developed for translationally symmetric systems as a starting point for the understanding of the properties of the distorted systems. However, if the concentration of the distortions is large, it is necessary to leave translational symmetry, and to develop new methods.

Starting from the ideal crystal, models of disorder may be constructed in various ways, as illustrated in figure 15. Models for glassy systems and amorphous semiconductors may be obtained by relaxing the lattice structure (structural disorder). A lattice with two or more different kinds of atoms distributed at random establishes the most simple model of an alloy (compositional disorder).

A simple model for structurally disordered systems, such as amorphous metals, metallic glasses or heavily doped semiconductors, is provided by the Hamiltonian

$$H = \frac{p^2}{2m} + \sum_{j=1}^{N} V_j(r - R_j) \tag{10}$$

where p is the momentum operator, m the effective mass of the particle and V_j the potential energy of an atom at the site R_j. The distribution of the atomic potentials in space may be described by a (normalized) probability density distribution function $P(\{R_j\})$. In the simplest case of a completely random distribution of N statistically independent atoms within a volume Ω this is given by

$$P(\{R_j\}) = \Omega^{-N}. \tag{11}$$

This model has been used extensively in the theory of *weak localization* (Lee and Ramakrishnan 1985, Bergmann 1984, Vollhardt 1987, Vollhardt and Wölfle 1992).

Also commonly used is the following model, which is defined on a lattice and is particularly convenient for the treatment of compositionally disordered solids, such as alloys:

$$H = \sum_{j\nu} \epsilon_{j\nu} |j\nu\rangle\langle j\nu| + \sum_{j\nu,k\mu} V_{j\nu,k\mu} |j\nu\rangle\langle k\mu|. \tag{12}$$

Here $\epsilon_{j\nu}$ are the energies associated with the states labelled by ν at the sites j of the lattice, and the non-diagonal elements $V_{j\nu,k\mu}$ denote the matrix elements between the states. The diagonal part of the Hamiltonian corresponds to the potential energy and the non-diagonal part to the kinetic energy in the continuous space description (10). Disorder is introduced by taking the site energies and/or the hopping matrix elements at random, and assuming
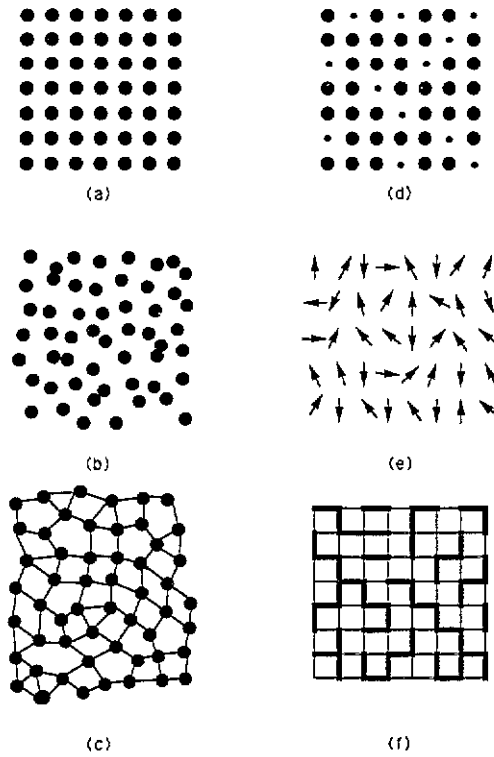
Figure 15. Different sorts of disorder that can be modelled by starting from the ordered limit (a). Identical atoms sitting at random positions represent structural disorder (b), when the position are such that the number of the nearest neighbours is constant the system is called topologically disordered (c). Two different kinds of atoms sitting at the sites of an ideally ordered lattice represent the simplest case of compositional disorder (d), while an assembly of randomly oriented spins is a typical example of an orientationally disordered system (e). A regular lattice of identical atoms connected with two different kinds of hopping matrix elements is the most simple case of a system with randomness in the kinetic energy, for instance induced by a random vector potential (f).

some probability distribution function for them. The simplest case is that of pure diagonal disorder,

$$P(\{\epsilon_{j\nu}\}, \{V_{j\nu,k\mu}\}) = P_\epsilon(\{\epsilon_{j\nu}\}) \prod_{j\nu,k\mu} \begin{cases} \delta(V_{j\nu,k\mu} - V) & j, k \text{ nearest neighbours} \\ \delta(V_{j\nu,k\mu}) & \text{otherwise} \end{cases}$$

(13)

with statistically independent site energies

$$P_\epsilon(\{\epsilon_{j\nu}\}) = \prod_{j\nu} p(\epsilon_{j\nu}).$$

(14)

Examples are the completely random two-component (A, B) alloy,

$$p(\epsilon_{j\nu}) = \tfrac{1}{2}[\delta(\epsilon_{j\nu} - \epsilon_A) + \delta(\epsilon_{j\nu} - \epsilon_B)]$$

(15)

and the Anderson model,

$$p(\epsilon_{j\upsilon}) = \frac{1}{W}\Theta(\tfrac{1}{2}W - |\epsilon_{j\upsilon}|) \tag{16}$$

which may also be considered as an alloy with infinitely many components. A common example of non-diagonal disorder is the *local gauge-invariant* model,

$$P(\{\epsilon_{j\upsilon}\}, \{V_{j\upsilon,k\mu}\}) = \prod_{j\upsilon}\delta(\epsilon_{j\upsilon})$$

$$\times \prod_{j\upsilon,k\mu} \begin{cases} \frac{1}{2}[\delta(V_{j\upsilon,k\mu} - V) + \delta(V_{j\upsilon,k\mu} + V)] & j, k \text{ nearest neighbours} \\ \delta(V_{j\upsilon,k\mu}) & \text{otherwise.} \end{cases} \tag{17}$$

Although primarily of theoretical interest because of its apparent simplicity, the local gauge-invariant model represents a physically rather peculiar and interesting situation. In order to see this, consider the case of nearest-neighbour non-diagonal matrix elements which fluctuate randomly between $+V$ and $-V$,

$$V_{jk} = V \exp(iS_{jk}\pi) \tag{18}$$

where $S_{jk} = 0, 1$ at random. As we shall explain below, the product of the phase factors of the bonds making up any closed loop gives the magnetic flux through that loop. For a single unit cell $\Phi = \alpha\Phi_0 = Ba^2$ (a is the lattice constant, B a fictitious magnetic field and $\Phi_0 = h/e$ the magnetic flux quantum). Thus, depending on whether the sum of the S_{jk} in the closed loop around a unit cell is even or odd, $\Phi = 0$ or $\Phi = 1/2\Phi_0$, respectively. Therefore, although the Hamiltonian is real and symmetric, it can be considered as describing the quantum mechanics of a spinless particle in the presence of a randomly oriented magnetic field, the magnitude of which fluctuates at random from unit cell to unit cell between the values $B = 0$ and $B = h/(2ea^2)$. Apparently the model is the most simple version of a so-called random phase model, where

$$V_{jk} = V \exp(i\phi_{jk}) \tag{19}$$

with $0 \le \phi_{jk} \le 2\pi$ at random. The latter model represents a system with a magnetic field which varies at random from unit cell to unit cell.

Let us now convince ourselves that it is indeed a magnetic flux which is represented by complex phase factors in the off-diagonal elements of the Hamiltonian. We start by asserting (Feynman 1965) that the transition probability amplitude between two sites x and y of a quantum particle of charge e, when moving under the influence of a magnetic field of flux density B, acquires a phase factor

$$V_{xy}(B) = V_{xy}(0) \exp\left(i(e/\hbar) \int_x^y \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}\right) \tag{20}$$

where $\boldsymbol{A}$ is the vector potential, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, and the line integral in the exponent runs over an arbitrary path between x and y. When the path is closed the integral can be written

$$\oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \iint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \Phi \tag{21}$$

where the surface integral represents the magnetic flux through the enclosed area. In the above case of a discrete Hamiltonian, the integral must be replaced by a discrete sum over lattice points. Hence, the relationship of the local gauge-invariant model discussed above in a (fictitious) magnetic field becomes obvious.

The special case of a homogeneous magnetic field is of particular interest since it represents a very simple experimental tool for changing the fundamental symmetry properties of a system, which are very important for the critical behaviour at the Anderson transition, as we shall explain below.

In the continuous model (10) the influence of a magnetic field may be incorporated by replacing p by $p - eA$ where A is the vector potential. For a homogeneous magnetic field B, in the z direction convenient choices are

$$
A = B \begin{cases} (0, -x, 0) & \text{Landau gauge} \\ 1/2(y, -x, 0) & \text{Symmetric gauge.} \end{cases} \tag{22}
$$

The corresponding tight-binding Hamiltonian can easily be deduced from (12) by using the discrete version of (20). In the Landau gauge, for instance, one obtains in three dimensions

$$
V_{lmnvl'm'n'\mu}(B) = V_{lmnvl'm'n'\mu}(0)
$$
$$
\times \{\exp(\pm 2\pi in\alpha)\delta_{l'l}\delta_{m',m\pm 1}\delta_{n'n} + \delta_{l',l\pm 1}\delta_{m'm}\delta_{n'n} + \delta_{l'l}\delta_{m'm}\delta_{n',n\pm 1}\} \tag{23}
$$

where $\{lmn\}$ denote the integer cartesian components of the lattice points of a simple cubic lattice, and $\alpha = Ba^2/\Phi_0$ is the number of magnetic flux quanta in a unit cell. The magnetic field is introduced here via the so-called Peierls substitution (Luttinger 1951).

Although looking rather specialized at first glance, Hamiltonians of the type of (12) have many applications in various areas of the physics of disordered systems. They may be used to describe the vibrational properties (Dean 1972) as well as the electronic properties of amorphous semiconductors (Kramer and Weaire 1979) and alloys (Elliott *et al* 1974) and spin glasses (Edwards and Anderson 1975). The *n-orbital model*, which is nothing but the Hamiltonian (12) with n states per lattice site, was used to establish the scaling theory of localization (Wegner 1979a, b, c, Oppermann and Wegner 1979).

The Anderson model has been extensively used in numerical studies of the localization problem. Most of the results that follow are based on this model Hamiltonian.

The most general single-particle Hamiltonian which may be used for the description of a disordered system is of the form

$$
H = H_{kin} + V \tag{24}
$$

where H_{kin} and V are the kinetic and the potential energies of the particle, respectively. In (10) H_{kin} corresponds to $p^2/2m$ and V is the (random) superposition of the potentials of the atoms.

A convenient choice of the distribution function of V is the generalized Gaussian

$$
P([V]) = C \exp\left(-\int V(r)K(r, r')V(r')\, dr\, dr'\right). \tag{25}
$$

The normalization constant C is determined by the functional integral $\int D[V]P[V] = 1$. The function K is the inverse of the correlation function of the potential (Papoulis 1984),

$$
\int dr'' K(r, r'')\langle V(r'')V(r')\rangle = \delta(r - r'). \tag{26}
$$

A special case is the Gaussian white noise potential specified by

$$\langle V \rangle = 0 \tag{27a}$$

$$\langle V(r)V(r') \rangle = \langle V^2 \rangle \delta(r - r') \tag{27b}$$

such that $K(r, r') = \langle V^2 \rangle \delta(r - r')$. The averages in (26) and (27) are defined as functional integrals,

$$\langle \ldots \rangle = \int_{-\infty}^{+\infty} D[V] P[V] \ldots . \tag{28}$$

This most convenient notation has been used in the derivation of the non-linear σ models (Wegner 1979b, 1985, Hikami 1981, Levine *et al* 1984, Pruisken 1984, Weidenmüller 1987).

4.2. Properties of the electronic spectrum

In the case of an ideal crystal a characteristic feature of the density of states is the occurrence of van Hove singularities, which are due to the long-range order. There are sharp band edges, for instance. A disordered system does not have any long-range order. There are no singularities in the density of states. This can be proven rigorously for the tight–binding model considered above with the diagonal elements being given by a smooth distribution function (Wegner 1979b). In particular, there are no sharp edges, but smooth band tails instead. A number of theoretical approaches deal with their analytic form. The exponential behaviour of the band tails

$$n(E) = A \exp \left(B[\pm(E_0^{\pm} - E)]^{-d/2} \right) \tag{29}$$

can easily be obtained for a random d-dimensional two-component alloy, and for the Anderson model with rectangular distribution of the site energies by using a famous argument that is due to Lifshitz (1965). $E_0^{\pm}$ are the true upper and lower (Lifshitz) bounds of the spectrum in these cases, and A, B are model-specific constants. As the Lifshitz argument is physically extremely instructive we repeat it here in brief.

We consider the above tight-binding model (12) with diagonal disorder and the distribution function (15). Only two values of energy, say $+\frac{1}{2}W$ and $-\frac{1}{2}W$, are allowed for the lattice sites, each of which is assumed to be surrounded by Z nearest neighbours. Then the electron states with energies close to the true band edges that are given by $E_0^{\pm} = \pm(ZV + \frac{1}{2}W)$ will be localized within a potential fluctuation of a sufficiently large volume $\Omega = L^d$. They will be standing waves with the (smallest) wavenumber $\sim \pi/L$. The lowest energy level is then given by (for the lower band edge, for instance)

$$E = E_0^- + \frac{\text{constant}}{2m} L^{-2}. \tag{30}$$

The value of the constant depends on the shape of the volume. Its minimum value is attained for a sphere. The probability for the appearance of such a spherical region is given by c^{Ω} if c is the probability of having $-\frac{1}{2}W$ at a lattice site (at the upper band edge it would be $(1 - c)^{\Omega}$). Expressing now the volume in terms of the energy of the lowest level one obtains the above result for the density of states.

The Lifshitz result has been rederived and extended to other distribution functions by various authors using variational (Zittarz and Langer 1966) and path integral methods
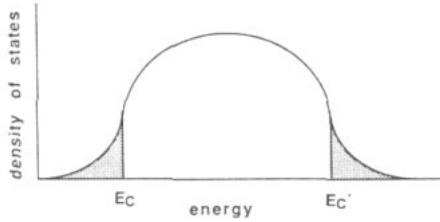
Figure 16. Qualitative picture of the density of states of the Anderson model. The states in the band tails are localized while those in the middle of the band are extended. E_C and E'_C denote the mobility edges.

(Friedberg and Luttinger 1975). For the Gaussian distribution one obtains (Houghton and Schäfer 1979)

$$n(E) = A \exp(-B|E|^{2-d/2}).$$ (31)

The density of states is shown qualitatively in figure 16 for a system with only one band.

Intuitively, and this can also be seen by using an argument due to Lifshitz, it is obvious that the states in the tails of the band are localized within finite regions (of diameter L) of the (infinite) system. However, this does *not* mean that the localization length, which will be later defined as the asymptotic decay length of the states, is given by L.

At energies far away from the band edges one may expect that a weak random potential cannot localize the states to finite regions of space. Instead, the amplitude of the states, although fluctuating more or less randomly, will be non-zero essentially everywhere. Consequently, these states will be called extended. Within the energy region of extended states no localized states can exist. To see this, let us assume that there is a localized state in the extended region for some configuration of the disorder. Then, by infinitesimally changing the disordered potential, coupling between the localized states and the extended states is introduced and the localized state hybridizes with the extended states into new extended states. As a consequence certain energies, denoted by E_c and E'_c in figure 16 must exist which separate the extended from the localized states. As we shall see later, localized states do not contribute to transport, even if they are situated energetically at the Fermi energy, whereas extended states do. Thus, E_c and E'_c are denoted as *mobility edges*. In general, the mobility edges depend on the disorder. If the latter is large enough they will merge into the centre of the band. The system becomes an insulator.

This picture is due to Anderson (1958). The transition from a metallic to an insulating system induced by the disorder is therefore called an Anderson transition. The concept of mobility edges was introduced by Mott (1968). There is no rigorous proof that extended states must exist near the centre of the band. Establishing such a proof is one of the main subjects of the *theory of localization*. Up to now, the only mathematically rigorous statement concerns the existence of extended states in one dimension, namely that for any finite amount of disorder *there are no extended states* in $d = 1$, a statement which was already made by Mott and Twose (1961). In two dimensions the present general belief is that there are also no extended states, even for infinitesimally small disorder. However, as this belief is based only on an approximate theory (the so-called one-parameter scaling theory, to be discussed later), this question must be considered to be still unsolved. The important lesson to be learnt at this stage is that the phenomenon of localization depends on the dimensionality. Below we shall see that this provides a possibility for experimental tests of the theory.

In order to understand the physical mechanisms leading to localization of the quantum states in random potentials better it is instructive to consider first a classical particle moving in a random potential $V(x)$ in one dimension (figure 17). For simplicity, we take the potential to be restricted to values smaller than E_0. Then one can decide by simple energy considerations whether or not the particle is *localized*. If the total energy E is smaller than E_0 the particle is confined to finite intervals, within the (accidental) potential wells. On the other hand, if $E > E_0$ then it can move along the whole x axis.

position
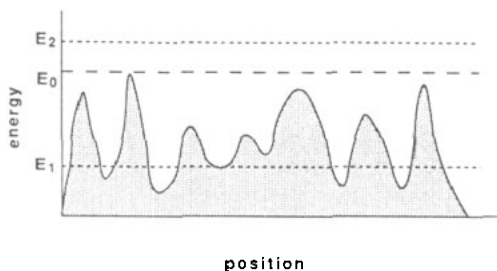
Figure 17. Classical particle in a 1D random potential. The motion of the particle is restricted to finite intervals for $E_1 < E_0$. For $E_2 > E_0$ the motion is unrestricted.

For a quantum mechanical particle it is more complicated to distinguish the character of the states. On the one hand, the potential barriers cannot absolutely confine the particle to a certain well because of tunnelling. This may even lead to complete delocalization of a classically localized particle. On the other hand, for $E > E_0$ repeated scattering at the potential fluctuations may eventually lead to a superposition of destructively interfering waves in such a way that a classically *extended* particle may become localized. Besides potential localization it is the competition between tunnelling and interference which determines whether or not a state is localized.

An example for localization via quantum interference is the above mentioned one-dimensional localization. An example for delocalization via tunnelling, although not related to disordered systems, are the Bloch states in a crystal, in particular those which correspond to the core states.

4.3. Quantities of interest, ensemble averages

The simplest quantity which may be considered is the spectral density of energy levels E_ν,

$$n(E) = \frac{1}{\Omega} \sum \delta(E - E_\nu).$$ (32)

This can be written in terms of the one-particle Green's function (Economou 1983)

$$G(r, r'; E^\pm) = \left\langle r \left| \frac{1}{E \pm i\eta - H} \right| r' \right\rangle$$ (33)

as

$$n(E) = \mp \frac{1}{\pi \Omega} \text{Tr}\{\text{Im}G(E^\pm)\}.$$ (34)

For the sake of simplicity we omit the explicit statement of the limit $\eta \to 0$, here, and in the following.

As mentioned above, the localization properties of the states influence the transport properties of the system. Therefore, the theory of localization is essentially a theory of the transport properties, i.e. the electrical conductivity σ. The latter is given by the Kubo formula,

$$\sigma(T, \omega) = \lim_{\Omega \to \infty} \frac{\pi e^2}{m^2 \omega \Omega} \sum_{\alpha} \sum_{\beta \neq \alpha} |\langle \alpha \mid p \mid \beta \rangle|^2 [f(E_\alpha) - f(E_\beta)] \delta(E_\alpha - E_\beta - \hbar\omega).$$

(35)

Here, $f(E_\alpha)$ is the Fermi function, $E_{\alpha,\beta}$ are the energy eigenvalues corresponding to the eigenstates denoted by $|\alpha\rangle$, $|\beta\rangle$, and T and ω are the temperature and the frequency of the electric field, respectively. p is the projection of the momentum operator onto the direction of the electric field. In the mean free path approximation one obtains the DC conductivity (Mott 1970),

$$\sigma_{dc} = \frac{e^2 k_F^2 \ell}{3\pi^2 \hbar}$$

(36)

where ℓ and k_F are the mean free path and the Fermi wavenumber of the uniform system, respectively. A first restriction on the mean free path *classical* result is given by the Ioffe–Regel criterion (Ioffe and Regel 1960) which states that ℓ should be greater than the wavelength in order for the expression (36) to be valid. At least, if ℓ is of the order of or smaller than the wavelength then a full quantum mechanical calculation of the conductivity is needed.

In terms of the above mentioned one-electron Green's function the conductivity may be written as

$$\sigma(T, \omega) = \lim_{\Omega \to \infty} \frac{e^2}{\pi \Omega m^2} \int_{-\infty}^{+\infty} dE \frac{f(E) - f(E + \hbar\omega)}{\omega}$$
$$\times \text{Tr}\{p\text{Im}G(E^+ + \hbar\omega)p\text{Im}G(E^+)\}.$$

(37)

In this form we will use it in the later sections of this review.

As we have replaced the system to be considered by a statistical ensemble by introducing a probability distribution in the preceding section all of the physical quantities have to be configurationally averaged. However, this will yield physically meaningful results only if the quantity considered is *self-averaging* in the usual statistical sense, i.e. if

$$\langle A \rangle = \lim_{\Omega \to \infty} A(\Omega).$$

(38)

In connection with localization this property is not trivially fulfilled, as we shall see below, even in the metallic limit where the conductivity is large, and may be approximated by the mean free path expression. If a quantity is not self-averaging, then, in principle, one has to consider its probability distribution function (Lerner 1991b), or, equivalently, all of its moments.

5. Definitions of localization

In this section we want to summarize some of the definitions of localization which have been used in recent years. Again, instead of being complete, we emphasize the main ideas.

5.1. Asymptotic behaviour of wavefunctions

The wavefunctions in a random medium may be physically characterized by various parameters such as their average diameter, the fraction of the volume of the system which they occupy, spatial correlations between their amplitudes and phases, and their asymptotic behaviour, and, indirectly, by their transport properties. As we shall see, the latter are closely related to the spatial correlation of the amplitudes and to the asymptotic behaviour.

The *asymptotic behaviour* of a localized state is usually described by the exponential decay length of its envelope, λ, the localization length, i.e.

$$\Psi(r) = f(r)e^{-r/\lambda} \tag{39}$$

where $f(r)$ is a randomly varying function. $\lambda \to \infty$ corresponds to an extended state. In practice this definition is not very useful, since its application would require the calculation of single eigenstates. It is more convenient to define the localization length in connection with the transport properties.

5.2. The inverse participation number

In order to decide whether or not a state is localized it is often sufficient to consider the second moment of the probability density (Wegner 1980),

$$P^{-1} = \sum_r |\Psi(r)|^4 \qquad \|\Psi\| = 1. \tag{40}$$

This is the *inverse participation number*. It is a measure of the portion of the space where the amplitude of the wavefunction differs markedly from zero. It may also be considered as providing a measure for an average diameter R of the state via $R = P^{1/d}$. For plane waves one obtains that $P = L^d$, i.e. equals the volume of the system, and diverges in the thermodynamic limit. Such a behaviour may be considered as being representative of extended states.

One may define the *fractal dimensionality* d^* of a state by (Aoki 1983c, 1986, Kramer *et al* 1988)

$$\lim_{L \to \infty} P = L^{d^*}. \tag{41}$$

For plane waves $d = d^*$. For a general extended state d^* may be different from the Euclidean dimensionality, i.e. $d^* \le d$. For a localized state P is proportional to the volume in which the state has a non-vanishing amplitude. This volume tends to a constant in the thermodynamic limit. Thus, the fractal dimensionality vanishes in the case of localized states.

Considering the higher moments of the probability density one may correspondingly define higher fractal dimensions. If the latter are not given by integer multiples of d^* the states show *multifractal behaviour* (Huckestein and Schweitzer 1991, Pook and Janssen 1991).

It is important to note that in general $R \ne \lambda$. This can be seen by considering the case of the states in the exponential tails of a band. As discussed in the preceding section these are due to occasional potential fluctuations within macroscopically large volumes. Thus, their average diameter is macroscopically large. On the other hand, the asymptotic behaviour of the states is determined by their exponential decay into the space outside the potential fluctuations. The corresponding length is inversely proportional to the square root of the energy. Since we consider the asymptotic regime of the density of states the energy is very large. Consequently, the exponential decay length becomes very small (figure 18).
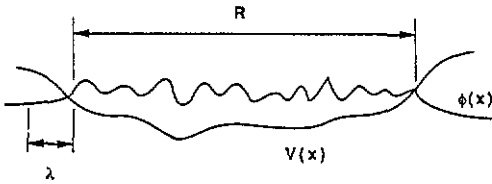
Figure 18. Localization of a state $\Phi(x)$ in an accidental potential well $V(x)$ in the Lifshitz tail of a band. The localization length λ is different from the 'diameter' R of the state.

5.3. Absence of diffusion

The character of the states near a given energy determine the transport properties of the system near the absolute zero of temperature. In particular, one has a vanishing zero-temperature DC conductivity σ_{dc} if there are only localized states near the Fermi energy E_F.

Inserting the momentum operator

$$p = \frac{m}{i\hbar}[H, r] \tag{42}$$

into the Kubo formula (37), and taking into account that

$$H = z - G^{-1} \tag{43}$$

yields

$$\sigma_{dc} = -\frac{2e^2}{h} \lim_{\eta \to 0} \lim_{\Omega \to \infty} \frac{1}{\Omega} \, \text{Tr}\{2\eta^2 \langle G^+(E)rG^-(E)r\rangle - i\eta r^2 \langle (G^+(E) - G^-(E))\rangle\}. \tag{44}$$

As previously in the case of the momentum operator (35), r is the projection of the position operator $\mathbf{r}$ onto the direction of the electric field.

The second term on the RHS is easily shown to be equivalent to $\text{Tr}\{\langle G^+G^-r^2\rangle\}$ by using spectral representation. Therefore, and because

$$\langle |\langle r|G^+(E)|r'\rangle|^2\rangle = \langle |G^+(|r - r'|; E)|^2\rangle \tag{45}$$

due to homogeneity and isotropy, we obtain

$$\sigma_{dc} = \frac{2e^2}{h} \lim_{\eta \to 0} 4\eta^2 \int d^d r r^2 \langle |G^+(r; E)|^2\rangle. \tag{46}$$

The expression on the RHS of this equation gives the square of the mean distance the electron–hole pair described by G^+G^- can diffuse within an infinite time interval (figure 19). If it is finite, the DC conductivity vanishes at zero temperature. The states are localized in a finite region of space. The convergence of the mean diffusion length in the localized regime can be used to define a characteristic length, the localization length, that characterizes the asymptotic behaviour of the electronic states at large distances.
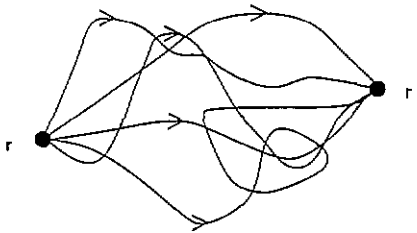
Figure 19. The transmission of an electron between sites r and r'. Classically there is one, and only one, path the electron can travel. It is given by the classical equations of motions and the initial conditions. Quantum mechanically all possible paths have to be taken into account.

5.4. Transmission through random potentials

The absolute square of a Green's function is the probability for the transition of an electron from the site r to the site r',

$$t(r, r'; E) = \langle \left| \langle r \left| G(E^+) \right| r' \rangle \right|^2 \rangle. \tag{47}$$

In the following $t(r, r'; E)$ is often called the transmission probability. If t decays exponentially for large distances then the mean diffusion distance in (46) is finite. The localization length λ may now be defined by using the exponential decay of the transmission probability as

$$\frac{2}{\lambda} = - \lim_{|r-r'| \to \infty} \frac{\ln t(r, r'; E)}{|r - r'|}. \tag{48}$$

If the states near the Fermi energy are asymptotically localized, as defined in (39), then the localization length is finite. This may be most easily seen by using the spectral representation for the Green's function.

5.5. Absence of diffusion and inverse participation number

The two-particle spectral function may also be related to the inverse participation number of (40). Defining the *average return probability* of a particle within a time interval η^{-1} by

$$A(r, E, \eta) = \frac{1}{\pi^2} \langle r \left| \delta_\eta(E - H) \right| r \rangle^2 \tag{49}$$

where δ_η is the Lorentzian definition of the δ function, and the last equality presupposes homogeneity of the system. The average inverse participation number is given by

$$\langle P^{-1} \rangle = \frac{1}{\rho(E)} \sum_{r,\nu} \langle |\Psi_\nu(r)|^4 \delta(E - E_\nu) \rangle \tag{50a}$$

$$= \frac{\pi}{\rho(E)} \lim_{\eta \to 0} \eta A(0, E, \eta) \tag{50b}$$

where $\rho(E)$ is the density of states. The inverse participation number corresponds to the probability that a particle will return to the same site after an infinite time interval (Thouless 1974, Wegner 1980, Weaire and Williams 1977).

If P remains finite in the thermodynamic limit, then the particle cannot diffuse away from a given point even within an infinite time interval.
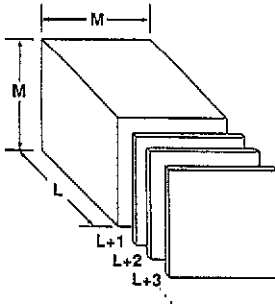
Figure 20. Quasi-one-dimensional system of the size $L \times M \times M$ used for the recursive calculation of the Lyapunov exponents.

5.6. Sensitivity to boundary conditions

Another possibility to investigate the localization properties of the states is the shift of the energy eigenvalues of a finite system due to small changes in the boundary conditions, as was proposed by Edwards and Thouless (1972). The average energy shift, δE, in second-order perturbation theory, is related to the conductivity by

$$\sigma L^{d-2} = \frac{e^2}{h} f \frac{\delta E}{\Delta E} = \frac{e^2}{h} f g(L) \tag{51}$$

where ΔE is the average energy spacing of the eigenvalues and f a numerical factor depending on the details of the model used (square or triangular lattice, for instance). $g(L)$ is called the Thouless number. The idea is that for localized states the mean energy shift will become very small for large system sizes such that $g(L)$ vanishes exponentially whereas in the metallic regime the boundary conditions will always influence the energy levels, even for infinite systems. Although the assumptions made in deriving (51), which concern the statistical behaviour of the energy differences in the system and the corresponding momentum matrix elements, do not seem to be very well satisfied in detail (Kühl 1980), the sensitivity to boundary conditions has turned out to be very useful in practical calculations, see e.g. Licciardello and Thouless (1975a, b, 1978), besides being conceptually important for the formulation of the scaling theory of localization (Abrahams *et al* 1979, Wegner 1976).

5.7. Lyapunov exponents

For 1D and quasi-1D systems the localization length may be calculated from the limiting behaviour of products of random matrices (Mehta 1967, Ishii 1973, Derrida *et al* 1987, Muttalib *et al* 1987, Pichard and Sanquer 1990, Pichard 1991a, b, Feng and Pichard 1991). The simplest derivation of this fact is obtained by writing the Schrödinger equation of the tight-binding Hamiltonian (12) for a bar shaped system (figure 20) as an initial value problem. For a system of dimensions $M^{d-1} L$

$$a_{L+1} = \mathbf{V}^{-1}(E - \mathbf{H}_L)a_L - a_{L-1}. \tag{52}$$

Here, $\mathbf{H}_L$ and $\mathbf{V}$ are $M^{d-1} \times M^{d-1}$ matrices, representing the Lth layer and the coupling matrix between two adjacent layers, respectively. a_L is the M^{d-1}-dimensional vector containing the coefficients of the states.

Using the transfer matrix notation

$$\mathbf{T}_L = \begin{pmatrix} E - \mathbf{H}_L & -\mathbf{V} \\ \mathbf{V} & 0 \end{pmatrix} \tag{53}$$

the evolution of the state is described by the matrix product

$$\mathbf{Q}_L = \prod_{n=1}^{L} \mathbf{T}_n \tag{54}$$

such that we can write

$$\begin{pmatrix} a_{L+1} \\ a_L \end{pmatrix} = \mathbf{Q}_L \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}. \tag{55}$$

The product matrix satisfies the theorem of Oseledec (1968), namely that there exists a limiting matrix

$$\Gamma = \lim_{L \to \infty} (\mathbf{Q}_L \mathbf{Q}_L^\dagger)^{1/(2L)} \tag{56}$$

with eigenvalues $\exp(\gamma_j)$, where γ_j denotes the characteristic Lyapunov exponents of $\mathbf{Q}_L$. The smallest of these eventually determines the slowest possible exponential increase of the state for $L \to \infty$. Therefore, it may be identified with the inverse of the longest exponential localization length in the quasi-1D system of cross-sectional area M^{d-1}. This definition gives the same result as (48) when applied to the present situation. If we replace the vectors a_n in (55) with $M^{d-1} \times M^{d-1}$ matrices $\mathbf{a}_n$, where $\mathbf{a}_1 = \mathbf{I}$ and $\mathbf{a}_0 = \mathbf{0}$, then $\mathbf{a}_{L+1}^{-1} = \mathbf{G}(1, L; E)$, the submatrix of the Green's function between the 1st and Lth slices. This may be seen by comparing (52) and (55) with the resolvent equation $(z - \mathbf{H})\mathbf{G}(z) = 1$.

The Lyapunov exponents may by used directly to calculate the conductance g of a quasi-1D system (in units of e^2/h) (Pichard 1984, Pichard and André 1986),

$$g = \mathrm{Tr} \frac{2}{\mathbf{Q}\mathbf{Q}^\dagger + (\mathbf{Q}\mathbf{Q}^\dagger)^{-1} + 2} \tag{57}$$

the trace being performed within the subspace of the states that correspond to the cross-sectional plane. Using the $M^{d-1} \equiv N$ eigenstates of the Lyapunov exponents one obtains

$$g = \sum_{j=1}^{N} \frac{2}{\cosh^2 \gamma_j L}. \tag{58}$$

This relation is very useful in numerical calculations of the conductance, and also for the investigation of its statistical properties.

6. One-dimensional localization

1D disordered systems play a key role in the understanding of the properties of solids since many features of the electronic states, and of the related transport properties, can be discussed rigorously. We concentrate in this section on the localization of the states, the behaviour of the conductivity of the infinite system, the conductance and the resistance of a

finite system at the absolute zero of temperature. The statistical properties of the transport quantities will be discussed in section 10.

We consider the *tight-binding* Hamiltonian (12), lattice spacing $a \equiv 1$ with only nearest-neighbour hopping and one state per lattice site, i.e.

$$H = \sum_{j=1}^{N} \{\epsilon_j a_j + V(a_{j+1} + a_{j-1})\} \tag{59}$$

where V is the constant hopping matrix element. a_j, $a_{j\pm1}$ are amplitudes of the wavefunctions associated with the lattice sites, and ϵ_j are the corresponding energies. Only the case of statistically independent site energies is considered here (see (14)). The influence of statistical correlations has been discussed in the literature. The main result is that although interesting quantitative effects occur, such as a decrease of the localization length with increasing correlation length in certain energy and disorder regions (Johnston and Kramer 1986, Kasner and Weller 1986), the asymptotic behaviour in the weak disorder region is not changed. These results strongly support the *common belief*, namely that the critical behaviour close to a mobility edge does not depend on the microscopic details of the randomness†.

Particularly convenient distributions of site energies are the Gaussian and box distributions. A measure of the disorder is the width W of the distribution function. All necessary ingredients for a non-trivial description of localization are incorporated in this, under the circumstances, extremely simple model.

6.1. The localization length

In one dimension all eigenstates of a random Hamiltonian are exponentially localized in the asymptotic sense. An example obtained by numerical diagonalization is shown in figure 21 (Czycholl and Kramer 1980).
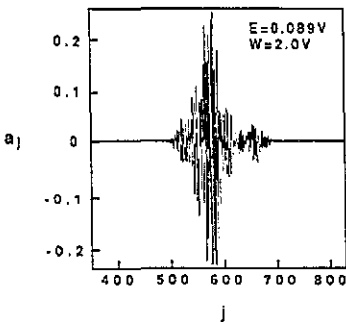

Figure 21. Amplitudes of a localized state, a_j, obtained by numerical diagonalization of the 1D Anderson model (59). Parameters are $L = 1000$, $E = 0.089V$, width of box distribution $W = 2V$. j denotes the lattice sites.

† Although there is no mobility edge in one dimension in the strict sense, one may consider the region near $W = 0$ as the critical region.

The inverse exponential decay length of the eigenstates may be calculated from

$$-2\gamma = \lim_{j \to \infty} \frac{1}{j} \ln(a_j^2 + a_{j+1}^2) \tag{60}$$

with $\{a_j\}$ the solutions of the time-independent Schrödinger equation. In general, it will depend on the particular realization of the random potential, $\gamma = \gamma(\epsilon_1, \ldots, \epsilon_L)$. However, as we shall discuss below, the localization length (and its inverse) are self-averaging quantities. Their configurational averages agree with the most probable values in the thermodynamic limit $L \to \infty$.

The calculation of the exponential increase of the wavefunction by means of the transfer matrix method (cf section 5.7) is in this case particularly simple. The transfer matrix reduces to a 2×2 matrix. There are only two Lyapunov exponents, which have the same magnitude but are of opposite sign. The eigenvalues of the limiting matrix Γ are thus inverses of each other. a_j is for $j \to \infty$ exponentially increasing with a characteristic length $\gamma(E)^{-1}$. Since this is again self-averaging one may assume continuity with E, and identify γ^{-1} with the inverse localization length defined in (60). This conjecture can also be justified by constructing the eigenstates by fitting particular solutions, which are exponentially increasing from the left and the right end of the system, to each other. The eigenvalues of H are then defined as those energies at which a continuous fit is possible (Mott and Twose 1961).

The identification of the localization length as a limiting property of a product of random matrices also immediately provides the proof that in a one-dimensional stochastic system all eigenstates are localized (Molcanov 1978, Kunz and Souillard 1980, Delyon *et al* 1983), independently of the magnitude of the disorder W ($\neq 0$), via the theorem of Fürstenberg (Ishii 1973).

Many calculations of the localization length have been performed using a variety of methods. Perturbational treatments (Thouless 1979, Kappus and Wegner 1981, Derrida and Gardner 1984, Lambert 1984, Johnston and Kramer 1986, Kasner and Weller 1986) were very successful in the weak disorder limit. More recently it has been possible to extend this well beyond perturbation theory to give an almost complete description (Pendry 1982a, b , 1986, 1987, Pendry and Kirkman 1984, 1986, Pendry *et al* 1986, Slevin and Pendry 1988, Pendry and Barnes 1989). Numerical procedures (Czycholl and Kramer 1980, MacKinnon 1980, Czycholl *et al* 1981, Thouless and Kirkpatrick 1981, Pichard 1986) were designed to treat even macroscopically large systems. We describe here briefly a recursive method which is very similar to the above described transfer matrix method and the result obtained by second-order perturbation theory.

Identifying r and r' in (47) with the first and the last site of the lattice underlying the 1D-tight-binding Hamiltonian we obtain for the inverse localization length (48) ($\gamma \equiv 1/\lambda$)

$$\gamma \equiv \frac{1}{\lambda} = -\lim_{\eta \to 0} \lim_{L \to \infty} \frac{1}{2L} \ln \left| G_{1L}^+(E) \right|^2 \tag{61}$$

where $G_{1L}^+(E)$ is the matrix element of $G^+(E)$ between the states that correspond to the first and Lth sites of the system. As mentioned above, it is also given by a_{L+1}^{-1}. $t(L, E) = |G_{1L}(E)|^2$ gives the transmission probability for a particle to be transferred from site 1 to site L. Since γ is always finite in one dimension, an infinitely long 1D disordered system cannot be transparent, even in the limit of very weak disorder.

Writing

$$G_{1L}^+(E) = \prod_{m=1}^{L} g_m^+(E) \tag{62}$$

where

$$g_m^+(E) \equiv G_{mm}^+(E) = \frac{1}{E + i\eta - \epsilon_m - V^2 g_{m-1}^+} \tag{63}$$

is the diagonal element of G at the end of a system of length m, we obtain

$$\gamma_L = \frac{L-1}{L}\gamma_{L-1} + \frac{1}{L}\ln|g_L|. \tag{64}$$

This recursive equation for the inverse of the localization length is especially useful for numerical purposes. It can also be easily generalized to higher dimensions by using the bar-shaped geometry described in the preceding section in connection with the Lyapunov exponents (MacKinnon and Kramer 1981, 1983a).

There is a relation between the spectral properties and the localization length which was discovered by Herbert and Jones (1971). It may readily be derived by calculating G_{1L} as the inverse of $E - \mathbf{H}$, and using (61):

$$\gamma(E) = \int_{-\infty}^{+\infty} \rho(x)\ln|E - x|\,\mathrm{d}x = \Re \int_{-\infty}^{E} \lim_{L\to\infty} \frac{1}{L}\sum_{j=1}^{L} G_{jj}(x)\,\mathrm{d}x \tag{65}$$

where $\rho(E)$ is the density of states. The diagonal elements of the Green's function may be evaluated in second-order perturbation theory (Thouless 1979). In the limit of small disorder one obtains ($|E| < 2V$)

$$\gamma(E) = \frac{W^2}{24(4V^2 - E^2)} \tag{66}$$

i.e. $\gamma(0) = W^2/96V^2$ for small W.

The numerical results which were obtained by using (64) are consistent with $\gamma(0) = W^2/105V^2$ (Czycholl *et al* 1981, Pichard 1986). The difference in the prefactor is due to an anomaly in the band centre which originates in resonance effects leading to a breakdown of second-order perturbation theory (Kappus and Wegner 1981, Derrida and Gardner 1984, Lambert 1984). As mentioned above, the result that the localization length of a 1D disordered system diverges at $W = 0$ as W^{-2} remains true also for the case of a spatially correlated potential.

6.2. Localization and transport

That the DC conductivity vanishes in 1D disordered systems has been shown analytically by Kunz and Souillard (1980) and numerically for the one-dimensional Anderson model with rectangular distribution of the site energies by MacKinnon (1980), Czycholl *et al* (1981), and Thouless and Kirkpatrick (1981) by using the DC Kubo formula (44) at the absolute zero of temperature in a recursive procedure similar to the one described above for the localization length. The results are plotted in figure 22.

Within the accuracy of the calculation the conductivity scales as (η imaginary part of the energy)

$$\sigma(\eta, W) = W^{-2}\sigma(\eta W^{-2}, 1). \tag{67}$$

It is only for large disorder ($W > 5$) that one observes deviations from this behaviour. The scaling law allows for the necessary extrapolation with $\eta \to 0$ for any fixed W. For large η one has $\sigma \sim \eta^{-1}$ (*classical*, Drude-like behaviour) whereas for small η the conductivity tends to zero linearly with η.
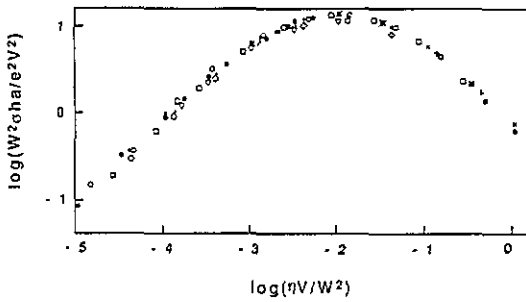
Figure 22. The DC conductivity of the one-dimensional Anderson model with rectangular distribution of site energies as a function of the disorder, W, and the imaginary part of the frequency, η.

6.3. Theory of the averages of resistance and conductance

Classically, the conductance $g(L)$ of a d-dimensional hypercube of volume L^d is related to the conductivity by

$$g(L) = \sigma L^{d-2}. \tag{68}$$

Since the conductivity vanishes in the localized regime, it is no longer a useful parameter when considering the transport properties of a macroscopic, but finite, sample. One should have a theory of the conductance (or, equivalently, of its inverse, the resistance) without referring to the conductivity.

A very simple equation for the resistance $R(L)$ of a 1D wire of length L has been given by Landauer (1970). The basic idea is to consider the charge transport through the disordered wire, which is thought to be connected to ideally conducting wires on the left and on the right, as a quantum mechanical transmission problem. The voltage is generated by a difference in the charge densities on the left- and the right-hand sides of the wire. The current is given by the total number of particles transmitted through the wire at the Fermi velocity. The result is

$$R(L) = \frac{h}{e^2} \frac{1 - t(L)}{t(L)} \tag{69}$$

where $t(L)$ is the transmission coefficient. Generalizations to quasi-1D systems (many transmission channels) have been discussed by several authors (Anderson *et al* 1980, Langreth and Abrahams 1981, Fisher and Lee 1981, Economou and Soukoulis 1981a, b, Thouless 1981, Anderson 1981, Landauer 1985, Büttiker *et al* 1985, Pendry 1982c, 1984, 1989a, b, Pendry and Castaño 1988, Pichard 1984, Pichard and André 1986, Carton *et al* 1986, Mašek and Kramer 1988, 1989).

A dimensionless resistance may be defined by $r(L) = R(L)e^2/h = t(L)^{-1} - 1$. As the logarithm of $t(L)$ is statistically a well-behaved quantity (it obeys the central limit theorem) and its configurational average is asymptotically equal to $-\gamma L$, one may expect that the resistance is an exponentially increasing function of L, on the average. In addition, the probability distribution of r will not fulfil the central limit theorem. The exponential increase of the resistance has been proven for a variety of models by several authors (Abrahams and Stephen 1980, Andereck and Abrahams 1980, Stone *et al* 1981, Kree and Schmid 1981,

Sak and Kramer 1981, Kirkman and Pendry 1984a, b, Mel'nikov 1981). In the following we reproduce briefly the elementary derivation by Abrahams and Stephen.

In order to calculate the average resistance of a system of length L one has to consider the configurational average

$$\mathbf{U}_L = \langle \mathbf{Q}^\dagger \mathbf{Q} \rangle = \langle \mathbf{T}_1^\dagger \ldots \mathbf{T}_L^\dagger \mathbf{T}_L \ldots \mathbf{T}_1 \rangle. \tag{70}$$

Since the transfer matrices are statistically independent this yields a recursion relation

$$\mathbf{U}_{L+1} = \langle \mathbf{T}_{L+1}^\dagger \mathbf{U}_L \mathbf{T}_{L+1} \rangle. \tag{71}$$

If one takes $\langle \epsilon_j \rangle = 0$ then $\mathbf{U}_L$ is diagonal, and can be cast into the form $\mathbf{U}_L = \mathbf{A}_N \tau_z + \mathbf{B}_L$ where τ_z is the Pauli matrix. The largest eigenvalue of the recursion relation (71) is

$$\gamma_1 = \tfrac{1}{2}\sigma_2 + (1 + \tfrac{1}{4}\sigma_2^2)^{1/2} \tag{72}$$

with $\sigma_2 = \langle \epsilon_j^2 \rangle$. Thus, asymptotically one has

$$\langle r(L) \rangle = \exp\{L \ln \gamma_1\} \approx \exp\{\sigma_2 L/2\} \equiv \exp\{\gamma_r L\} \tag{73}$$

where $\gamma_r \equiv \ln \gamma_1 \approx \sigma_2/2$ is the inverse of a *localization length* which is characteristic for the exponential increase of the average of the resistance.

The dimensionless conductance, $g(L)$, is defined as the inverse of the dimensionless resistance, $r(L)$. The calculation of the corresponding configurational average is much more complicated than in the case of the resistance. It has been performed by Abrikosov and Ryzhkin (1978) in the limit of weak disorder. Kirkman and Pendry (1984b) have treated the general case. Elaborate numerical calculations have been done by Markoš and Kramer (1993a). The result for weak disorder, and in the centre of the band ($E = 0$), is

$$\langle g(L) \rangle \propto (L\sigma_2)^{-3/2} \exp(-\tfrac{1}{16}\sigma_2 L) \tag{74}$$

in the limit of large L. Thus, $\gamma_g = \sigma_2/16$ is the inverse of the *localization length* which is characteristic for the exponential decrease of the average conductance.

By comparison of (66) and (72)–(74) one observes that in the weak disorder limit there are relationships between the various localization lengths, namely

$$\ln\langle g \rangle = \tfrac{1}{4}\langle \ln g \rangle \tag{75a}$$

$$\ln\langle r \rangle = 2\langle \ln r \rangle \tag{75b}$$

$$\langle \ln r \rangle = -\langle \ln g \rangle. \tag{75c}$$

That the three lengths do not agree with each other reflects the fact that the resistance and the conductance are *not* self-averaging in 1D disordered systems (Sak and Kramer 1981).

In addition to the fundamental statistical effects in the DC transport properties, one can expect interesting and novel statistical features in time- and frequency-dependent transport. First studies were done for one dimension (Pendry *et al* 1986, Mašek and Kramer 1988) and in the weakly disordered metallic regime (Falko 1989).

7. Weak localization and quantum interference

The limit of a weak random potential was considered as rather unimportant for the localization problem until it was rediscovered (Abrahams *et al* 1979, Gorkov *et al* 1979) that there exists a certain class of diagrams, the *maximally crossed diagrams* (Langer and neal 1966), in the perturbation expansion of the conductivity that could be summed exactly for non-interacting time-reversal invariant systems, and gives logarithmic corrections to the low-temperature and low-frequency conductivity in two dimensions. The conclusion was that *the zero-temperature and zero-frequency conductivity of disordered systems without interactions and for zero magnetic field always vanishes for $d \leq 2$*, and consequently all quantum states must be localized. A considerable body of quantitative theoretical and experimental work followed this discovery. One of the important points was that the diagrams could be interpreted physically as quantum interference processes that give rise to an enhanced backscattering. This point of view could be verified directly using a type of Aharonov–Bohm configuration but with normal metal cylinders instead of superconductors (Aronov and Sharvin 1987). We will not repeat here extensively the formal theory. It has already been treated very elaborately in many review papers (Vollhardt and Wölfle 1992, Lee and Ramakrishnan 1985, Altshuler and Aronov 1985, Fukuyama 1985, Kawabata 1985, Bergmann 1984). Instead, in the first instance, we will stress the physical point of view of the quantum interference in deriving the main results.

7.1. Quantum interference

Let us consider the limit of a weak random potential $V(r)$. Then the Born series may be used to evaluate the Green's functions. One notes that the zero-temperature conductivity can be written as an infinite sum of terms of the form

$$t(r, r'; E) = \langle \left| \sum_n A_n^+(r, r') \right|^2 \rangle = \sum_n \langle |A_n^+|^2 \rangle + \sum_{n \neq m} \langle A_n^+(A_m^+)^* \rangle \qquad (76)$$

where

$$A_{r_1 \ldots r_n}^{\pm}(r, r') = G_0^{\pm}(r, r_1) V(r_1) G_0^{\pm}(r_1, r_2) V(r_2) \ldots V(r_n) G_0^{\pm}(r_n, r') \qquad (77)$$

is the probability amplitude for a transition from r to r' when n scattering events at the random potential $V(r)$ at the sites $r_1 \ldots r_n$ are encountered during the process.

That (76) is a valid representation of the average transmission probability may readily be seen from (47) by considering the resolvent expansion of the one-particle Green's function (33)

$$\mathbf{G} \equiv \mathbf{G}^{\pm}(E) = (E^{\pm} - \mathbf{H})^{-1} = (E^{\pm} - \mathbf{H}_0 - \mathbf{V})^{-1} \qquad (78a)$$

$$\mathbf{G}^{\pm}(E) = \mathbf{G}_0^{\pm} + \mathbf{G}_0^{\pm} \mathbf{V} \mathbf{G}_0^{\pm} + \mathbf{G}_0^{\pm} \mathbf{V} \mathbf{G}_0^{\pm} \mathbf{V} \mathbf{G}_0^{\pm} + \ldots \qquad (78b)$$

where $\mathbf{G}_0^{\pm} = (E^{\pm} - \mathbf{H}_0)^{-1}$ is the resolvent of the unperturbed system, and $V(r)$ denotes the random perturbation. Writing (78) in position representation one obtains

$$\langle r|\mathbf{G}^{\pm}|r'\rangle = \sum_{n=0}^{\infty} \sum_{r_1 \ldots r_n} A_{r_1 \ldots r_n}^{\pm}(r, r') \qquad (79a)$$

$$\equiv \sum_{n=0}^{\infty} A_n^{\pm}(r, r') \qquad (79b)$$

from which (76) follows immediately. The symbolic summations over $r_1 \ldots r_n$ have to be evaluated as integrals if working in continuous space. When using the lattice Hamiltonian (12) the summations are over the sites of the lattice. The total probability amplitude (79) may be interpreted as a superposition of all of the amplitudes corresponding to the specific paths denoted by $r_1 \ldots r_n$ in (77). The transmission probability may be considered as a superposition of all possible paths between r and r' (figure 23).
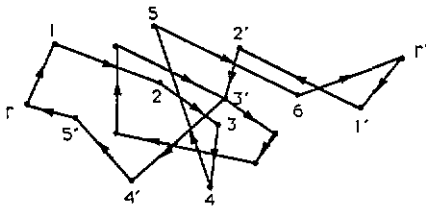

Figure 23. Graphical representation of the transition probability of an 'electron–hole pair' between the sites r and r'. The electron (A_6) and the hole (A_5^*) are scattered at the sites 1, 2, 3, 4, 5, 6, and $5', 4', 3', 2', 1'$, respectively.

The first term in (76) represents classical diffusion from r to r'. Loosely speaking the *electron*, represented by the *retarded propagator* $\mathbf{G}_0^+$, and the *hole*, represented by the *advanced propagator* $\mathbf{G}_0^-$, are scattered at the same intermediate sites (figure 24(a)). The summations include the number of scatterings as well as the specific paths of the electron–hole pair. The second term in (76) is due to quantum mechanical interference between different paths.

(a)   (b)
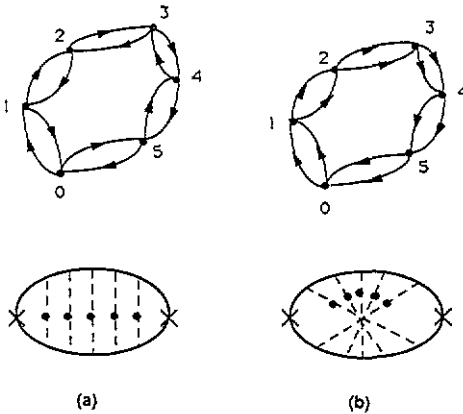
Figure 24. *Diffusons* (a) and *cooperons* (b) are the most important contributions to the total quantum mechanical return probability in weakly disordered systems. In the example shown in the figure electron and hole are assumed to be scattered at the same sites (1, 2, 3, 4, 5). However, in the case of the diffuson electron and hole travel in the same (clockwise) direction, whereas for the cooperon they travel in opposite directions. Diffusons describe therefore scattering in the 'electron–hole channel', cooperons in the 'electron–electron channel'. Position space (top) and reciprocal space (bottom) diagrams are related by Fourier transformation.

Due to the configurational average most of the terms in (76) are vanishingly small except for those in which the sites $r_1 \ldots r_n, r_1' \ldots r_n'$ are close to each other. For a white noise potential (27) they must be at least pairwise equal. If $r \neq r'$ the interference terms are always smaller than the diffusion terms. For closed paths, $r = r'$, however, there is a class of interference terms which contribute to the total transmission probability with an amount equal to that of the corresponding diffusion terms if the system is time-reversal invariant, i.e. does not contain any magnetic effects. These interference terms are characterized by the condition that $r_1 = r_n'$, $r_2 = r_{n-1}'$, ..., $r_n = r_1'$ (figure 24(b)). Physically they describe an electron and a hole which move in *opposite* directions around the loop whereas the corresponding diffusion terms represent an electron and a hole moving in the *same* direction. As a consequence of the presence of the former, backscattering is strongly enhanced and forward scattering is reduced. Thus there is a correction to the diffusive (classical) conductivity (36) which is due to quantum interference and which *reduces* the conductivity.

Whereas the diffusion terms ($\propto |A_n|^2$) describe the motion of electron–hole pairs (*diffusons*), the interference terms involving the same intermediate sites can be considered to describe the motion of an electron–electron pair, the second electron being nothing but the hole moving along the time-reversed path $r_n' \ldots r_1'$. Therefore, these latter terms are often called *cooperons* (Vollhardt and Wölfle 1980a, b, 1992). In momentum space the *diffusons* correspond to the so-called *ladder diagrams*, to be discussed in more detail below. The *cooperons* are represented by the *maximally crossed* diagrams already discovered in 1966 by Langer and Neal (1966) (figure 24(b)). We will consider the momentum space formulation of weak localization in more detail in the following. The position space formulation was treated elaborately by Stone (1992).

7.2. Diagrammatic expansions

In this section we sketch the essential steps for the quantitative calculation of the weak localization corrections to the DC conductivity (Edwards 1958, Langer and neal 1966, Gorkov *et al* 1979, Abrahams *et al* 1979, Vollhardt and Wölfle 1980a, b, 1992, Hikami *et al* 1980) and provide, using as an example the one-particle Green's function, the basic tool for their evaluation: the language of Feynman diagrams.

The starting point is the expression (46) for the DC conductivity at zero temperature. For convenience it is transformed from position into momentum space by using

$$G^{\pm}(r, r') = \sum_{kk'} \exp(ik \cdot r - ik' \cdot r') G^{\pm}(k, k') \tag{80}$$

and

$$\int d^d r \exp(iq \cdot r) r^2 = -\nabla_q^2 \delta(q). \tag{81}$$

We obtain

$$\sigma_{dc} = \frac{2e^2}{h} \lim_{\eta \to 0} 4\eta^2 \lim_{q \to 0} \nabla_q^2 \sum_{kk'} \langle G^+(k, k' + q) G^-(k', k - q) \rangle. \tag{82}$$

The technique used in evaluating the configurationally averaged product of two Green's functions is called *partial summation*. It is based on the *resolvent expansion* of the one-particle Green's function (78) as a power series in the random potential (Born series),

$$\langle \mathbf{G}^{\pm} \rangle = \mathbf{G}_0^{\pm} + \mathbf{G}_0^{\pm} \langle \mathbf{V} \mathbf{G}^{\pm} \rangle \equiv \mathbf{G}_0^{\pm} \sum_{n=0}^{\infty} \langle (\mathbf{V} \mathbf{G}_0^{\pm})^n \rangle. \tag{83}$$

Formally one can decompose the configurational average of the right-hand side

$$\langle \mathbf{G}^{\pm} \rangle = \mathbf{G}_0^{\pm} + \mathbf{G}_0^{\pm} \mathbf{S}^{\pm} \langle \mathbf{G}^{\pm} \rangle \tag{84}$$

such that

$$\langle \mathbf{G}^{\pm} \rangle = (E^{\pm} - \mathbf{H}_0 - \mathbf{S}^{\pm})^{-1}. \tag{85}$$

$\mathbf{S}^{\pm}$ is the operator of the *self-energy* which plays the role of a complex-valued, average, effective potential for the motion of the particle. For completeness we add that via

$$\langle \mathbf{G}^{\pm} \rangle = \mathbf{G}_0^{\pm} + \mathbf{G}_0^{\pm} \langle \mathbf{T}^{\pm} \rangle \mathbf{G}_0^{\pm} \tag{86}$$

the resolvent is related to the scattering matrix $\mathbf{T}^{\pm}$ of the random potential

$$\langle \mathbf{T}^{\pm} \rangle = \langle \mathbf{V} \rangle + \langle \mathbf{V}\mathbf{G}_0^{\pm}\mathbf{T} \rangle \equiv \langle \mathbf{V} \sum_{n=0}^{\infty} (\mathbf{G}_0^{\pm}\mathbf{V})^n \rangle. \tag{87}$$

From the Born series (83) one readily obtains a power series for the configurationally averaged product of the two one-particle resolvents. Formally, this leads to a Bethe–Salpeter equation

$$\langle \mathbf{G}^+ \mathbf{G}^- \rangle = \langle \mathbf{G}^+ \rangle \langle \mathbf{G}^- \rangle + \langle \mathbf{G}^+ \rangle \langle \mathbf{G}^- \rangle \mathbf{U} \langle \mathbf{G}^+ \mathbf{G}^- \rangle \tag{88}$$

with an *irreducible vertex operator* $\mathbf{U}$ which plays the same role with respect to the motion of the two particles as the self-energy operator does for one particle.

In order to obtain relations which are useful in practice we consider the momentum representation. The random potential is written as a superposition of impurity potential $v(r)$ located at random positions $R_1 \dots R_N$

$$V(r) = \sum_{j=1}^{N} v(r - R_j). \tag{89}$$

In momentum space,

$$V(k, k') \equiv \langle k | \mathbf{V} | k' \rangle = \sum_{j=1}^{N} \exp[\mathrm{i}(k - k') \cdot R_j] v(k, k'). \tag{90}$$

For further simplification we shall assume in the following that $v(r)$ is very weak and well localized in position space so that

$$\langle |V(k, k')|^2 \rangle = \frac{N}{\Omega} |v(k - k')|^2 \equiv \begin{cases} u_0^2 & k, k' \le K_0 \\ 0 & \text{otherwise} \end{cases} \tag{91}$$

where we have introduced a very large but finite cut-off wavenumber K_0 in order to formally avoid divergences of integrations. Also, for convenience, the zero of energy is chosen at $v(k = 0)$ which corresponds to the spatial average of the impurity potential.

In order to avoid clumsy multiple sum notations in the Born series conventionally the language of Feynman diagrams is used. As a simple exercise we consider the calculation of the averaged one-particle Green's function. The nth-order term in the Born series contains

multiple sums over the n positions of the impurities. Several of the latter may be identical, thus describing multiple scattering at a given impurity. In the configurational average these multiple scattering terms have to be treated separately such that the formally simple structure of the Born series is eventually transformed into a complicated assembly of mathematical expressions. Figure 25 shows the Feynman diagrams up to the fourth order for $\langle G^+ \rangle$. The rules for the evaluation of the diagrams are

$$\begin{array}{ll}
\xrightarrow[\;k\;]{} & \hat{=}\ \mathbf{G}_0^+(k) & \text{free-particle Green's function} \\[2em]
\vdots \atop {p=k-k'} & \hat{=}\ v(p) & \text{potential scattering} \\[2em]
\underset{p_2}{\overset{p_1 \quad\quad p_m}{\bigwedge}} & \hat{=}\ \dfrac{N}{\Omega}\delta(p_1 + p_2 + \ldots + p_m) & m\text{-fold scattering.} \quad (92)
\end{array}$$
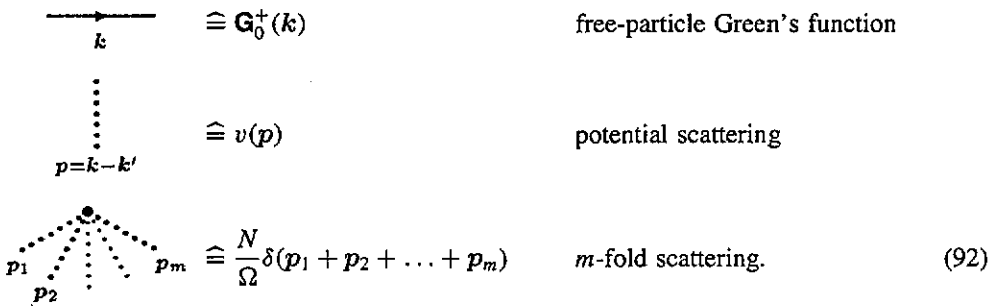
A k summation is associated with each free-particle line that connects two scattering lines. The configurationally averaged one-particle propagator is a sum of all possible diagrams containing n potential scattering lines and $(n+1)$ free-particle propagators. Each nth-order term is decomposed into multiple scattering contributions that are characterized by the number of connected scattering lines.

Generally we can distinguish two classes of diagrams. First, there are diagrams which can be completely disconnected by cutting single free-particle lines. These are called *reducible diagrams*. They represent simple algebraic products of more complicated diagrams. The latter are represented by the second class of diagrams, which cannot be factorized and are therefore called *irreducible*. It is easy to see that the irreducible diagrams are diagonal in k and that the average propagator is

$$\langle G^\pm(k, k') \rangle = \frac{\delta(k - k')}{E^\pm - \hbar^2 k^2 / 2m - \Sigma^\pm(k)} \tag{93}$$

where $\Sigma^\pm(k)$ denotes the sum of all irreducible terms plus all terms of the same topological structure (skeleton diagrams) but with all of the internal free-particle propagators replaced by the full averaged one-particle Green's function. Whereas the real part of $\Sigma^\pm(k)$ gives the shift of the unperturbed energy $\hbar^2 k^2 / 2m$ due to the perturbation, its imaginary part can be interpreted as an inverse mean free time τ

$$\hbar/2\tau \equiv \pm\mathrm{Im}\,\Sigma^\pm(k). \tag{94}$$

Since we have incorporated the mean of the impurity potential in the zero of energy the lowest non-trivial order of the self-energy is given by

$$\Sigma_2^\pm(k) \equiv \overset{\bullet}{\underset{}{\bigtriangleup}} = u_0^2 \sum_k G_0^\pm(k) \tag{95}$$

and its imaginary part is

$$\gamma \equiv \mp\mathrm{Im}\,\Sigma_2^\pm(k) = \pi u_0^2 n(E) \tag{96}$$
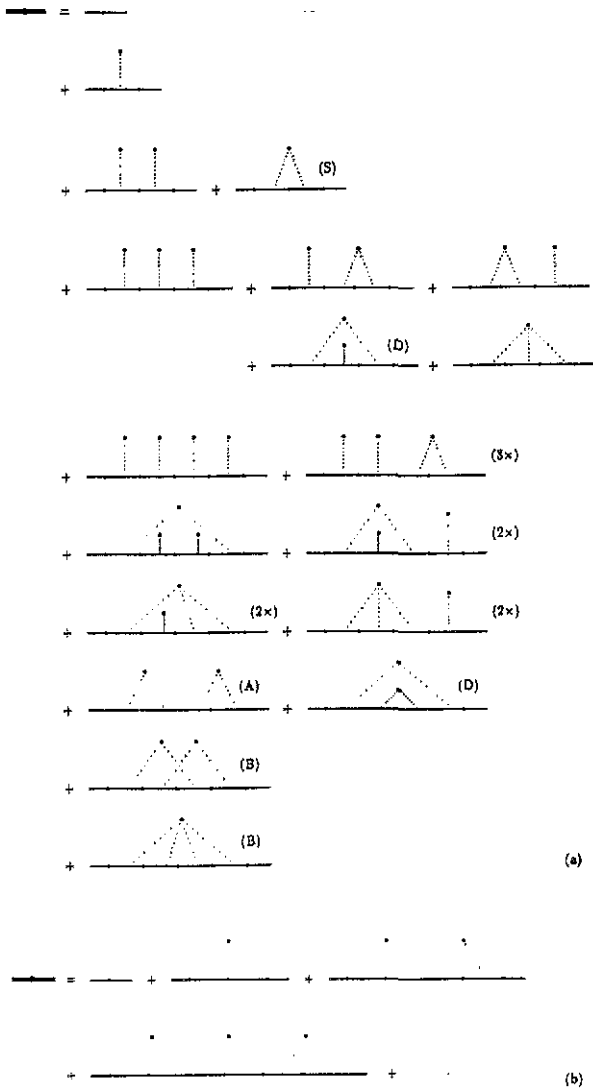
Figure 25. (a) The Feymann diagrams for the one-electron Green's function in the presence of disorder up to fourth order in the random potential. Diagrams that can be factorized into independent parts by cutting lines that correspond to the Green's function of the free electron are called *reducible diagrams* (A). If this is not possible, the diagram is called irreducible (B). Skeleton diagrams do not contain terms appearing in the averaged Green's function between two scatterings at the same atom. For instance, diagrams (D) can be reduced to the same skeleton (S) by removing all of the internal scatterings. (b) Geometrical series representing the configurationally averaged Green's function in lowest non-trivial order of perturbation theory where the electron is successively scattered only twice at each of the impurity potentials.

with the density of states $n(E) = \sum_k \delta(E - \hbar^2 k^2/2m)$. The mean free time is thus

$$\tau = \hbar/(2\pi u_0^2 n(E)) \tag{97}$$

and the corresponding mean free path ℓ is obtained by multiplication with the Fermi velocity $v_F = \sqrt{2E/m}$

$$\ell = v_F\tau. \tag{98}$$

For a metallic system the Fermi energy is situated well inside a region of high density of states, where γ is approximately constant. For the purposes of demonstrating the effect of weak localization it is therefore sufficient to consider the limit of very weak scattering where the dominant effect of the disorder on the one-particle propagator is to introduce a finite mean free path at energies well inside a band. Thus it is justified to assume in the following that the average one-particle Green's function has the form (figure 25(*b*))

$$\langle G^{\pm}(k)\rangle = \frac{1}{E^{\pm} - \hbar^2 k^2/2m \pm i\gamma}. \tag{99}$$

The Bethe–Salpeter equation for the configurationally averaged two-electron Green's function (88) can be written as

$$\begin{aligned}
\langle G^+(k+q, k'+q)G^-(k', k)\rangle &= \langle G^+(k+q)\rangle\langle G^-(k)\rangle\delta(k-k') \\
&+ \langle G^+(k+q)\rangle\langle G^-(k)\rangle\sum_{k''} U(k, k'')\langle G^+(k''+q, k'+q)G^-(k', k'')\rangle.
\end{aligned} \tag{100}$$

7.3. The diffuson approximation

7.3.1. Electron–hole diagrams. As for the averaged one-electron Green's function, the diagrammatic formulation can also be used for the average of the product of two Green's functions required for the calculation of the DC conductivity, $\langle G^+(k, k'+q)G^-(k, k-q)\rangle$. In order to distinguish between the retarded and advanced Green's functions, G_0^+ and G_0^-, respectively, opposite directions are assigned to the corresponding free-particle lines. Typical examples of the diagrams are shown in figure 26.

We identify several classes. The first contains only terms in which scattering lines are connected solely within the retarded (electron) or the advanced (hole) channel, respectively, but not between the channels. These are easily seen to be summable formally. They yield the product of the averaged one-particle Green's functions on the right-hand side of equation (100). In the second class, scattering lines are connected to both the electron and the hole channels. Physically, these diagrams describe the coherent scattering of the electron–hole pair at the potentials of the impurities. Effectively, the coherent pair scattering events introduce an interaction between the electron and the hole. They represent the physical origin of the vertex operator **U** in (88), and prevent the decomposition of the averaged product into a product of the averages of the electron and the hole propagators. It is precisely this vertex operator which provides the physical scattering mechanism for the weak localization correction to the classical DC conductivity. Before calculating this correction we shall discuss the structure of the diagrams in more detail, and derive the classical mean free path expression for σ_{dc}.

Firstly, we note that the diagrams can again be separated into reducible and irreducible classes. The diagrams belonging to the former can be factorized by cutting single-particle lines into products of expressions which are given by the irreducible diagrams. The sum of all irreducible diagrams constitutes the vertex operator in (100) in momentum representation. Furthermore, there are infinitely many of each vertex diagram of a given topology. They can be obtained by replacing the free-particle propagators in a given vertex diagram structure by all possible diagrams that contribute to the average one-particle Green's function. Partial summation then leads eventually to the replacement of all of the free-particle propagators by the averaged one-particle propagator.
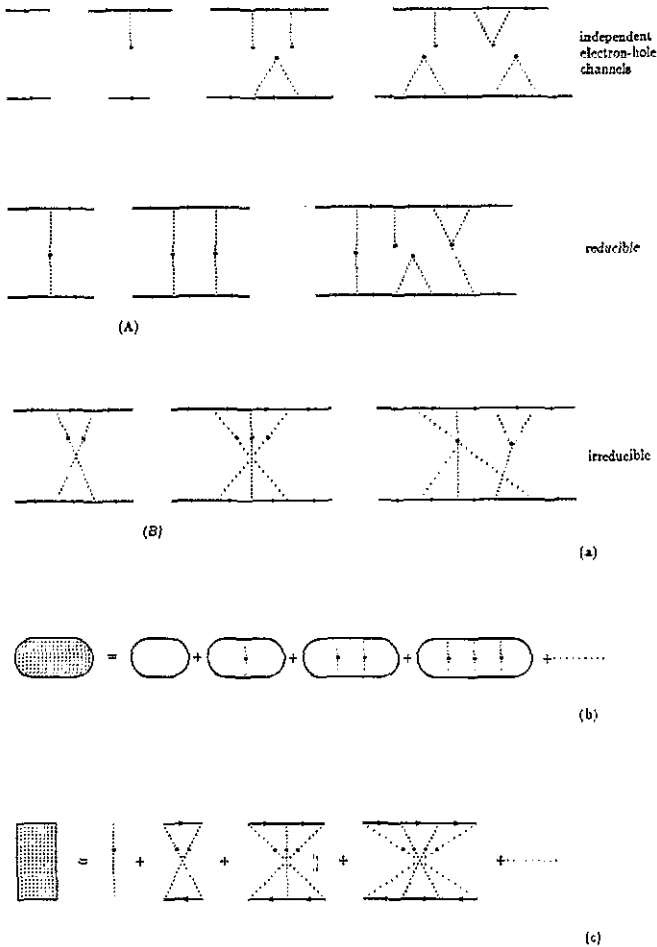
Figure 26. (*a*) Feynman diagrams representing the configurationally averaged two-electron Green's function. Diagrams of the type (A) lead to the classical result for the DC conductivity (ladder approximation (*b*)), while those of the type (B) represent maximally crossed quantum interference terms that lead to the lowest-order quantum correction to the classical conductivity (cooperon approximation (*c*)).

7.3.2. *The ladder approximation.*

In lowest approximation the irreducible vertex operator is given by the square of the impurity potential

$$
\begin{aligned}
{}^{k_1 k_2}_{\vdots \atop k_2' k_1'} \;=\; & \sum_R \frac{1}{\Omega} \int \mathrm{d}^d R \, \exp\{\mathrm{i}(k_1 - k_2 - k_1' + k_2') \cdot R\} v(k_1 - k_2) v(k_2' - k_1') \\
=\; & \frac{N}{\Omega} |v(k_1 - k_2)|^2 \, \delta(k_1 - k_2 - k_1' + k_2') \\
=\; & u_0^2 \delta(k_1 - k_2 - k_1' + k_2').
\end{aligned}
\tag{101}
$$

By inserting (101) into (100), and replacing the free-particle propagators by the averaged one-particle Green's function (99), a geometrical series for $\angle G^+(k, k' + q)G^-(k', k - q)\rangle$

is obtained, which can be summed (figure 26(b)). The result of this *ladder approximation* is

$$\sum_{kk'} \langle G^+(k, k' + q)G^-(k', k - q)\rangle = \frac{f(q)}{1 - u_0^2 f(q)} \tag{102}$$

with

$$f(q) = \sum_k \langle G^+(k + q)\rangle \langle G^-(k)\rangle \tag{103}$$

which has to be evaluated for small q (cf (82)). Using

$$\langle G^+\rangle \langle G^-\rangle = \frac{\langle G^-\rangle - \langle G^+\rangle}{\langle G^+\rangle^{-1} - \langle G^-\rangle^{-1}} \tag{104}$$

and the lowest-order result (99) for $\langle G^\pm(k)\rangle$ one obtains

$$f(q) = \frac{\pi n(E)}{(\eta + \gamma)} - \frac{S_d \hbar^2}{2dm} \frac{\pi n(E)E}{(\eta + \gamma)^3} q^2 \tag{105}$$

with $S_d = 4\pi, 2\pi, 2$ for $d = 3, 2, 1$, respectively. By inserting (105) into (102) and (82) and then performing the derivatives with respect to q and the limits $q \to 0$ and $\eta \to 0$ in the correct order, one eventually obtains the conductivity

$$\sigma_0 = \frac{e^2}{h} \frac{\hbar^2 \pi S_d}{dm} \frac{En(E)}{\gamma} \tag{106}$$

which, for $d = 3$, can be shown to be precisely the result (36) when γ is replaced by the mean free path (98).

7.4. The cooperon approximation

The classical conductivity obtained in the previous section is the zeroth-order term of an expansion in powers of $(k_F\ell)^{-1}$. The first-order correction can be obtained by considering a summable class of irreducible vertex diagrams, namely the maximally crossed diagrams of figure 26(c). It is straightforward to see that the summability of the *maximally crossed diagrams* is a consequence of time-reversal invariance in the absence of a magnetic field. Hence, the sum can be obtained from the sum of the ladder diagrams (102) by replacing k' by $-k'$ in the *hole channel*; thus replacing a hole with momentum $\hbar k'$ by an electron with momentum $-\hbar k'$. In the following, the essential steps will be performed to obtain the correction to the classical conductivity (106) due to the *weak localization* induced by these diagrams. The effect of this *cooperon correction* corresponds to the enhanced backscattering described above.

7.4.1. The Ward identity.
Firstly we note that, since we are going to change the approximation for the vertex operator **U**, it is not possible to use the same approximate form for the averaged one–electron Green's function as before, (99). This is due to an exact relation between the one–electron self-energy $\Sigma^\pm(k)$, and the irreducible vertex function $U(k, k')$ defined in (101), namely

$$\Delta\Sigma(k) \equiv \Sigma^+(k) - \Sigma^-(k) = \sum_{k'} U(k, k')(\langle G^+(k')\rangle - \langle G^-(k')\rangle). \tag{107}$$

This identity was proven earlier (Vollhardt and Wölfle 1980b) and accounts for the fact that the irreducible vertex can be generated systematically from the self-energy $\Sigma^+(k)$ by differentiation with respect to $\langle G^\pm(k)\rangle$. As a consequence, we have to allow for the self-energy to be different from $\gamma = \pi n(E)u_0^2$ in (99), i.e. $\langle G^\pm(k, E)\rangle$ to be given by (93).

7.4.2. The irreducible vertex function. The calculation of the sum of the terms visualized in figure 26(c) is straightforward. A geometrical series is obtained, which can be summed. The result is

$$U(k, k') = \frac{u_0^2}{1 - f(k + k')u_0^2} \tag{108}$$

where $f(k + k')$ is defined in (103). As $U(k, k') \equiv U(k + k')$ diverges for $\eta \to 0$ and $|k + k'| \to 0$, $k' \approx -k$ yields the dominant contribution to the solution of (100). Thus it is indeed sufficient to consider the lowest orders of $f(k + k')$. Inserting (105) into (108), and assuming in lowest order that $\Sigma^{\pm}(k) \approx \mp i\gamma$, we obtain

$$U(k + k') \approx \frac{u_0^2 \gamma}{\eta + \hbar D_0(k + k')^2} \tag{109}$$

where we have used the Einstein relation for the classical conductivity (106), $\sigma_0 = e^2 n(E) D_0$, such that

$$D_0 = \frac{1}{2} \frac{\hbar E S_d}{dm\gamma}. \tag{110}$$

7.4.3. The configurationally averaged two-electron Green's function. From the Bethe–Salpeter equation (100) one can obtain the averaged two-electron Green's function by inserting (109) and iterating. The resulting geometrical series may again be summed to give

$$\langle G^+(k + q, k' + q) G^-(k', k) \rangle = \frac{\langle G^+(k + q) \rangle \langle G^-(k) \rangle}{1 - \sum_Q U(Q) \langle G^+(Q - k + q) \rangle \langle G^-(Q - k) \rangle}. \tag{111}$$

The product of the averaged one-electron Green's function may be written as

$$\langle G^+(k + q) \rangle \langle G^-(k) \rangle = \frac{\langle G^-(k) \rangle \langle G^+(k + q) \rangle}{2i\eta - \hbar^2(k \cdot q)/m - \Delta\Sigma(k)}. \tag{112}$$

Power expansion with respect to q yields in lowest approximation

$$\langle G^+(k + q) \rangle \langle G^-(k) \rangle \approx 2i\rho(k, E) \frac{1}{2i\eta - \Delta\Sigma(k)} \left(1 + \frac{1}{2i\eta - \Delta\Sigma(k)} \frac{\hbar^2}{m}(k \cdot q) \dots \right) \tag{113}$$

where

$$\rho(k, E) = \frac{\eta + \gamma}{(E - \hbar^2 k^2/2m)^2 + \gamma^2} \tag{114}$$

is approximately, for sufficiently small γ, the spectral function corresponding to (93). It is strongly peaked at $E = \hbar^2 k^2/2m$ such that we can write

$$\sum_k \rho(k, E) h(|k|) \approx \pi n(E) h(\sqrt{E}) \tag{115}$$

where $n(E) \equiv \sum_k \rho(k, E)/\pi$ is the density of states and $h(|k|)$ is a smoothly varying function of $|k|$.

7.4.4. The weak localization correction to the conductivity. In order to evaluate the weak localization correction to the conductivity we have to insert (113) into (111), perform the differentiation with respect to q, and take the limit $q \to 0$ retaining only terms $\propto \eta^2$. The result is

$$\lim_{q \to 0} \frac{d^2}{dq^2} \langle G^+(k+q, k'+q)G^-(k', k) \rangle = 2\rho(k, E) \frac{\hbar D_0}{\gamma^2} \sum_Q U(Q) \frac{1}{4\eta^2}. \tag{116}$$

In the derivation we have used the Ward identity (107) and the fact that, in lowest order, $\Delta \Sigma(k) \approx -2i\gamma$ as well as that the sum over Q is dominated by the contribution from $Q \approx 0$. Finally, the correction to the Drude conductivity is given by

$$\delta\sigma = -\frac{4e^2}{h} \hbar D_0 \sum_Q \frac{1}{\eta + \hbar D_0 Q^2}. \tag{117}$$

Note that the sum on the right-hand side of this equation diverges for $\eta \to 0$. Therefore, in order to guarantee that $\delta\sigma \ll \sigma_0$, η may not be taken very small. Furthermore, although we have used the sum of all the diagrams in figure 26(*b*), including the lowest-order diagram which on its own yields σ_0, nevertheless our calculation only gives $\delta\sigma$ since we have considered only the $Q \approx 0$ contribution.

7.4.5. The temperature dependence of the weak localization correction. The quantum mechanical correction $\delta\sigma$ of (117) allows us to discuss the low-temperature dependence of the DC conductivity of weakly disordered metallic systems. We introduce phenomenologically a phase coherence time τ_ϕ induced by inelastic phase-breaking processes, such as electron–phonon and electron–electron scattering (Thouless 1977, 1979, 1980, Anderson *et al* 1979, Altshuler and Aronov 1985) by assuming

$$\eta = \hbar/\tau_\phi \tag{118}$$

where τ_ϕ depends on the temperature, for instance, as $\tau_\phi \propto T^{-p}$ (*p* of order unity). The corresponding diffusion length is then given by

$$\ell_\phi^2 = D_0 \tau_\phi. \tag{119}$$

Introducing these definitions into (117) we obtain

$$\delta\sigma(T) = -\frac{4e^2}{h} \frac{1}{(2\pi)^d} \int d^d Q \frac{1}{\ell_\phi^{-2} + Q^2}. \tag{120}$$

Evaluation of the integral gives

$$\delta\sigma(T) = -\frac{2e^2}{h} \begin{cases} \ell_\phi(T) - \ell & d = 1 \\ (1/\pi) \ln(\ell_\phi(T)/\ell) & d = 2 \\ (1/\pi^2)(\ell^{-1} - \ell_\phi^{-1}(T)) & d = 3. \end{cases} \tag{121}$$

It is seen that at low temperatures the quantum interference correction decreases the conductivity in one and two dimensions, whereas in three dimensions it eventually becomes unimportant when $\ell_\phi/\ell \gg 1$. The striking logarithmic correction in two dimensions has been clearly identified in many experiments, and was the starting point of the scaling theory of localization at the Anderson transition (cf sections 2, 3 and 8).

7.5. Negative magnetoresistance

The influence of a magnetic field on the transport can be studied quantitatively by using the diagrammatic approach. The qualitative features are most easily obtained by employing the real space picture (cf figure 23). As a starting point consider the transition amplitudes, which acquire additional phase factors in the presence of a magnetic field:

$$A(r, r') = A^0(r, r') \exp\left(\frac{ie}{\hbar} \int_r^{r'} A \cdot ds\right). \tag{122}$$

In the case of a closed loop the phase is $\phi = \int A \cdot ds = 2\pi \Phi/\Phi_0$. Φ is the magnetic flux through the loop and Φ_0 the flux quantum h/e. The return probability for a given path n, $t_n(r, r)$, is now

$$t_n(r, r) = t_n^0(r, r)[1 + \cos(4\pi \Phi/\Phi_0)]. \tag{123}$$

The second term arises from the interference between the two time-reversed paths discussed above. For small fluxes the cosine may be expanded. The return probability decreases with increasing magnetic field. Consequently, the transmission probability between two points must increase, and this induces an increase of the conductivity. This is the mechanism for the negative magnetoresistance observed for a long time but explained only in the course of the development of the theory of weak localization by Kawabata (1980) and Altshuler et al (1980).

    The complete quantitative theory of the magnetoresistance of 2D systems, including the effects of spin–orbit scattering, was formulated by Hikami et al (1980), and by Altshuler et al (1980), and belongs to the few items in the theory of localization that are repeatedly verified quantitatively in many experiments (see section 3) (Dumpich and Carl 1991, Carl et al 1989a, b, 1990, 1993).

7.6. Oscillations of magnetoresistance

Equation (123) shows an interesting feature. For a given path the contribution to the magnetoresistance oscillates as a function of the magnetic flux with a period that is given by $\Delta \Phi = h/2e = \Phi_0/2$. If it were possible to select experimentally only those paths whose areas are the same when projected onto a plane perpendicular to the direction of the magnetic field these oscillations should be observable. This effect was predicted to occur in thin metallic cylinders, when placed in a magnetic field parallel to the cylinder axis, by Altshuler et al (1981). Experimentally, the oscillations were first observed in thin-walled Mg and Li cylinders by Sharvin and Sharvin (1981) (see section 3). The experiment was repeated on Mg cylinders and Al cylinders by Gijs et al (1984a, b). A few additional results fitted quantitatively to the theoretically obtained formulae are shown in figures 27 and 28).

    Another possibility of selecting paths of a given area is to use small metallic rings (Webb et al 1985a, b, Chandrasekar et al 1985) and networks (Pannetier et al 1983, 1984a, b, Pannetier 1991, Licini et al 1985). Whereas the dominant period in the cylinder experiments is $\Phi_0/2$, the ring and network experiments also show periods of Φ_0. This can easily be understood by evaluating, according to (122), the phase difference between two paths entering a ring structure on one side and leaving on the other, but following the left and the right branch, respectively (figure 29). In the cylinder geometry the oscillations with period Φ_0 are suppressed by averaging along a cylinder. Each contribution involving two
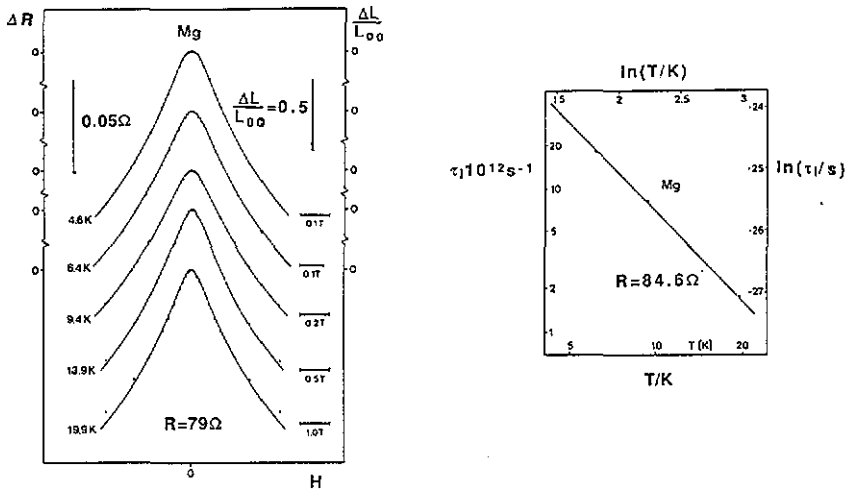
Figure 27. The negative magnetoresistance ΔR (left scale) as observed in a thin Mg film at different temperatures (left-hand part of the figure). Different temperatures correspond to different inelastic scattering times τ_i. They can be extracted from the experimental data by fitting to the theoretical results (right-hand side of the figure) (Bergmann 1984).
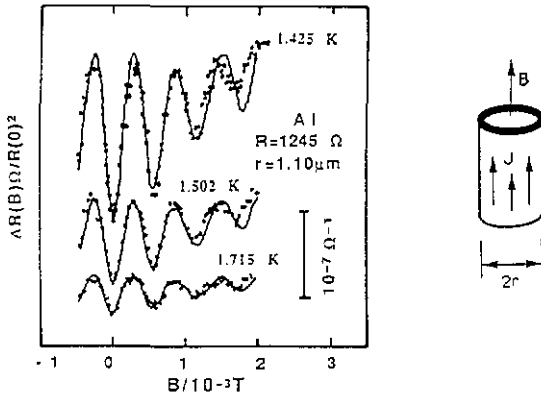
Figure 28. The oscillations of the magnetoresistance ΔR of an Al cylinder at different temperatures. Dots denote experimental data, curves are the fits to the theory of Altshuler *et al* (1981) (after Gijs *et al* (1984a)).

different sites (r, r') sustains an additional phase factor $\exp(\phi_{rr'})$ which, when averaged over (r, r') gives zero.

The important point in all of these interference experiments is that the electrons must be able to behave coherently around the circumference of the cylinder or ring. Therefore, the mean distance $L_\phi = \sqrt{D_\phi}$ between phase-breaking processes must be of the order of or larger than this circumference. This condition restricts the experiments to low temperatures and diameters of the order of 1 μm. As L_ϕ enters the theory as a parameter, the phase-breaking length and its temperature dependence may be determined by fitting to the experimental data.

Reviews of the work up to now have been given by Washburn and Webb (1986), Aronov
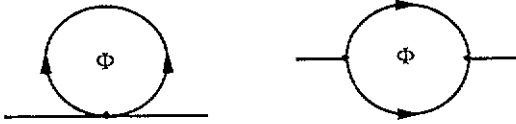
Figure 29. Paths that lead to magnetoresistance oscillations with periodicities of $\frac{1}{2}\Phi_0$ (left) and Φ_0 (right).

and Sharvin (1987), and van Haesendonck *et al* (1991).

8. The scaling approach

The basic problem in the evaluation of the critical properties of the conductivity and the localization length at the Anderson transition is that the closer one approaches the critical point, the larger the system size has to be chosen, in order to obtain meaningful results. More precisely, the calculation of the critical properties requires us to perform a thermodynamic limit in a controlled way. One possibility to achieve this goal is to apply scaling laws. That there are scaling laws in the present problem which can be successfully exploited can be seen by considering the limit of weak localization discussed in the previous section. The quantum correction to the mean free path result of the conductivity turned out to depend on the various parameters only via ℓ/ℓ_ϕ. By considering the phase coherence length ℓ_ϕ as an effective system size and noting that the mean free path ℓ is a measure of the disorder, one observes that changes in the size of the system can be compensated by changes in the disorder. On the other hand, in the asymptotic limit of strong localization (see section 6) the conductance turned out to depend only on the ratio L/λ, the ratio of the geometrical system size and the localization length.

It is natural to assume, as a hypothesis, that, as in the asymptotic regions, scaling behaviour can also be found in the vicinity of the transition. If this were the case, information about the critical behaviour could readily be obtained by exploiting the corresponding scaling law, as we shall see below. A much more difficult question to answer is whether or not scaling is in fact a valid concept. Up to now this latter problem, and the related problem of universality, has only been attacked, with some success, by using numerical methods.

8.1. Single-parameter scaling

Thouless (1974) introduced an argument which suggests that the conductance G_{2L} of a block of size $(2L)^d$ is related solely to the conductance G_L of the 2^d blocks of size L which are combined to build the larger block. We discuss here the most general form of such an argument, in order to derive some general conclusions and to arrive at some idea of the limitations.

We consider a set of properties of a system of size L^d which we represent by a vector a_L, whose elements are chosen to be dimensionless. It is often useful to think of L as an effective system size, such as the inelastic scattering length L_i or even the resolution with which an external observer could measure the system. We assume that the set a is complete in the sense that we can write

$$a(bL) = F(a(L), b). \tag{124}$$

The set a may well have to be infinitely large. In fact, in order to describe the distribution of values required for a full description of a disordered system it must be so, except in special cases.

At this point we make the first approximation, namely that equation (124) can be rewritten in differential form

$$\frac{\mathrm{d}a}{\mathrm{d}\ln L} = f(a). \tag{125}$$

This approximation may not always be valid. Indeed, in the closely related problem of quasi-crystals the fact that the mapping remains discrete is fundamental to the physical properties (Kohmoto *et al* 1987). Nevertheless, many of the results are independent of this approximation and the use of a discrete mapping would only lead to unnecessary complications.

Equations such as (125) have been extensively studied in the context of non-linear dynamics, chaos, etc, from which much of the language is derived, as well as in more conventional phase transitions. The most important property of such equations is that for increasing L the vector a tends towards a simple subspace, often a line or even a point. Such subspaces or points, a^*, are called *attractors* or *fixed points*, respectively. They are defined by the condition $f(a^*) = 0$. Asymptotically we can describe the behaviour of the system completely in terms of the properties of the attractor. It is important to remember, however, that this is really a description of the behaviour of almost infinite systems. It is not clear *a priori* whether any given real or numerical experiment is sufficiently close to the attractor that the deviations from it may safely be ignored.

Close to a fixed point (125) may be linearized to

$$\frac{\mathrm{d}a}{\mathrm{d}\ln L} = \mathbf{f}'(a - a^*) \tag{126}$$

where $\mathbf{f}'$ is the matrix of derivatives of f at a^*. The solutions of (126) are of the form

$$a = a^* + \sum_i (L/\xi_i)^{f_i'} a_i \tag{127}$$

where f_i' and a_i are the eigenvalues and eigenvectors of $\mathbf{f}'$. The constants of integration have been written in terms of length scales ξ_i which contain all the information about scattering rates, energy, etc. This guarantees that all terms are dimensionless as originally assumed. Consider now the effect on a of changing some parameter τ from its value at the fixed point τ^*. To linear order the second term in (127) must obey $\delta a \sim (\tau - \tau^*)$, which implies that ξ must obey

$$\xi_i \propto |\tau - \tau^*|^{-\nu_i} \tag{128}$$

where $\nu_i = -1/f_i'$. The exponents ν_i can be divided into two groups, termed *relevant* and *irrelevant*, according to whether $f_i' \gtrless 0$, and the flow in the corresponding direction is away from the fixed point or towards it, respectively. In fact, for large L the behaviour will be dominated by the component with the largest f_i' (figure 30).

For the moment we consider only the case where there is a single exponent $f_i' > 0$. This corresponds to the case where (125) has a 1D attractor. A theory based on this property is often called a *one-parameter scaling theory*. We then have a single relevant ν which we can identify as the critical exponent.
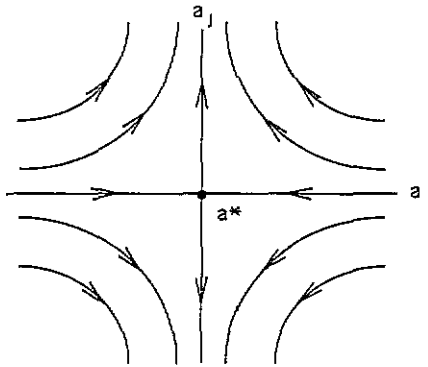
Figure 30. 2D flow diagram of a scaling procedure. The fixed point a^* is attractive and repulsive with respect to components a_i and a_j of the scaling variable, respectively.

The attractive line may be parametrized in several different ways. Conventionally the original suggestion of Thouless is followed and the single parameter used is the dimensionless conductance, $g = (h/e^2)G$, or more precisely its arithmetic mean, $\langle g \rangle$. In the following paragraphs we will use the abbreviation g instead of $\langle g \rangle$.

The equation describing the flow along the line is (Abrahams *et al* 1979)

$$\frac{\mathrm{d} \ln g}{\mathrm{d} \ln L} = \beta(\ln g). \tag{129}$$

On the other hand, most numerical simulations use the ratio of the localization length on a 1D strip to the width of the strip, $\Lambda = \lambda_M/M$ (MacKinnon and Kramer 1981).

This approach will only be valid as long as the effective system size L is larger than any other length scales associated with the irrelevant contributions in (127). In particular, the mean free path ℓ constitutes a lower limit for L.

Once we have identified a single parameter several results follow from quite general considerations. Firstly, the general solution of the 1D scaling equation (129) has the form

$$g = g(L/\xi). \tag{130}$$

Everything is defined in terms of a single length scale ξ. Note, however, that the function $g(x)$ may be multi-valued. There may be several values of g corresponding to the same value of x. Secondly, for very strong disorder, $g \ll 1$, we expect exponential localization, i.e.

$$g \sim \exp(-2L/\lambda) \qquad \beta(\ln g) \sim \ln g. \tag{131}$$

By comparing (130) with (131) we identify ξ as the localization length λ. Thirdly, for very weak disorder we expect classical Ohmic conductivity,

$$g \sim \sigma L^{d-2} \qquad \beta(\ln g) \sim d - 2. \tag{132}$$

Again, by comparing (130) with (132) we find that ξ is related to the conductivity σ by

$$\xi \sim \sigma^{-1/(d-2)}. \tag{133}$$

The β function can now be sketched as shown in figure 3. Note in particular that, for $\beta > 0$, g tends towards case (133), extended behaviour, whereas for $\beta < 0$, g tends towards the localized state. $\beta = 0$ represents a fixed point. In one dimension β is almost certainly always negative. The flow is always towards small g, i.e. localized states. In three dimensions β must cross zero. There is always a fixed point and a metal–insulator transition. Two dimensions represents the marginal case. It is impossible to tell whether β crosses zero without further information.

It follows from (128) that the exponent ν is independent of which side of the fixed point it is derived from. Hence, by using (133) we obtain a relationship between the conductivity exponent s in

$$\sigma \sim |\tau - \tau^*|^s \tag{134}$$

and the localization length exponent ν in

$$\lambda \sim |\tau - \tau^*|^{-\nu} \tag{135}$$

of the form

$$s = (d - 2)\nu. \tag{136}$$

This relation, originally derived by Wegner (1976), was earlier interpreted by many authors as a prediction of a minimum metallic conductivity, $s = 0$, for $d = 2$. Note, however, that this interpretation presupposes the existence of a fixed point on the 1D attractor.

8.2. Perturbation Theory

As described in the previous section, it is possible to calculate corrections to these results using diagrammatic perturbation theory. These corrections are equivalent to taking into account enhanced backscattering. From the quantum corrections to the classical conductivity one can derive the following form of the β function for the conductance:

$$\beta(\ln g) = (d - 2) - b/g. \tag{137}$$

Abrahams *et al* (1979) noted that this implies that β is always negative in two dimensions and hence that all states are localized even for infinitesimal disorder.

Using (137) and ignoring higher order terms in $1/g$ it is found that

$$s = \nu = 1 \tag{138}$$

in three dimensions. In fact, in the case of systems containing magnetic impurities or spin–orbit coupling (Schäfer and Wegner 1980, Hikami 1980, 1981, 1984a, b, 1986, Oppermann and Jüngling 1980, Jüngling and Oppermann 1980) the coefficient b in (137) may be zero. This leads to a more general result. When the leading term in an expansion in $1/g$ is $(1/g)^n$ then, again by ignoring all higher-order terms, we find

$$s = \nu = \frac{1}{(d - 2)n}. \tag{139}$$

In two dimensions on the other hand the solution of (137) can be written as

$$g = g_0 - b \ln L \qquad \text{or} \qquad - b \ln(L/\xi). \tag{140}$$

Clearly both forms are equivalent to that given for two dimensions in (121). The weak logarithmic dependence is often termed *weak localization*. It was discussed in the previous section.

In one diemnsion it is well known that all states are exponentially localized (see section 6). By combining this with the formula derived by Landauer (1970) connecting the conductance with the transmission coefficient (cf equation (69))

$$g = T/(1 - T) \tag{141}$$

where $T = \exp(-\gamma L)$, it is possible to derive a complete expression for the β function:

$$\frac{d \ln g}{d \ln L} = -(1 + g) \ln \left(\frac{1 + g}{g} \right). \tag{142}$$

The perturbation theory results are expected to be valid as long as the effective system size L, usually the inelastic scattering length, is larger than the mean free path ℓ. However, in one and two dimensions the correction to the classical Ohm's law behaviour increases with L so that eventually the theory must break down. We expect the length scales for this breakdown to be of the order of the localization length, which can be written as $\xi_{loc}^{1D} \approx \pi \ell$ and $\xi_{loc}^{2D} \approx \ell \exp(\frac{1}{2} \pi k_F \ell)$. These are perturbative estimates (Lee and Ramakrishnan 1985). In three dimensions the correction to σ goes as $(\ell^{-1} - L^{-1})$ which is small but significant since statistical finite-size fluctuations go at least as $L^{-3/2}$.

8.3. Bound for the critical exponent

There have been various attempts to estimate bounds for the critical exponents at the Anderson transition. Mott(1976, 1981), arguing from a model exhibiting minimum metallic conductivity, concluded that $v < 2/d$. Otherwise, fluctuations of the wavefunctions would smear out the transition. On the other hand, Harris(1974) used similar arguments and came to the conclusion that $v > 2/d$ in order for fluctuations to be irrelevant. Recently Chayes *et al* (1986), starting from the assumption of the validity of a one-parameter scaling law for a somewhat abstractly defined 'scaling' event, derived rather rigorously for a system with statistically uncorrelated randomness that $v > 2/d$. Note the contradictions in the sign of the inequality.

In the following we shall give an explicit derivation of the lower bound starting again from the assumption of the existence of a one-parameter scaling law, but for a self-averaging scaling variable, and presuming randomness which may be statistically correlated (Kramer 1993).

In order to generalize the lower bound it is assumed that a positive scaling variable $\Lambda(M) = \Lambda(M, \epsilon_1, \dots, \epsilon_N)$ exists. It is supposed to depend on a set of $N = N(M)$ random variables. M denotes the size of the system. Physically, the random variables may represent the values of a potential energy at certain sites, the positions of impurities, the values of exchange or bond matrix elements, or, more generally, N of the matrix elements of the Hamiltonian of the system when taken in a complete orthonormal basis.

$\{\epsilon_1, \dots, \epsilon_N\}$ are the members of a statistical ensemble described by a normalized distribution function $P(W, \epsilon_1, \dots, \epsilon_N) = W^{-N} f(\epsilon_1/W, \dots, \epsilon_N/W)$. W is the disorder parameter. P is assumed to be sufficiently rapidly decaying at infinity for each of the variables to be continuously differentiable, and bounded, such that all of its moments exist. For simplicity, it is also assumed that the odd moments vanish, and that the second moment of each of the variables is given by W^2.

The random variables may be statistically correlated. Specifically, the correlation function

$$K(j, j') = W^{-2} \int \cdots \int \prod_i d\epsilon_i \, \epsilon_j \epsilon_{j'} P(W, \epsilon_1, \ldots, \epsilon_N) \equiv W^{-2} \langle \epsilon_j \epsilon_{j'} \rangle \tag{143}$$

is supposed to be homogeneous, $K(j, j') = K(j - j')$, and to decay sufficiently rapidly for $|j - j'| \to \infty$ so that $\sum_{j'} K(j - j')$ exists. Consequently,

$$P(W, \epsilon_1, \ldots, \epsilon_N) = p_j(\epsilon_j) P_{N-1}(W, \epsilon_1, \ldots, \epsilon_{j-1}, \epsilon_{j+1}, \ldots, \epsilon_N) \tag{144}$$

whenever $|\epsilon_j - \epsilon_i| \to \infty \quad (i = 1, \ldots, N, i \neq j)$.

Consistent with experience from numerical scaling studies (MacKinnon and Kramer 1981, 1983a, Kramer *et al* 1990) the scaling variable is assumed to be self-averaging, i.e. its configurational average

$$\Lambda(M, W) \equiv \langle \Lambda(M, \epsilon_1, \ldots, \epsilon_N) \rangle \tag{145}$$

is identical to its most probable value in the thermodynamic limit. This implies that the configurational average of a function of Λ is the same as the function of its configurational average.

A further crucial assumption is that there is a one-parameter scaling function

$$\Lambda(M, W) = h(\xi(W)/M) \tag{146}$$

which can be expanded near the critical point, W_c, $\Lambda(M, W) = \Lambda_c - a(W - W_c)M^y$, ($\Lambda_c$ and a positive). The scaling parameter must diverge at W_c as $|W - W_c|^{-\nu}$ with $\nu = 1/y$.

From (145) $d\Lambda(M, W)/dW$ may be expressed by the derivative of the distribution function. Since Λ is positive, and self-averaging, an upper bound for $|d\Lambda(M, W)/dW|$ at W_c can be derived by using the Cauchy–Schwartz inequality,

$$\left| \frac{d\Lambda}{dW} \right|_{W=W_c} \leq \frac{\Lambda_c}{W} \left(N - \sum_{jj'} \langle \frac{\partial^2 \log P}{\partial \epsilon_j \partial \epsilon_{j'}} \epsilon_j \epsilon_{j'} \rangle \right)^{1/2}. \tag{147}$$

The first term on the RHS is the result for independent variables. The second is due to statistical correlations. If the latter does not increase faster than N

$$\left| \frac{d\Lambda}{dW} \right|_{W=W_c} \leq B \Lambda_c (N)^{1/2} \tag{148}$$

with a constant $B < \infty$. Proportionality with respect to N is ensured by homogeneity, convergence with respect to the summation over j' by the above requirements for the properties of the distribution function.

The statement of (148) is that, independently of the nature of the randomness, and of whether or not the system is interacting, there is an upper bound for the derivative of the scaling function at the critical point, if it exists. It is proportional to the square root of the number of random variables, provided the distribution function is bounded, continuously differentiable, with all of its moments existing, and the correlations are of finite range.

Near the critical point the dependence of the scaling variable on the size of the system is given by M^y. On the other hand, the number of the random variables must increase with

M, say as M^κ. In general, one cannot assume κ to be smaller than d, the dimensionality of the system, otherwise the 'concentration' N/M^d would vanish in the thermodynamic limit. Because of (148) we have $y \leq \frac{1}{2}\kappa$, or, equivalently, $\nu \geq \nu_0 \equiv 2/\kappa$, consistent with the earlier result for the uncorrelated randomness and $\kappa = d$.

It should be noted that the above derivation is valid as long as $a|W - W_c|M^y \ll \Lambda_c$. This will automatically be satisfied for the above bound close to W_c, with the choice $M \leq M_W = |W - W_c|^{-2/\kappa}$.

All of the presently available numerical data on the critical behaviour at the Anderson transition are consistent with this lower bound. On the other hand, the perturbative results obtained by using the non-linear σ model in connection with the ϵ expansion seem to suffer from extremely bad convergence properties of the series in ϵ ($= d - 2$). When only the corrections to the lowest-order results ($\propto \epsilon$) are taken into account the exponent does not satisfy the lower bound (Wegner 1989, Hikami 1990, 1991).

8.4. Field theoretical formulation

Shortly after the pioneering works on the scaling hypothesis (Wegner 1976, Abrahams *et al* 1979) considerable effort by many researchers was devoted to a formal justification of the underlying assumptions. Certain analogies between the present problem of the Anderson transition, an essentially quantum mechanical phenomenon, and classical phase transitions led to a considerable body of formal field theoretical approaches which culminated in a non-linear sigma model. Here we attempt only the briefest introduction to these ideas. For a full discussion we refer the reader to the review by Efetov (1983) and the lecture notes by Wegner (1979c).

Consider the integral form

$$\int_{-\infty}^{+\infty} D\phi^\dagger D\phi \exp\{i(\phi^\dagger[E + i\eta - \epsilon]\phi)\} = \frac{i\pi}{E + i\eta - \epsilon}. \tag{149}$$

Note that this is a 2D integral, because we integrate over ϕ and $\phi^\dagger$ separately. Note, also, that the integral only converges when $\eta > 0$ because the real part of the exponent is $-\eta\phi^\dagger\phi$. (149) can be generalized,

$$\int_{-\infty}^{+\infty} \prod_{ij}^{N} D\phi_i^\dagger D\phi_j \exp\{i(\phi_i^\dagger[E + i\eta - H_{ij}]\phi_j)\} = \frac{(i\pi)^N}{\det[E + i\eta - H_{ij}]}. \tag{150}$$

The proof of this relation is most easily performed by transforming to the complete system of eigenstates of the Hamiltonian, $H|\alpha\rangle = E_\alpha|\alpha\rangle$,

$$\phi_j \equiv \langle j \mid \phi\rangle = \sum_\alpha \langle j \mid \alpha\rangle\langle\alpha \mid \phi\rangle \tag{151}$$

and recalling that the N-dimensional volume elements in the integral are invariant under unitary transformations. Using (149) we see that the integral is proportional to $\prod_\alpha^N (E + i\eta - E_\alpha)^{-1}$, which is nothing but the result (150) since the determinant is also invariant under unitary transformations.

Using the same technique it is possible to show that the following relation for the matrix elements of the Green's function holds:

$$\int_{-\infty}^{+\infty} \prod_{ij}^{N} D\phi_i^\dagger D\phi_j \phi_m \phi_n^\dagger \exp\{\ldots\} = G_{mn}\frac{-i(i\pi)^N}{\det[E + i\eta - H_{ij}]}. \tag{152}$$

Note that this is nothing but the statistical average $\langle \phi_m \phi_n^\dagger \rangle$ where the integral in (150) represents a thermodynamic partition function.

The disorder enters the left-hand side of (152) only via the exponential function. Performing a configurational average would therefore be relatively straightforward, at least as long as the distribution of the matrix elements of the Hamiltonian is Gaussian (cf section 4). Unfortunately, such an average would not give the average of the Green's function since the right-hand side of (152) contains the determinant as well.

There are various tricks for getting rid of this problem (Schäfer and Wegner 1980, Wegner 1979a, Efetov *et al* 1980, Efetov 1980, 1983, Houghton *et al* 1980, McKane and Stone 1981). The *replica trick* (Schäfer and Wegner 1980, Wegner 1979a) consists in applying the above procedure to m equivalent systems (replicas). The result will include the unwanted factor raised to the power m. If we then take the limit $m \to 0$ we obtain $(\det[\ldots])^0 = 1$ and are left with the desired configurational average of the Green's function.

In the *supersymmetric* method (Efetov *et al* 1980, Efetov 1980, 1983) one introduces *Grassman, anticommuting* or *fermionic* variables. When the integral is evaluated using Grassman instead of normal *bosonic* fields, the determinant appears in the numerator and there are no problems with the convergence of the integrals (see below). By using fields with both commuting and anticommuting parts, *supersymmetric* fields, the problem of the determinant can be completely eliminated (Efetov 1983).

A calculation of the conductivity requires quantities like $G^+ G^-$ (see section 5), which cannot be calculated from the same partition function because the G^- part violates the condition for convergence of the integral. Instead, one calculates $\langle \phi_k^1 \phi_l^{\dagger 1} \phi_m^2 \phi_n^{\dagger 2} \rangle$ using the exponent

$$\sum_{\alpha=1}^{n} i(\phi_{i\alpha}^{\dagger 1}[E + i\eta - H_{ij}]\phi_{j\alpha}^1 - \phi_{i\alpha}^{\dagger 2}[E - i\eta - H_{ij}]\phi_{j\alpha}^2)$$

$$= i\left([E - H_{ij}]\sum_{\alpha=1}^{n}[\phi_{i\alpha}^{\dagger 1}\phi_{j\alpha}^1 - \phi_{i\alpha}^{\dagger 2}\phi_{j\alpha}^2] + i\eta\sum_{\alpha=1}^{n}[\phi_{i\alpha}^{\dagger 1}\phi_{i\alpha}^1 + \phi_{i\alpha}^{\dagger 2}\phi_{i\alpha}^2]\right)$$

(153)

which can be written as $\mathcal{H}_0 + \mathcal{H}_1$. $\mathcal{H}_0$ is invariant under a global transformation which leaves

$$I = \sum_{\alpha=1}^{n}[\phi_{i\alpha}^{\dagger 1}\phi_{j\alpha}^1 - \phi_{i\alpha}^{\dagger 2}\phi_{j\alpha}^2]$$

(154)

unchanged. $\mathcal{H}_1$ is invariant under the unitary group $U(2n)$ but, because of the minus sign, which comes from the requirement that the integrals converge, the symmetry group of $\mathcal{H}_0$ is the non-compact group $U(n, n)$, sometimes termed *pseudo-unitary*. Note that η plays the role of a symmetry-breaking field, analogous to a small magnetic field in a ferromagnet. The field conjugate to η is

$$\langle \phi_{i\alpha}^{\dagger 1}\phi_{i\alpha}^1 + \phi_{i\alpha}^{\dagger 2}\phi_{i\alpha}^2 \rangle = i\langle G(E + i\eta) - G(E - i\eta) \rangle$$

$$= 2\pi\rho(E)$$

(155)

where $\rho(E)$ is the density of states. Note the peculiar aspect of the analogy with the ferromagnet. In the problem of the mobility edge the order parameter is finite on both sides of the transition.

When the Hamiltonian is real the real and imaginary parts of the ϕ are independent variables. This additional symmetry means that instead of the group $U(n, n)$ we have the pseudo-orthogonal group $O(2n, 2n)$. If spin-flip scattering is introduced in the form of a constant or a random magnetic field, time-reversal symmetry is broken and the symmetry group reverts to $U(n, n)$. However, when the spin-flip scattering is due to spin–orbit coupling time-reversal symmetry is preserved and the group becomes symplectic $Sp(n, n)$. Thus the Hamiltonians may be classified into four *universality classes* (see table 1).

Table 1. Universality classes and symmetry groups

	Without spin	With spin
Time-reversal symmetry	Orthogonal $O(2n, 2n)$	Symplectic $Sp(n, n)$
No time-reversal symmetry	Unitary $U(n, n)$	Unitary $U(n, n)$

8.5. Results of the non-linear σ model

After considerable algebra (153) is transformed into the form

$$\tilde{\mathcal{L}}(\tilde{\mathbf{Q}}) = \int dr \{ \tfrac{1}{2} g \mathrm{Trg}(\nabla \tilde{\mathbf{Q}} \nabla \tilde{\mathbf{Q}}) + \eta \mathrm{Trg}(\tilde{\mathbf{Q}} \Lambda) \} \tag{156}$$

where the $\tilde{\mathbf{Q}}$ are $2n \times 2n$ matrices, which are derived from the combinations of the ϕ fields of the form $\phi_{i\alpha}^{s} \phi_{i\beta}^{\dagger s'}$, and Λ is a diagonal matrix with 1s in the positions corresponding to $s = 1$ and -1s elsewhere. The function $\tilde{\mathcal{L}}(\tilde{\mathbf{Q}})$ is the *effective Lagrangian* for the so-called *non-linear σ model*. It has been the subject of a large body of work by field theorists and its properties are generally well understood. However, it cannot be solved exactly, so that the results are effectively expressed in terms of an expansion in the coupling constant g^{-1} and $\epsilon = d - 2$ (compare (137)).

Until recently the situation was as follows. To leading order the critical exponents are $s = \nu = 1$ for the orthogonal case and $s = \nu = \tfrac{1}{2}$ for the unitary and symplectic cases (Brézin *et al* 1980, Hikami *et al* 1980, Wegner 1981, Jüngling and Oppermann 1980, Hikami 1980, 1983) in three dimensions. These exponents show a superficial agreement with experiment (see section 3), where exponents 1 or $\tfrac{1}{2}$ are found. However, it is difficult to understand how the degree of compensation of a semiconductor could be related to the presence of spin–orbit or magnetic effects. There have, however, been attempts (Lee *et al* 1987, Milovanovic *et al* 1989) to explain the effect in terms of the magnetic properties of a half-filled band.

In two dimensions the result of (137) survives for the orthogonal case. However, in the symplectic case (spin–orbit coupling) the coefficient b in (137) is negative (Jüngling and Oppermann 1980, Oppermann and Jüngling 1980) so that (140) now takes the form

$$g = g_0 + |b| \ln L. \tag{157}$$

Thus the conductance (or conductivity) increases logarithmically with increasing L. This is called weak anti-localization and has been confirmed experimentally by Bergmann (see section 3).

Until recently it appeared that these results remain unchanged by higher-order terms (Hikami 1983, 1985) until fourth-loop order. This includes the strange result that there is no fixed point (and hence no transition and no localization) for the 2D symplectic case. Evangelou and Ziman (1987) investigated this point numerically and found a small fraction of localized states. Liebert (1989) showed that the mobility edge coincides with a very rapid, possibly discontinuous, fall in the density of states, which has a tiny tail out to the Lifshitz limit (Englisch and MacKinnon 1990). Recently there have been some attempts to calculate the critical exponent for the 2D symplectic case (MacKinnon 1990, Ando 1989, Fastenrath 1992a, b).

However, it has been shown that the field theoretical result is flawed (Bernreuther and Wegner 1986, Wegner 1989) and that the corrections to fourth-loop order are of the same order as the previous terms. This was confirmed by Hikami (1990, 1991) in a different formulation by using *superstring theory*. Thus it seems that the series has clearly not yet converged.

Doubts about the validity of the one-parameter non-linear σ-model formulation of the metal–insulator transition problem, which is based on a theory for the average of the conductance, have also been raised by using more general arguments (Kravtsov and Lerner 1984, 1985, Kravtsov *et al* 1988, 1989, Shapiro 1986, 1987, Lerner 1991a, b). Since the statistical distribution of the conductance does not fulfil the *central limit theorem* (see section 10) the average of the conductance is not representative of the statistical ensemble. At least all of the higher moments of the distribution must be considered in order to formulate a scaling theory, which could then be of multi-parameter type. There is also an explicit calculation for an exactly solvable model (Bethe lattice) which seems to indicate that the one-parameter non-linear σ-model formulation of the problem of the disorder-induced metal–insulator transition is incorrect (Efetov *et al* 1980, Efetov 1984a, b, 1987a, b, 1990, Zirnbauer 1986a).

8.6. Results from numerical scaling calculations

Using the Anderson model with diagonal disorder, the present authors formulated a numerical scaling procedure for the average of the *logarithm of the quantum mechanical transmission probability*, $\log t$, through d-dimensional strips and bars of finite cross section and essentially infinite lengths (MacKinnon 1980, MacKinnon and Kramer 1981, 1983a). The transmission probability corresponds to the conductance through the system. Since the $\log t$ is a statistically well-behaved quantity and its configurational average is representative of the ensemble (see section 10), the above mentioned problems with the higher moments are avoided in the thermodynamic limit.

The method of computation is a generalization of equation (64) to matrices. Equivalently, one can use the transfer matrix formulation (see section 5.7) (Pichard and Sarma 1981a, b, MacKinnon and Kramer 1983a).

The systems considered are quasi-one-dimensional. For any finite cross section the localization length, $\xi_M = \lim_{L \to \infty} -2L/\log t$, is finite, and its statistical accuracy can be controlled by increasing the length L of the system. A set of raw data obtained for the 2D and 3D Anderson models with rectangular distribution of the site energies is shown in figure 31.

In order to be able to extrapolate to infinite system size ($M \to \infty$) it is necessary to investigate the scaling behaviour of ξ_M. In the centre of the band, and *close enough* to the transition, it turned out to be possible to establish a scaling function within the accuracy of
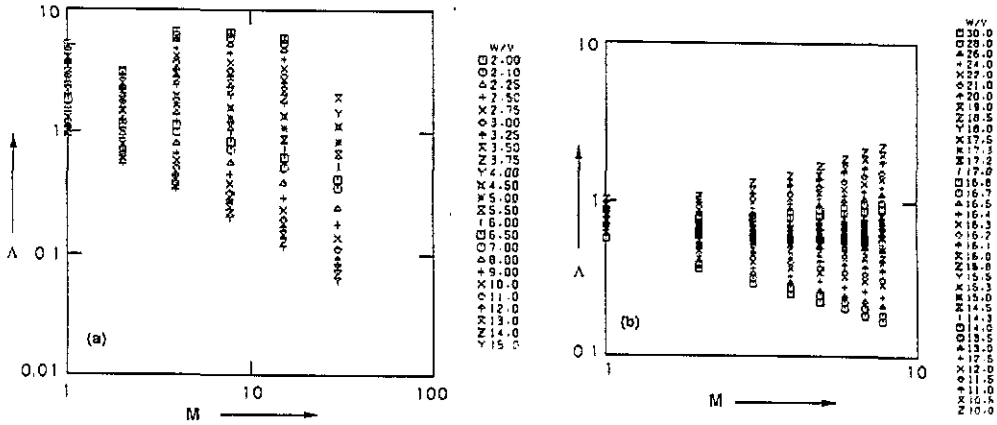
Figure 31. Raw data for the renormalized localization length, $\Lambda = \xi_M/M$, of an infinitely long bar of cross-sectional area M^{d-1} (cf figure 20) described by the Anderson Hamiltonian with box distribution of the diagonal elements. (*a*) Strip-like (2D), (*b*) bar-shaped (3D) geometry. Values of the disorder (widths of the distribution function) are as indicated. Energy is in the band centre.

the raw data,

$$\frac{\xi_M}{M} = f_d\left(\frac{\xi_\infty}{M}\right) \tag{158}$$

that depends on the dimensionality for a given universality class, and not on the disorder. The scaling function has the properties that were claimed in section 8.1 to be necessary for one-parameter scaling. Examples for the numerically determined scaling functions are shown in figure 32. The scaling parameter ξ_∞ is a function of the disorder. It corresponds to the localization length, and to the inverse of the conductivity in the localized and the metallic regimes, respectively.
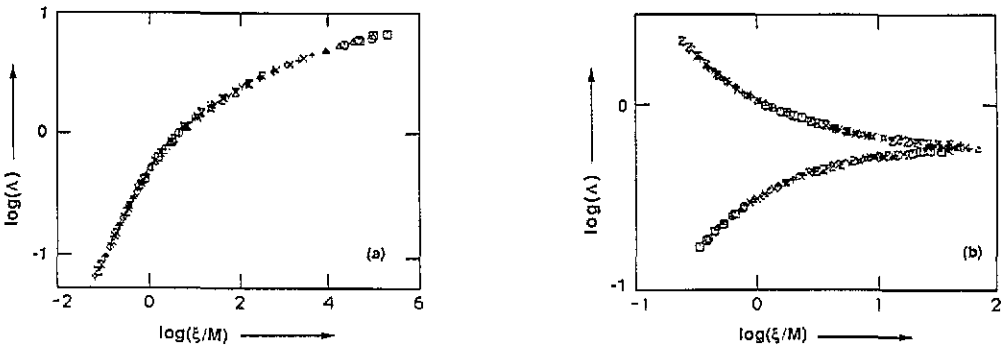

Figure 32. Numerically determined scaling functions for the 2D (*a*) and the 3D (*b*) Anderson models with rectangular distribution of the site energies. $\xi_\infty = \xi(W)$ is the scaling parameter necessary to scale the raw data of figure 31 onto the same curves.

For several orthogonal models, including the 2D and the 3D Anderson models with rectangular distribution of the site energies, and for the 3D Anderson model with Gaussian

distribution of the site energies, the scaling function, the phase diagram of localization (figure 33), and the critical exponents for the localization length and the conductivity were determined (MacKinnon and Kramer 1981, 1983a, MacKinnon 1985a, 1990, 1993, Bułka *et al* 1985, 1987, Schreiber *et al* 1989, Kramer *et al* 1990) (figure 34). Calculations far away from the band centre were also performed (Schreiber and Kramer 1987, Kramer *et al* 1990) (figure 35).
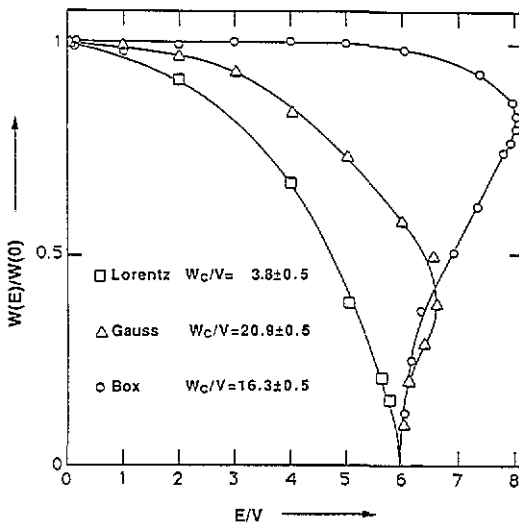
Figure 33. The phase diagram of localization, $W(E_C)$, for the Anderson model with box, Gaussian and Lorentzian distributions of site energies. Energy E is in units of V, the off-diagonal element of the Hamiltonian. $W(0) \equiv W_C$ are the critical disorders for the Anderson transition.

Figure 34. The scaling parameters ξ of the 3D Anderson model with box (left curve) and Gaussian distribution (right curve) of the site energies as functions of the disorder W/V.

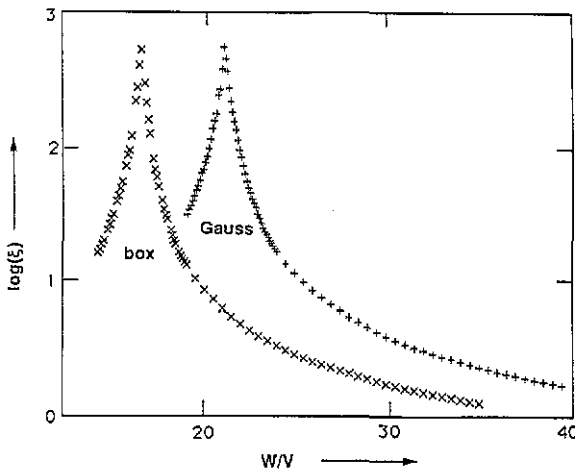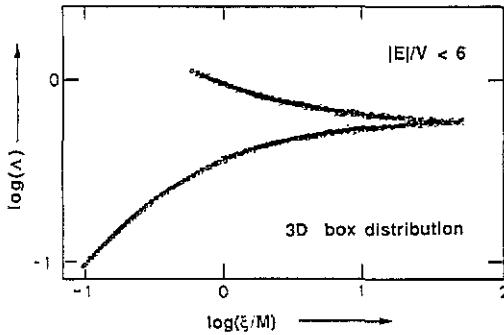**Figure 35.** The scaling function of the 3D Anderson model with diagonal disorder and energies $|E| \leq 6V$.

The results may be summarized as follows.

1. Within the statistical accuracy of the raw data (1%), in almost all of the cases the one-parameter scaling function can be established when sufficiently close to the transition. There is a quantitative criterium for the validity of the scaling behaviour.
2. The method yields complete localization in two dimensions with an essential singularity at zero disorder, and an Anderson transition in three dimensions.
3. Box and Gaussian distributions of the site energies yield the same scaling function. This is the explicit demonstration of the universality of scaling in the orthogonal class.
4. The localization phase diagrams show re-entrant behaviour near the unperturbed band edges. There are two transitions. One, at low disorder, corresponds to delocalization due to a strong increase in the density of states when increasing the disorder for a given energy. The other, at higher disorder, corresponds to localization due to quantum interference when increasing the disorder.
5. The critical exponents near the centre of the band are $s = \nu = 1.5 \pm 0.1$ for the box and the Gaussian distributions†. This is consistent with other numerical estimates (Sarker and Domanyi 1981, Economou *et al* 1985, Pichard and André 1986) but disagrees with the results of perturbation theory (Vollhardt and Wölfle 1982, Kroha *et al* 1990, Kroha 1990) and ϵ expansions (Wegner 1985) discussed above.

These are at present the only available *quantitatively controlled* results for the critical behaviour of the orthogonal class. They are consistent with one-parameter scaling for the average of the *logarithm of the conductance* instead of the average of the conductance *close to the transition*. At present there exists no direct theoretical possibility of treating the average of the logarithm of the conductance theoretically with the methods described above. Thus the numerical scaling method establishes, at least until now, the only source of information when the critical behaviour at the Anderson transition is to be investigated, and when comparisons with experiments have to be done. It seems, at least at this stage, that none of the experimentally observed metal–insulator transitions, provided that the analyses

---

† The disagreement between the exponents for the box and the Gaussian distributions reported previously (Kramer *et al* 1990) has been shown to be caused by the statistical error bars later obtained for the Gaussian model. Recent calculations in which the statistical error was taken as 0.1% confirmed that the models with diagonal disorder have the same exponents within the errors (MacKinnon 1993, Hofstetter and Schreiber 1993, Grußbach and Schreiber 1992).

of the critical exponents were correctly performed (Möbius 1988, 1989) can be explained by the orthogonal model without taking into account additional physical effects.

9. Localization in a magnetic field

The motivation to study localization in a magnetic field is two-fold. First of all, the application of an external magnetic field constitutes the most simple and controllable method of changing the universality class of a system. As a consequence, experimental investigations of the metal–insulator transition in the presence of a magnetic field were performed (Biskupski and Briggs 1988, Chen *et al* 1989, Katsumoto *et al* 1987, Kobayashi *et al* 1980, Hopkins *et al* 1989) (cf figure 10(*c*)). Secondly, in order to understand the quantized Hall effect (cf figure 36) (von Klitzing *et al* 1980, von Klitzing 1986), some knowledge of the basic localization features of the electronic states in 2D disordered systems is unavoidable (Aoki 1987).
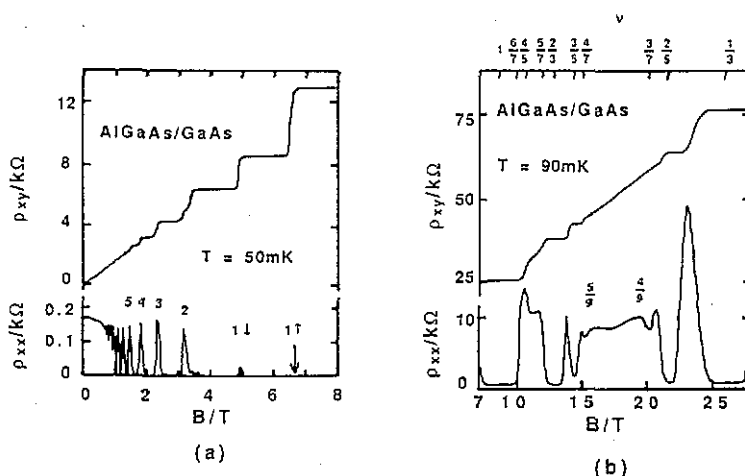

Figure 36. The quantized Hall effect as observed in AlGaAs/GaAs heterostructures of lower (*a*) and higher (*b*) electron mobility. The Hall resistivity ρ_{xy} as a function of the magnetic field B shows, at low temperature, distinct plateaux that are given by integer (*a*) and rational (*b*) fractions of h/e^2. Simultaneously, the dissipative part of the resistance, ρ_{xx}, shows pronounced minima. Changing the magnetic field is equivalent to changing the filling factor $\nu = neB/h$. The Hall plateaux occur at integer (*a*) and rational (*b*) filling factors (after Paalanen *et al* (1982) (*a*) and Chang *et al* (1984)).

The presence of a magnetic field introduces as an additional complication the non-trivial nature of the energy spectrum, and the states, in the limit of vanishing disorder. In the simplest case of an electron described in the effective mass approximation and moving in a 2D plane we have the degenerate spectrum of the discrete Landau levels and the corresponding Landau states. The discrete model (23) has an even more complicated spectrum. The magnetic subbands are broadened by the inherent periodic potential (Harper 1955) and show chaotic and self-similar features depending on whether the number of flux quanta per unit cell is rational or irrational (Hofstadter 1976, Wannier *et al* 1979). In three dimensions the bands are additionally broadened due to the kinetic energy in the direction of the magnetic field.

9.1. Summary of results for two dimensions

The random potential is specified as described in section 4. Especially in connection with the Landau model it is useful to consider a Gaussian correlated potential energy

$$\langle V(r)V(r')\rangle = \langle V^2\rangle \exp(-|r - r'|^2/a^2). \tag{159}$$

The special case of a Gaussian white noise potential ($a = 0$) as well as the more general case of a spatially correlated potential ($a \neq 0$) has been used in perturbation theories, and in the recently developed field theoretical treatments. In the asymptotic limit where $a \gg \ell_c$, where $\ell_c = (h/e|B|)^{1/2}$ is the cyclotron length, the electronic problem is equivalent to a classical percolation problem. In this case one can use the results of percolation theory in order to discuss the localization properties of the states. However, as we shall see later, it is imperative to study the influence of quantum mechanical effects (tunnelling and interference). One possibility is to put in these effects by hand, as has been done by Chalker and Coddington (1988) in a very interesting study. It would certainly be more satisfactory to have a model which is able to cover both of the limits $a/\ell_c \to 0$ (white noise limit) and $a/\ell_c \to \infty$ on an equal footing. A first attempt to develop such a model has been made by Huckestein and Kramer (1989, 1990), Miek (1990), Ono *et al* (1991) and Huckestein (1992).

Besides the size of the system which has to be taken to be infinite anyway, the physical situation described by the Hamiltonian is characterized by the magnetic length ℓ_c, and the spatial correlation length, a, of the potential. In the lattice model, (23), there is still another length scale, namely the lattice distance, which is usually taken as unity. *A priori*, it is not obvious whether or not these additional length scales are important. The validity of the one-parameter scaling hypothesis would imply that the spatial correlation of the potential energy is no longer of importance when the critical regime is approached (Sajeev and Stephen 1983, Johnston and Kramer 1986). One could then start without loss of generality from the Gaussian white noise potential. For non-vanishing magnetic field it was not clear until 1990 whether or not this is allowed. The fact that numerical data available at that time seemed to indicate a breakdown of the one-parameter scaling theory, according to the conclusions of the authors (Ando and Aoki 1985a, b, MacKinnon and Kramer 1983b, Ando 1985, MacKinnon 1989a, b, 1992) made it necessary to investigate the influence of a finite correlation length. In addition to the disorder W it was necessary to take a/ℓ_c as a physical parameter (Pruisken 1988).

The main results for the problem of localization in a strong magnetic field in two dimensions are obtained from perturbational, numerical and field theoretical treatments. The conductivity, the localization length, the Thouless number and the participation number were considered. In the limit of extremely high magnetic field percolation arguments have been used (Kramer *et al* 1989).

9.1.1. Perturbational approach. Most elaborate studies have been carried out using perturbation theory (Ono 1982a, b, 1983, 1984, 1985, Hikami 1984a, b, 1986, Hikami and Brézin 1985, 1986, Singh and Chakravarty 1986). The starting point was the density–density correlation function

$$\chi(r, r'; t) = i\theta(t)\langle[\rho(r, t), \rho(r', t)]\rangle \tag{160}$$

which can be evaluated systematically as a perturbation series for weak randomness by using diagrammatic techniques. The localization length is obtained from the static density

response function $\chi(q, 0)$ (Fourier transformed density correlation function) according to

$$\chi(q, 0) = n(E_F)\frac{q^2}{q^2 + \lambda^{-2}}. \tag{161}$$

The explicit result for the critical behaviour of the localization length for the case of δ-*function-like impurity potentials* is

$$\lambda(E) = \text{constant} \times \exp(A/(E - E_N)^2). \tag{162}$$

The constant A is of the order of the square of the disorder-induced bandwidth, and E_N is the energy of the unperturbed Landau level. There is an essential singularity of λ in the band centre in this approximation. Although there are a number of assumptions in this approach which are not easy to justify, this was the first hint obtained from theory that extended states might exist only at a single energy in each of the Landau bands in the quantum mechanical high-magnetic-field limit. The limit of high Landau level index was considered in extensive studies stressing the importance of quantum interference corrections for the diffusive transport in the quantum Hall effect regime (Benedict and Chalker 1986, Carra 1987, Chalker *et al* 1988, Carra *et al* 1989).

9.1.2. Thouless number study. One of the earliest numerical studies of localization was performed by Ando (1984, 1985) by applying the Thouless number criterion to the random Landau model (23). Calculations were done for δ-function potentials, as well as for impurity potentials of a finite range and for as many as three Landau bands. An inverse localization length $\gamma(E)$ was defined via the exponential behaviour of $g(L)$ (see section 5.6),

$$g(E, L) = g_0 \exp(-\gamma(E)/L). \tag{163}$$

In order to be consistent with the definition in the previous section we have to identify γ with $2/\lambda$. The results for the case of short-range scatterers were consistent with a divergence of the localization length only at the centres of the Landau bands, but $\gamma(E) \propto |E - E_N|^\nu$ with $\nu < 2$ in contrast to the essential singularity obtained from the renormalized perturbation expansion. In the case of finite-range potentials $(a = O(\ell_c))$ no conclusive result about the critical exponent could be obtained due to the error bars in the numerical data being too large.

9.1.3. The percolation limit. For $|B| \to \infty$ the quantum mechanical problem can be replaced by a classical percolation problem provided that the random potential has a finite correlation length. The basis of the percolation argument has been formulated by Tsukada (1976). It can be shown that the Schrödinger equation in the single-band approximation, which is justified in the high-field limit, can be written as

$$V(X, Y)C(X) = (E - E_0)C(X) \tag{164}$$

where X, Y are the centre coordinates of the cyclotron motion defined by $X = k\ell_c^2$ and $Y = -i\partial/\partial k$, respectively. k is the wavenumber of the Landau state and $C(X)$ is the expansion coefficient of the eigenstate with respect to the Landau basis. (164) holds in the limit of $\ell_c \ll a$. It is seen that all Landau states that correspond to the equipotential line $V(X, Y) = (E - E_0)$ contribute with an equal amplitude to the eigenstate at energy
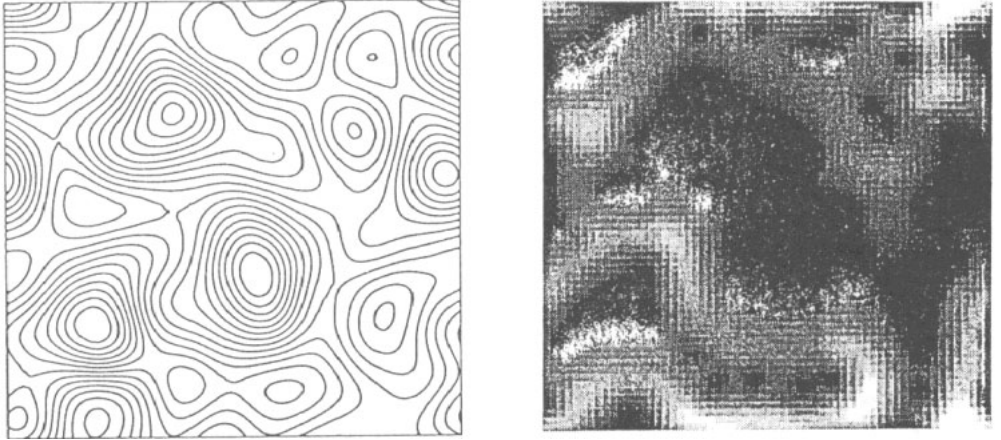
Figure 37. Numerically determined electronic eigenstate near the centre of a Landau band of a model with a Gaussian correlated random potential with correlation length $a = \ell_c$. Equipotential lines are shown on the left and the probability density of the eigenstate is shown on the right in a grey scale representation. Dark areas correspond to very small and light areas to high probability density. It is seen that the regions of high probability density are essentially given by the equipotential lines (courtesy of T Ohtsuki).

E (figure 37) (Tsukada 1976, Iordansky 1982, Kazarinov and Luryi 1982, Ono 1982a, Trugman 1983, Luryi and Kazarinov 1983, Apenko and Lozovik 1985).

The question whether or not an eigenstate is localized may be decided by investigating the percolation problem for the equipotential lines (Trugman 1983). An intuitive insight is obtained by considering the analogy with a hilly landscape (potential energy) that is gradually filled with water, the water level corresponding to the energy E. For very low water levels the water runs into the deep valleys. All shore lines (equipotential lines) are closed paths, i.e. the states are localized. The same is true for very high water levels. Only a few mountains are high enough to reach above the water level. The shore lines are again closed paths. It is clear that in between there must be one water level at which it is possible to travel either by boat or on foot from one side of the system to the other. This corresponds to an extended state. Percolation theory says that there is exactly one percolating path. The length t of an equipotential line is given by

$$t = |E - E_0|^{-1/\sigma} \tag{165}$$

with $\sigma = 36/91$. The diameter ξ of the area covered is

$$\xi = |E - E_0|^{-\nu} \tag{166}$$

with $\nu = \frac{4}{3}$ (Kramer *et al* 1988).

Equations (165) and (166) may be used to estimate the critical behaviour of the states. If we identify the localization length with ξ then the corresponding critical exponent is ν. Identifying the area covered by the cluster with $t\ell_c$ gives the possibility of deriving the critical behaviour of the participation number.

9.1.4. The participation number. The (inverse) participation number (ratio) has been studied for the random Landau model in different geometries using a variety of methods (Kramer *et al* 1988, Hikami 1986, Hikami and Brézin 1986, Aoki 1983b, 1984a, 1986, Kunz and Souillard 1982, Zirnbauer 1986b, Ohtsuki and Ono 1989). The shape of the wavefunctions were studied numerically (Aoki 1983a, 1984b, 1986, Ohtsuki *et al* 1992, Ono *et al* 1989, 1991). The spatial properties of the states turned out to be rather peculiar showing a remarkable self-similar network structure (figure 37). Fractal behaviour has been inferred near the critical energy. The fractal dimension d^* of the states near the centres of the Landau bands was estimated from the behaviour of the amplitudes of the wavefunction, $d^* \approx 1.6$ (Aoki 1986), and from the behaviour of the inverse participation number upon changing the system size (Ono *et al* 1989). It turned out that it was rather complicated to draw definite conclusions because of the limitations in the sizes of the systems considered. In any case, the fractal dimension of the states near the band centre seemed to be much smaller than $d = 2$. The extended states, if they existed, had therefore to be of the *spaghetti type*, i.e. not space filling. This conclusion was supported by percolation theoretical arguments (Kramer *et al* 1988). In this case one can derive the fractal dimension exactly, $d^* = 1.75$.

In a recent numerical study multi-fractal behaviour of the states near the centres of the magnetic bands was established for both the disordered Landau model and the Peierls model (figure 38) (Pook and Janßen 1991, Huckestein and Schweitzer 1991, 1992a, Fastenrath *et al* 1992).
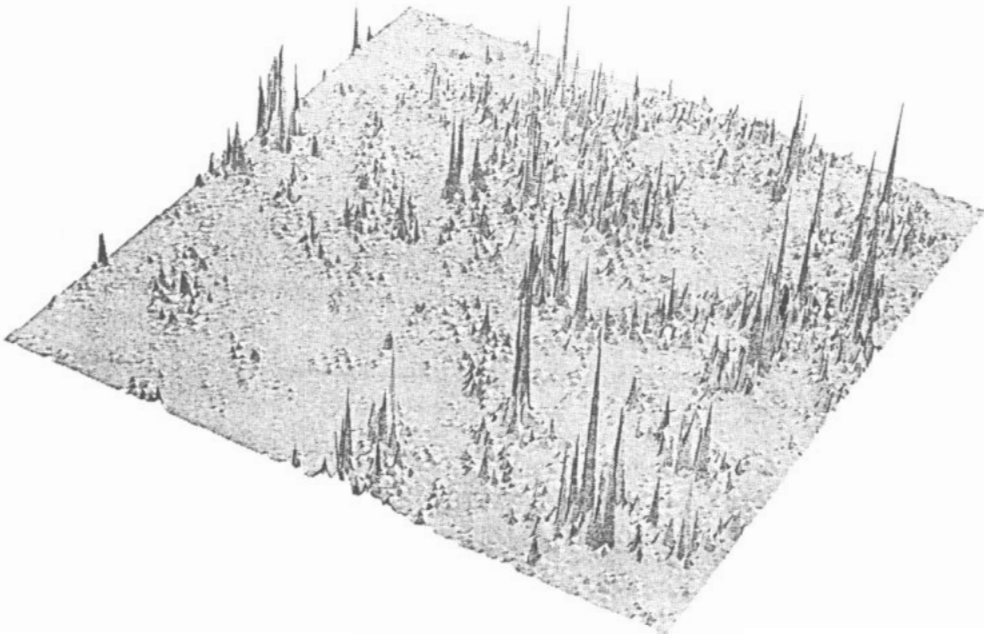

Figure 38. 3D plot of an eigenstate of the disordered Peierls model. Again the probability density is shown. It can be characterized as a multi-fractal (courtesy of B Huckestein and L Schweitzer).

The critical behaviour of the inverse participation number has been systematically studied for the lowest Landau level by Hikami using large-order perturbation theory (Hikami

1986, Hikami and Brézin 1986). Taking

$$P^{-1} = A |E - E_0|^\pi \tag{167}$$

he obtained $\pi = 3.8 \pm 0.4$. Assuming P to be given by the localization length, i.e. $P \approx \lambda^d$, this can be used to estimate $\nu = 1.9 \pm 0.2$. However, as mentioned above, this relation between the participation number and the localization length is not necessarily fulfilled.

9.1.5. Non-linear σ model. Field theoretical methods were successful in treating the localization problem $B = 0$. For $B \neq 0$ these methods have been used for the white noise limit (Pruisken 1984) and for the correlated potential (Zirnbauer 1986a, Weidenmüller 1987). The results are qualitatively consistent with the existence of singularities in the localization length as a function of the energy near the centres of the Landau bands. The essential point here is that an *effective Lagrangian* is derived by evaluating the configurational average over the randomness exactly (this is possible due to the assumption of a Gaussian distribution), and subsequently performing the remaining integration over the fermion fields approximately by using the saddle-point method after a transformation to boson coordinates. The effective Lagrangian contains two terms, one being related to the magneto-conductivity, and the the other to the Hall conductivity. The second of these is due to the *topological properties* of the model, and, according to the authors, not accessible to perturbational treatments. The two-parameter scaling picture (Khmelnitskii 1983, Pruisken 1984, Chalker 1987, Chalker and Daniell 1988), which can be derived by considering the two conductivity components as scaling variables, eventually leads to singularities in the centres of the Landau bands which are identified with a non-vanishing magneto-conductivity and divergences of the localization length.

9.2. Numerical scaling in the quantum Hall regime

The scaling properties of the asymptotic exponential decay length of the modulus of the Green's function, (33), with respect to energy, disorder and the width of the system have been studied for both the lattice model (Schweitzer *et al* 1984, MacKinnon *et al* 1984, MacKinnon and Kramer 1983b, Kramer and MacKinnon 1984) and the random Landau model. In the latter, δ-function potentials as well as Gaussian potentials have been considered (Ando 1982, 1983, 1985, 1987a, b, 1988, Aoki 1982, 1983c, 1985, 1987, 1988, Ando and Aoki 1985a, b, Aoki and Ando 1987). In a recent study the equivalent in Landau space of a Gaussian white noise potential was considered in a high-precision numerical scaling study (Huckestein and Kramer 1989, 1990,Miek 1990, Huckestein 1990, 1992). Numerically, the latter have been the most elaborate studies of the localization problem. The main results are described in the following.

9.2.1. Universal one-parameter scaling. The early attempts to apply this method to the 2D localization problem in the presence of a magnetic field were based on the lattice model. Attempts were made to find a one-parameter scaling relation of the type (158). This turned out to be successful to a certain degree. However, due to the small system sizes, localized states could only be established in the outermost parts of the magnetic subbands. In the centres of the bands finite energy regions were found where the states are larger than the diameter of the systems considered. No serious attempt was made at that time to extract the critical exponent, although the data seemed to be consistent with results obtained from the Thouless number study. There are theoretical reasons for the belief that the one-parameter scaling relation, (158), has to be replaced by a two-parameter scaling law in the presence of

a magnetic field. An attempt has been made to establish such a relation on the basis of more refined numerical data for the lattice model (MacKinnon 1989a, b). The data seemed to be consistent with a singularity of the localization length near the band centres, and $\nu = 2$. There are, however, still inconsistencies that have to be removed by future work.

The application of the scaling method to the random Landau model yielded the most precise and reliable information on the nature of the singularity in the centre of the band, although here the situation is more complicated due to the large number of coupling matrix elements between the Landau states. For the equivalent in the Landau space of the Gaussian white noise model, the attempt to establish a one-parameter scaling relation

$$\frac{\xi_M}{M} = f\left(\frac{M}{\xi(E)}\right) \tag{168}$$

for the logarithm of the transmission probability through a system of length k ($\to \infty$) and width M

$$\xi_M^{-1} = -\lim_{k \to \infty} \frac{\pi}{k} \log |G(E; 0, k)| \tag{169}$$

was successful (Huckestein and Kramer 1990) (figure 39). This was achieved due to the possibility of treating larger system sizes than before and using a new evaluation procedure for the data, borrowed from metrology (Huckestein 1990). The critical exponent for the lowest Landau band was extracted with an accuracy of better than 2%, $\nu = 2.34 \pm 0.04$. Comparison of the data with those obtained from different models (Huo and Bhatt 1992) including the quantum mechanically treated quasi-classical percolation limit (Chalker and Coddington 1988, Milnikov and Sokolov 1989) indicates that this result being independent of the microscopic details of the potential is truly universal. Thus, these numerical studies represent the second case in which the universality of the critical properties could be explicitly demonstrated.
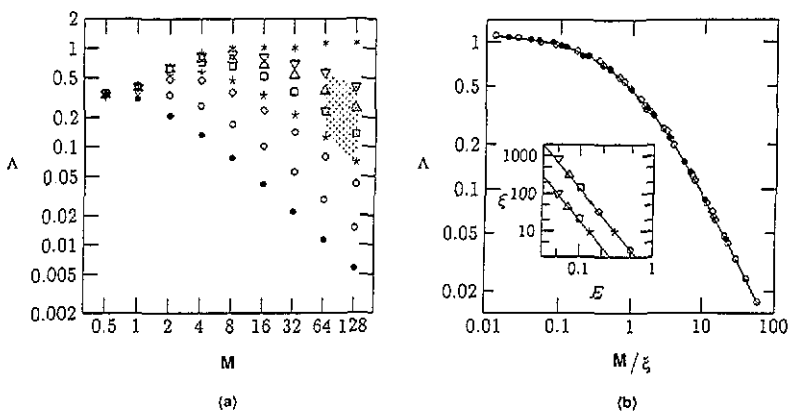

Figure 39. (*a*) Scaling in the quantum Hall regime. The renormalized exponential decay length, λ_M/M, is shown for a correlation length of the potential $a = \ell_c$ and various energies E. (*b*) λ_M/M for $a = 0$, $a = \ell_c$ and $a \to \infty$ (Chalker and Coddington 1988) scaled onto a single function. The inset shows the scaling parameters $\xi(E)$ for $a = 0$ and $a = \ell_c$ necessary to achieve the fit. The critical exponent is $\nu = 2.35 \pm 0.03$ (Huckestein *et al* 1991a).

9.2.2. Comparison with experiment. The results of the scaling approach can be compared with experimental data when identifying the size of the system with the temperature-dependent phase coherence length L_ϕ (see sections 3 and 7). This idea (Wei *et al* 1988, 1990), although not being particularly well supported theoretically for systems in the localized regime, yields a striking agreement with the results of temperature- and frequency-dependent measurements of the transport properties in the quantized Hall regime (Ebert *et al* 1982, Wei *et al* 1988, Lim *et al* 1990).

Since the scaling function of (168) depends only on the variable $x = EM^{1/\nu}$ near the centre of the band, the derivatives $d^n f/dE^n \propto M^{n/\nu}$ near $E = 0$. Near the band centre $E \propto B - B^*$ for fixed particle density, where B^* is the magnetic field at half-integer filling. Letting $M \equiv L_\phi(T) \propto T^{-p/2}$, it follows that if the nth-order derivative of the scaling function has extrema at B^* they must diverge according to $T^{-n\kappa}$, where $\kappa = p/2\nu$. Assuming further that the magnetoresistance and the Hall resistance depend on the same variable x, then by comparison with the measured temperature dependences (figure 40) which can be fitted with $\kappa = 0.42$, one obtains $p = 2.0 \pm 0.2$.
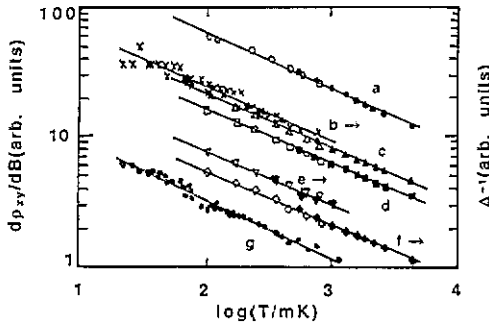

Figure 40. The derivative of the Hall resistivity of an InGaAs/InP sample, $d\rho_{xy}/dB$, and inverse of the widths of the corresponding peaks in ρ_{xx}, Δ, as a function of the temperature T. The double logarithmic plot shows linear behaviour of all of the data with the same slope, $\kappa \approx 0.4$. Lines b and g are obtained from an AlGaAs/GaAs sample at filling factor $\frac{1}{3}$. All other lines are for integer filling factors (after Wei *et al* (1988) and Engel *et al* (1990)).

The result for p is presently theoretically not understood, but experimentally it seems to be sample dependent. The exponent ν of the localization length was recently confirmed directly experimentally (Koch *et al* 1991).

Experimentally, the above value of κ was obtained for a variety of magnetoresistance peaks corresponding to integer filling factor as well as a number of fractional fillings (Engel *et al* 1990). This strongly suggests that the above scaling picture may also be valid in the regime of the fractionally quantized Hall effect.

The temperature dependence of the width of the plateaux corresponding to integer quantization of the Hall resistance has also been determined (Huckestein *et al* 1991b). Noting that the phase coherence length introduces effective (temperature-dependent) mobility edges into the system (figure 41), the plateau widths as a function of the temperature can be calculated from the number of states between the two successive mobility edges belonging to two successive Landau bands (Huckestein and Kramer 1990). The result is shown in figure 42.
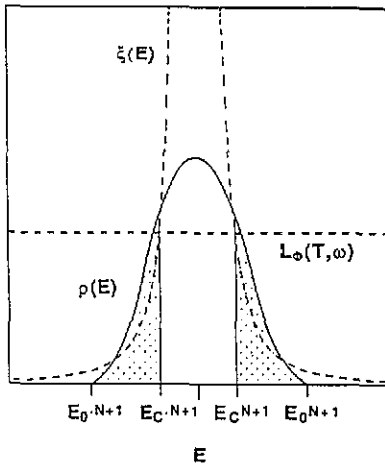
Figure 41. Model for the explanation of the temperature dependence of the Hall plateaux. The phase coherence length $L_\Phi(T, \omega)$ introduces effective mobility edges E_C and E_C' into the system. They are defined by the condition $\xi(E) = L_\Phi(T, \omega)$. Only electrons with $\xi > L_\Phi$ contribute to transport. $\rho(E)$ is the schematic density of states.
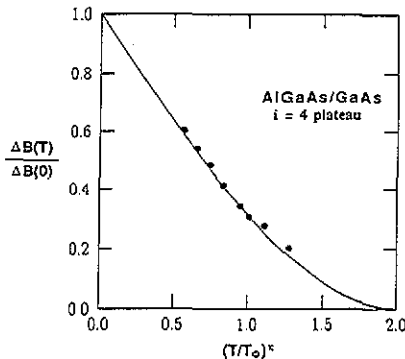

Figure 42. Width of the $i = 4$ Hall plateau in AlGaAs/GaAs as a function of the temperature T. Experimental data are represented by dots. The theoretical curve was obtained by taking $\kappa = 0.4$ and adjusting T_Φ (Huckestein *et al* 1991b).

It seems that a first step towards a quantitative and predictive understanding of the quantized Hall effect has been made—*via the, at the first glance, rather abstract realms of the scaling picture.*

9.3. Magnetic-field-driven metal–insulator transition

As mentioned above, quite a variety of experimental efforts have been devoted to the investigation of the MIT in 3D systems in the presence of a magnetic field. Clear experimental evidence has been produced for the existence of an MIT driven by the magnetic field (Biskupski *et al* 1984, Spriet *et al* 1986, Biskupski and Briggs 1988). It has been demonstrated that the mobility edge trajectory shows re-entrant behaviour for small B. There are two transitions, insulating to metallic and metallic to insulating, when B is

increased. Some of the experiments seemed to indicate a change in the critical exponent for $B \neq 0$ (Shafarman *et al* 1986, Castner *et al* 1987). Others were consistent with a critical exponent that remained constant when switching on the magnetic field (Mansfield *et al* 1985, Katsumoto *et al* 1987, 1989, Katsumoto 1991). As mentioned previously (see section 8) field theory, on the basis of the ϵ expansion, yielded that s should change from 1 (orthogonal class) to $\frac{1}{2}$ (unitary class) when a magnetic field is applied. To make the story complete, in the presence of interactions the corresponding scaling theories predicted, again on the basis of the ϵ expansion, that $s(B) = s(0) = 1$ (Finkelstein 1984c, Castellani *et al* 1984, Raimondi *et al* 1990). Thus, whether or not the critical exponent changes when applying a magnetic field could be an indication of whether or not interactions are important at the MIT.

Using the numerical version of the scaling theory, the scaling properties of the logarithm of the transmission probability of a model that is a combination of a Landau model (for the motion of the electron within 2D planes) and a tight-binding model for the electronic motion perpendicular to the planes, in the direction of the magnetic field, was performed (Ohtsuki *et al* 1992, 1993). It turned out that again, when close enough to the critical point, the various raw data obtained for different combinations of the system size, the energy and the disorder obey a one-parameter scaling law (figure 43) that is very similar, even in its quantitative aspects, to the one obtained for $B = 0$. The phase diagram for localization turned out to be consistent with re-entrant behaviour for not too large B that can be traced back to density-of-states-induced delocalization, as obtained previously for the Anderson model (see section 8). Moreover, the critical exponent obtained, $s = \nu = 1.3 \pm 0.2$, was, within the errors, the same as obtained for the orthogonal model. This result was corroborated in a recent study for the 3D Anderson–Peierls model (Henneke *et al* 1993).
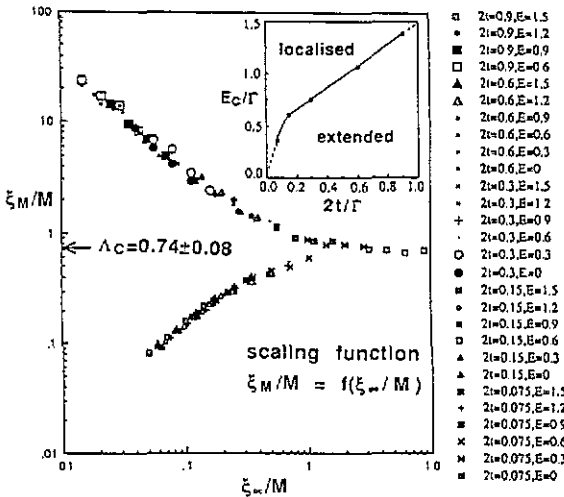

Figure 43. The scaling function of a 3D disordered system in the presence of a strong, quantizing magnetic field. $\xi_\infty = \xi_\infty(E, t)$ is the scaling parameter, t denotes the off-diagonal element of the Hamiltonian parallel to the magnetic field. The inset shows the phase diagram of localization, $E_C(t)$, close to the edge of the band. Γ is the bandwidth induced by the disorder alone (Ohtsuki *et al* 1992).

Thus, *if scaling exists and is universal for $B = 0$ there is no reason to believe that the non-interacting orthogonal and unitary classes give different critical exponents* on the basis of

the presently available numerical data, in contrast to the analytical results mentioned above. However, as the latter suffer from the fact that the approximations involved (perturbation expansion, ϵ expansion) are at present not very well controlled (Wegner 1989), and that the validity of the scaling approach seems questionable for the average of the conductance (Lerner 1991a, b), there is, in our opinion, presently a small imbalance in favour of the numerical scaling method. Although still suffering from relatively large errors ($\approx 20\%$) in the determination of the exponent, it has been successful in establishing scaling—but not for the conductance—and is in principle, as well as in practice, quantitatively controllable.

If the numerical results are correctly interpreted, namely that they indicate no essential difference between the critical exponents in the orthogonal and the unitary classes, then there is a high probability that the experimental data that are consistent with $s(B) = s(0)$ (Katsumoto *et al* 1987, 1989, Katsumoto 1991) indicate the presence of a generic disorder-induced MIT in the $Al_xGa_{1-x}As$ system. It should then be of extraordinary interest to measure the critical behaviour of doped semiconducting systems like Si:P in the presence of a magnetic field (Stupp 1992).

10. Fluctuations

We have seen above that the average conductance and the inverse of the average resistance of 1D disordered systems do not agree with each other (75). They are not self-averaging in the sense of statistical physics, namely, that their relative fluctuations vanish in the thermodynamic limit as the square root of the number of degrees of freedom. As in one dimension, the localization length is always finite, irrespective of the energy and disorder, and it is tempting to conclude that non-self-averaging of transport is an intrinsic property of the localized regime in two and three dimensions also.

In the metallic limit the conductivity may be calculated from a configurational average of a quantum mechanical transition probability, as we have seen in section 7. Due to the configurational average, most of the interference terms do not contribute to the transition probability (76). If we consider the conductance of a given sample, however, the interference terms dominate the total transition probability in a random manner. As a consequence, microscopic changes of the random potential in an impure metal of finite size should result in large changes of the (coherent) conductance. Thus, even in metals it is by no means guaranteed that quantum coherent transport is self-averaging.

It is also not clear what the consequences of non-self-averaging for the nature of the Anderson transition would be. The study of the distribution functions of the physical quantities of interest is therefore of crucial importance for a thorough understanding of Anderson localization.

10.1. The statistics of transport in 1D disordered systems

The central limit theorem for the localization length has been shown to be valid for the disordered harmonic chain by O'Connor (1975). Approximate treatments for the electronic problem have been given by Anderson *et al* (1980) using a Landauer-type approach, and by Mel'nikov (1981) by estimating the distribution function of the resistance, and calculating from that the distribution function of its logarithm. The case of a Gaussian white noise potential has been treated by Abrikosov (1981), and by Kree and Schmid (1981). Numerical results have been obtained by Andereck and Abrahams (1980), Sak and Kramer (1981), and by Kantor and Kapitulnik (1982). Tankei and Takano (1986) present somewhat different

results in the limit of large disorder. The method of O'Connor can be directly applied to the Anderson model by identifying

$$\ln(a_{j+1}^2 + a_j^2) = \ln(t_{j+1}^2) \equiv x_{j+1} \tag{170a}$$

$$a_j/a_{j-1} = \tan\theta_j \equiv y_j. \tag{170b}$$

x_j, y_j obey the recursion relations

$$x_{j+1} = x_j + \ln\left(\frac{y_j^2}{1+y_j^2}(1+y_{j+1}^2)\right) \tag{171a}$$

$$y_{j+1} = (E - \epsilon_j) - y_j^{-1}. \tag{171b}$$

This yields the most important result that the logarithm of the resistance (or of the conductance, alternatively) is a statistically well-behaved quantity, its limiting distribution function being approximately a Gaussian with a finite variance. The corresponding relative average fluctuations can be calculated explicitly by using approximative methods (Kree and Schmid 1981, Mel'nikov 1981, Abrikosov 1981, Tankei and Takano 1986), and by numerical procedures (Andereck and Abrahams 1980, Kantor and Kapitulnik 1982). They *decrease* with increasing length of the system, i.e.

$$\frac{\langle(\Delta \ln r)^2\rangle^{1/2}}{\langle \ln r \rangle} = (2/\gamma N)^{1/2} \qquad \gamma N \gg 1 \tag{172}$$

where the notation is as in section 6. This holds for small γ (Sak and Kramer 1981, Abrikosov 1981, Tankei and Takano 1986). In the limit of large γ, however, there seem to be deviations from this behaviour (Sak and Kramer 1981, Tankei and Takano 1986, Shapiro 1990, Slevin and Pendry 1990, Slevin 1991). The reasons are not yet fully understood, but they seem to be associated with the localization length becoming smaller than the lattice constant (Slevin and Pendry 1990, Slevin 1991).

As the distribution of $\ln r$ is asymptotically well behaved, it is intuitively clear that the resistance as well as the conductance must have statistical distributions which yield asymptotically divergent fluctuations. This can be verified by considering the average of the square of the resistance $\langle r(L)^2\rangle$ which is given by

$$\langle r(L+1)^2\rangle \propto \langle(\mathrm{Tr}\, \mathbf{U}_{L+1})^2\rangle \equiv \langle \mathrm{Tr}\, \mathbf{U}_{L+1}^{(2)}\rangle. \tag{173}$$

$\mathbf{U}_{N+1}^{(2)}$ is given by the recursion relation

$$\langle \mathbf{U}_{L+1}^{(2)}\rangle = \langle(\mathbf{T}_{L+1}^\dagger \otimes \mathbf{T}_{L+1}^\dagger)\langle \mathbf{U}_N^{(2)}\rangle(\mathbf{T}_{L+1} \otimes \mathbf{T}_{L+1})\rangle. \tag{174}$$

The asymptotic behaviour of the average square of the resistance is, as in the case of the average resistance, determined by the largest eigenvalue of this recursion relation, λ_2, i.e.

$$\langle r(L)^2\rangle \propto \exp(L \ln \lambda_2). \tag{175}$$

For small disorder a straightforward calculation gives

$$\lambda_2 = 1 + \sigma_2\sqrt{3}. \tag{176}$$

Thus, the second moment of the resistance grows more rapidly with the length of the system than the average resistance itself (Abrahams and Stephen 1980, Stone and Joannopulos 1982). Similar conclusions can be drawn for the higher moments of the resistance and for large disorder. A full account of the asymptotic behaviour of all the moments of the resistance and the conductance has been given by Kirkman and Pendry (1984a, b), and by Abrikosov (1981) and Mel'nikov (1981) by using approximative methods.

The result (175), together with (176), implies that the root mean square (RMS) fluctuations of the resistance grow exponentially with the length of the system. The resistance is not self-averaging. The physical reason for this behaviour is the exponential increase of the resistance in the localized regime when increasing the size of the system. Small statistical fluctuations of the localization length in the exponent will thus cause exponentially large changes of the resistance. Similar statements hold also for the conductance.
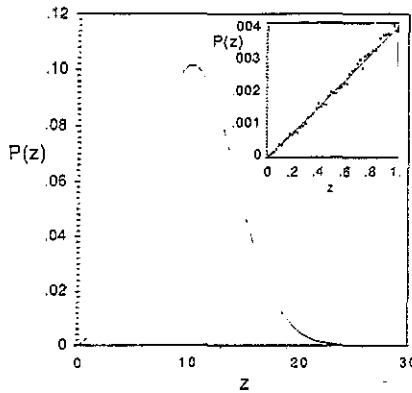

Figure 44. The probability distribution of the inverse localization length, $P(x)$, $x = 2\gamma L$, of a one-dimensional disordered system. It behaves approximately as a Gaussian around the most probable value and is linear for $x \to 0$ (inset) (Markoš and Kramer 1993a).

Recently, it has been shown that the fluctuations are governed by the equation (Pendry *et al* 1990, 1992)

$$\lim_{L \to \infty} \frac{\langle (tt^\dagger)^n \rangle}{\langle tt^\dagger \rangle} = C_n \tag{177}$$

where L is the length of the system and t is the (current amplitude) transmission coefficient. This result has been interpreted in terms of *maximal fluctuations*. Naively, C_n should tend to zero. However, if $tt^\dagger$ only takes the values 1 or 0 then C_n would be unity. That C_n is finite suggests that this is in fact a very good approximation to the truth, and that the fluctuations are the maximum compatible with $\langle tt^\dagger \rangle$. The result (177) may also be obtained by employing a different point of view, starting from the numerical observation (Markoš and Kramer 1993a) that the probability distribution of the inverse localization length $\gamma(L)$ (cf section 6) for large L is given by (figure 44)

$$P_\gamma^{(L)}(\gamma) \propto \gamma \exp\left(-\frac{(\gamma - \gamma_0)^2}{2\Delta^2}\right) \tag{178}$$

where

$$\gamma_0 = \langle\gamma\rangle(1 + O(1/L)) \tag{179}$$

and the variance Δ^2

$$\Delta^2 = \frac{2\gamma_0 a^2}{L}(1 + O(1/L)) \tag{180}$$

where a is a constant of order unity. Evaluation of the configurational average of the moments of the conductance

$$g \equiv \frac{2e^2}{h}\frac{1}{\cosh^2(\gamma L)} \tag{181}$$

(Pichard 1984) by the method of steepest descent around the maximum $\gamma_m \propto L^{-1}$ of the integrand ($a^2 > \frac{1}{2}$)

$$P_\gamma^{(L)}(\gamma)\cosh^{-2n}(\gamma L) \tag{182}$$

yields

$$\langle g^n \rangle = C_n \frac{1}{\langle\gamma\rangle L^{3/2}}\exp\left(-\frac{\langle\gamma\rangle L}{2a^2}\right) \tag{183}$$

with

$$C_n \propto n^{-1/2}\cosh^{-2n}(2). \tag{184}$$

This agrees for $n = 1$ with the result obtained analytically originally for weak disorder (Abrikosov and Ryzhkin 1978) and later more generally (Kirkman and Pendry 1984a, b, Roberts 1992) and confirms (177) for $n > 1$. Thus the 'maximal fluctuations' are seen to be related to the linear vanishing of the probability density of the inverse localization length for $\gamma \to 0$ since $\gamma_m \propto L^{-1}$. This strikingly demonstrates that the moments of the conductance are only determined by a vanishingly small fraction of the samples of the ensemble. These samples are very unrepresentative and have a conductance which is orders of magnitude larger than the most probable value, $g_0 \propto \exp(-2\langle\gamma\rangle L)$, which represents *the experimentally accessible value*.

   Although there are important quantitative aspects to be resolved in the future there is a probability that the reproducible conductance fluctuations observed originally in quasi-1D confined inversion layers in Si-MOSFETs (see figure 13) (Fowler *et al* 1982, Kaplan and Hartstein 1988, Hartstein 1988) can be identified with the fluctuations induced by strong localization described above.

   If the theoretically predicted resistance and conductance fluctuations exist in the regime of strong localization they should also be observable in hopping transport (see section 3) (Lee 1984, Serota *et al* 1986, Medina *et al* 1989, Medina and Kardar 1992, Nguen *et al* 1985a, b, 1986, Raikh and Ruzin 1987, 1989, 1991, Shapir and Wang 1987). In recent experiments done on short but wide inversion layers in GaAs field-effect transistors (Orlov *et al* 1989a, b), and on In$_2$O$_{3-x}$ films (Millikan and Ovadyahu 1990) reproducible fluctuations that are consistent with the hopping picture have indeed been detected, and an attempt was made to analyse them quantitatively. They are, however, not yet very well understood, especially in their quantitative behaviour (figure 45) (Kramer *et al* 1992, Markoš and Kramer 1993a, b).
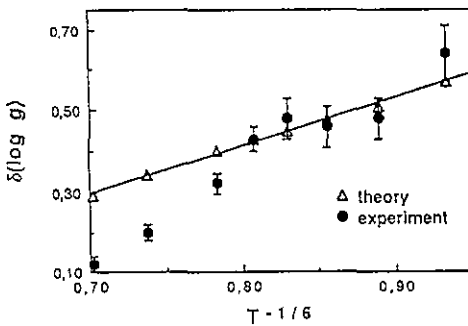
Figure 45. The fluctuations of the logarithm of the conductance in the hopping regime of a GaAs field-effect transistor as a function of the temperature. Data points are taken from figure 6 of Orlov *et al* (1989a). Triangles denote theoretical results obtained by Orlov *et al* using a model of inhomogeneous barrier structures (Raikh and Ruzin 1989). Dots are the experimental data. The straight line is a fit obtained by assuming that the effective system size is given by the Mott hopping length for $d = 2$ (see section 3) such that $\delta(\log g) \propto \sqrt{L} \propto T^{-1/6}$.

10.2. Fluctuations in the metallic limit

The experimental and theoretical investigations of recent years have revealed a most surprising feature of the transport properties of metallic systems which are defined by the condition that the mean free path ℓ is much smaller than the system diameter L, with L much smaller than the localization length (Wheeler *et al* 1982, Umbach *et al* 1984, Skocpol *et al* 1984, 1986, Licini *et al* 1985, Washburn and Webb 1986, Webb *et al* 1988, Popovic *et al* 1991, Gao *et al* 1989, Mailly *et al* 1989, Mailly and Sanquer 1991, Caro *et al* 1991). At very low temperatures, when inelastic scattering processes are frozen out to such a degree that in a sample of finite size almost no phase randomization takes place, sample specific statistical fluctuations of the conductance occur as a function of the Fermi level, an applied magnetic field, or the configuration of the impurities. They were larger than expected. They were *well reproducible* for a given sample. For a slightly modified sample (for instance by heating up and cooling down again) their behaviour changed in its details (figure 46). However, the root mean square deviation turned out to be approximately a constant,

$$\delta g = f = O(1) \tag{185}$$

and universal within certain limits in the sense that it did not depend on the average conductance of the sample.

There is a simple theoretical argument, borrowed from the theory of random matrices (Mehta 1967), which not only gives the correct order of magnitude of these 'universal conductance fluctuations' (UCF) but also sheds some light on their physical origin (Imry 1986b).

The basic idea is easily understood. The starting point is the generalization of the relation between the conductance and the quantum mechanical transmission properties, equation (181), to the metallic limit (Pichard 1984),

$$g = \frac{2e^2}{h} \sum_{j=1}^{N} \frac{1}{\cosh^2 \gamma_j L}. \tag{186}$$

$\exp(\gamma_j L)$ are the eigenvalues of the product of random transmission matrices, $\mathbf{Q Q}^\dagger$ (see sections 5 and 6), of a quasi-1D system of length L and finite cross section $\propto N$. For
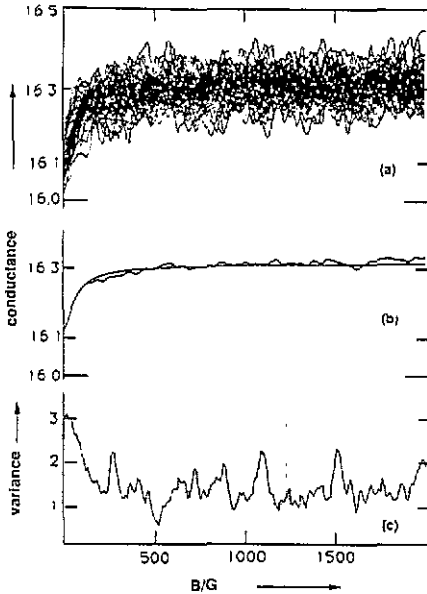
Figure 46. (*a*) Reproducible fluctuations of the magnetoconductance in units of e^2/h at $T = 45$ mK of a SiGaAs wire after heating and cooling down again (46 cycles). (*b*) The mean value of the 46 curves and a 1D weak localization fit ($L_\phi = 3$ μm). (*c*) The variance of the 46 curves in units of $10^{-3}(e^2/h)^2$ (after Mailly and Sanquer (1991)).

very weak disorder, which can safely be assumed in the metallic limit, only a finite, but nevertheless large, number of the Lyapunov exponents γ_j, say N_{eff}, contribute to the conductance. Terms for which $L \gg \gamma_j^{-1}$ are exponentially small. On the other hand, if $L \ll \gamma_j^{-1} \cosh^2 \gamma_j L \approx 1$. Therefore in the metallic regime to a good approximation

$$g \approx N_{\text{eff}}. \tag{187}$$

If all of the non-vanishing contributions to g were statistically independent one could expect that the relative fluctuations of g behave as $N_{\text{eff}}^{-\frac{1}{2}}$ for large N_{eff}. This is apparently not the case. Therefore the contributions of the γ_j cannot be statistically independent.

The matrix $\mathbf{QQ}^\dagger$ as a product of random matrices is again a random matrix, and one may ask whether or not the theory of random matrices is applicable in this situation where the matrix elements are not uncorrelated, as required in standard random matrix theory. However, there are strong analytical arguments (Imry 1986b, Mello 1987, Pichard 1991a) and numerical evidence (Markoš and Kramer 1993a, b) that this is indeed the case to some extent, at least in the asymptotic regions of weak and strong localization. In the metallic limit γ_j are not statistically independent but *strongly correlated* (figure 47). As a consequence, the change in the conductance induced by a small microscopic change in the randomness, e.g. due to the change in position of a single impurity, cannot be arbitrarily small. Either N_{eff} is unchanged, and the conductance will be the same, or N_{eff} changes by unity such that $\delta g = 1$.

The exact theory yields that there is only a small dependence of the fluctuations on the dimensionality and the geometrical shape of the sample. There is also a striking dependence
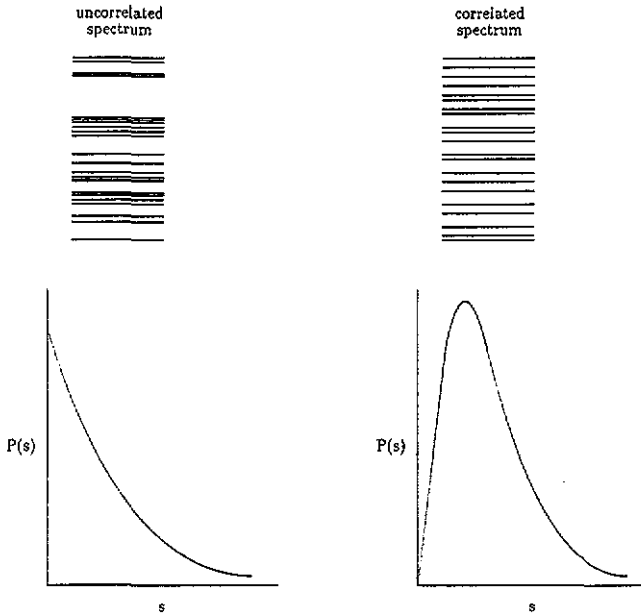
Figure 47. Qualitative level spacing distributions $P(s)$ for spectra of uncorrelated (Poissonian distribution, left) and correlated energy levels (Wigner surmise, orthogonal case, right).

of the magnitude of the fluctuations on the universality class of the system. These very interesting aspects of the UCF are well accessible to experimental investigations. In the presence of a magnetic field the amplitude is reduced by a factor $\frac{1}{4}$ for a system without spin–orbit scattering (figure 48). All of these findings were well supported by numerical and analytical calculations (Stone 1985, 1988, Lee and Stone 1985, Stone and Lee 1985, Altshuler 1985, Altshuler and Khmelnitskii 1986, Imry 1986a, b, Muttalib *et al* 1987, Zanon and Pichard 1988, Giordano 1987, 1988, Mello 1987, 1988, Mello *et al* 1988a, b, 1989, 1991, Kamien *et al* 1988, Mašek and Kramer 1988, Kramer and Schreiber 1989, Pichard and Sanquer 1990, Pichard *et al* 1990a, Pichard 1991a, Iida *et al* 1990a, b). The conductivity is related to the conductance by the classical relation $g = \sigma L^{d-2}$, therefore the relative fluctuations vanish only according to $\delta g/g \propto L^{2-d}$ in contrast to what one would expect from classical statistical physics, $\delta g/g \propto L^{-d/2}$. This means that even in 3D metallic systems the zero-temperature conductance and resistance (and hence conductivity and resistivity) are not, strictly speaking, self-averaging when the system is coherent.

10.3. Fluctuations and one-parameter scaling

From the results of the studies of both of the asymptotic limits we can conclude that a complete theory of Anderson localization must necessarily be a theory of the distribution functions of the relevant quantities, and not only the configurational averages. This viewpoint has been stressed during the last few years by an increasing number of researchers (Kravtsov and Lerner 1984, Altshuler *et al* 1986, 1989, 1990, Kumar and Jayannavar 1986, Shapiro 1986, 1987, Efetov 1987a, b, 1988, Lerner 1988, 1991b, Cohen *et al* 1988, Kravtsov *et al* 1988, 1989, Chase and MacKinnon 1987, Schreiber and Kramer 1987, 1988). The theory is presently far from being complete.
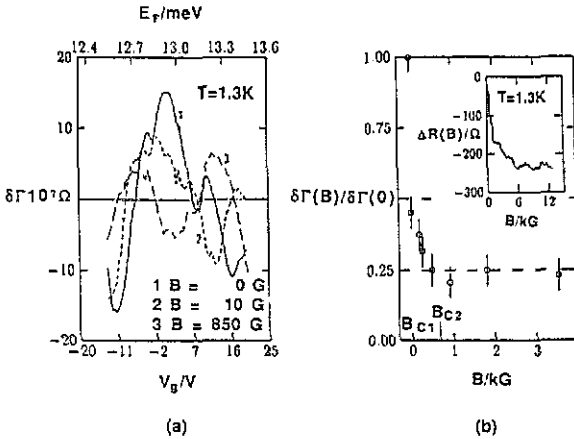
Figure 48. The conductance fluctuations $\delta\Gamma$ in a modulation doped AlGaAs/GaAs heterostructure. (a) $\delta\Gamma$ as a function of the voltage V_g applied to a gate for three different magnetic field strengths. The change of V_g is equivalent to changing the Fermi energy E_F. (b) Variance of $\delta\Gamma(B)$ as a function of the magnetic field. Inset: 'magneto-fingerprint' for $V_g = 10$ V (after Debray *et al* (1989)).

Numerically, the results obtained so far indicate that the *logarithm* of the conductance, $\ln g$, is self-averaging at least in the localized regime (figure 49). It is distributed according to a Gaussian with a variance Δ that is a unique function of the average of $\ln g$ (figure 50), at least in the limit $-\langle\ln g\rangle \rightarrow 0$. The statistical behaviour of the scaling variable introduced in section 8 is thus determined only by one quantity, namely its average. In this sense one-parameter scaling theory is valid even in the metallic limit where the distribution of $\ln g$ becomes universal, i.e. independent of the disorder and the system size (figure 51) (Markoš and Kramer 1993a, b).

Unfortunately, this does not tell us anything about the behaviour of the average of the conductance except that its distribution function must have extremely long tails. In the localized regime the distribution of g must be asymptotically of the form

$$P(g) \propto g^{-1} \exp[-(\ln g - \overline{\ln}g)^2/2\Delta^2]. \tag{188}$$

From the similarity of the distribution function we expect that the relation between the averages of $\ln g$ and g is similar to the one obtained in one dimension (section 6). $g(L)$ is exponentially decreasing but with a decay length that is larger than the localization length (Markoš and Kramer 1993a, b).

In the metallic regime we observe (figure 51)

$$P(g) \propto g^{-1} \exp(|\ln g - \overline{\ln}g|) \tag{189}$$

at least approximately within a certain region. This universality of the distribution of g bears a close resemblance to the universal conductance fluctuations discussed above and to the distribution discussed recently by Shapiro (1990).

An alternative view is the generalization of the concept of maximal fluctuations introduced in (177) above. In more than one dimension (177) becomes

$$\lim_{L\to\infty} \frac{\langle\mathrm{Tr}(\mathbf{tt}^\dagger)^n\rangle}{\langle\mathrm{Tr}(\mathbf{tt}^\dagger)\rangle} = C_n. \tag{190}$$
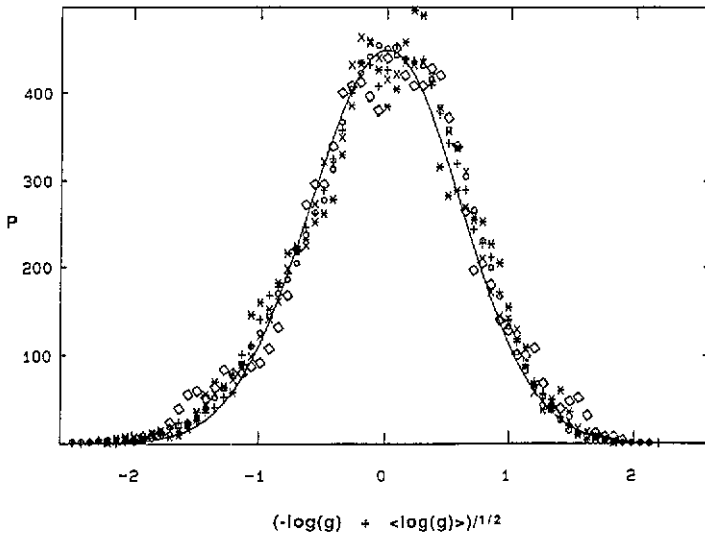
Figure 49. Distribution P of the logarithm of the transmission probability, $\log g$, of bar-shaped systems (see section 8) described by the Anderson Hamiltonian in the localized regime. The diagonal elements of the Hamiltonian are distributed according to a box function with width $W = 20$ V. System sizes $M \times M \times L$ are $M = 8$ ($\bullet$), 16 ($+$), 32 ($\times$), 64 ($*$), 128 ($\diamond$). The data are scaled such that they can be fitted to a Gaussian (full curve). Since the data scale within the limit of accuracy $\log g$ is self-averaging (Kramer and Schreiber 1989).
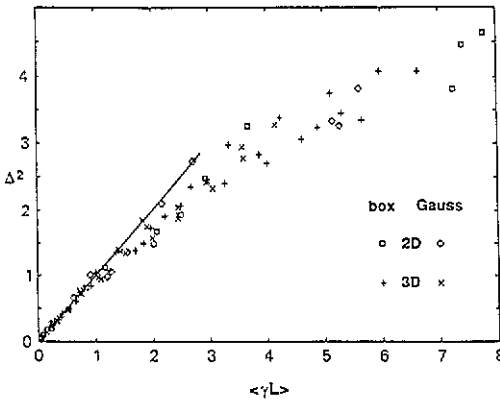

Figure 50. The variance of the logarithm of the transmission probability, Δ^2, as a function of its average, $\langle \gamma L \rangle$, for box and Gaussian distributions of the diagonal elements of the Anderson Hamiltonian in two and three dimensions with widths $W = 2, 5, 10, 20$ (2D) and $W = 10, 12,$ $14, 16, 18, 20, 22, 24, 26, 28$ (3D). System sizes are $M \times L$ with $L = 10$, $M = 6, 10, 20, 40, 60$ (2D, for $W = 20$ only $M = 20, 40, 60$ are considered), and $L \times M \times M$ with $L = 10$, $M = 8, 10,$ 13 (3D). Energy and length units are nearest-neighbour hopping matrix elements and the lattice distance, respectively (Kramer *et al* 1992). The universal asymptotic behaviour for small $\langle \gamma L \rangle$ is indicated by the straight line with the slope 1. At larger values of $\langle \gamma L \rangle$ deviations from the universal behaviour are observed.

Again this may be interpreted in terms of *open* and *closed* channels as in one dimension
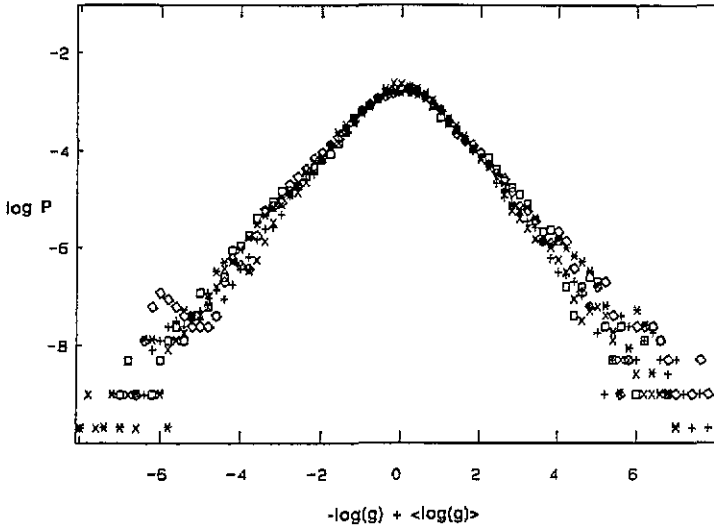
Figure 51. The logarithm of the distribution P of the logarithm of the transmission probability $\log g$ for the Anderson model at the band centre, $E = 0$, in the metallic regime. Disorder and system sizes $L \times M \times M$ ($M = 10$) $W = 2$, $L = 10$ ($\times$); $L = 20$ ($\square$); $W = 6$, $L = 10$ ($+$), $L = 20$ ($\diamond$); $W = 10$, $L = 10$ ($*$). The fact that the data scale indicates universality (the scatter of the data in the asymptotic regimes is a purely numerical effect) (Kramer and Schreuber 1989).

(Pendry *et al* 1992). This behaviour can be readily understood by considering the disordered system repeated many times to form a 1D crystal with a large unit cell. In this case $\mathbf{tt}^\dagger = 1$ in a band and $\mathbf{tt}^\dagger = 0$ in a gap (Chase and MacKinnon 1987, MacKinnon 1991). Figure 52 shows that this concept continues to be valid even in the case of a single disordered system with perfect leads.
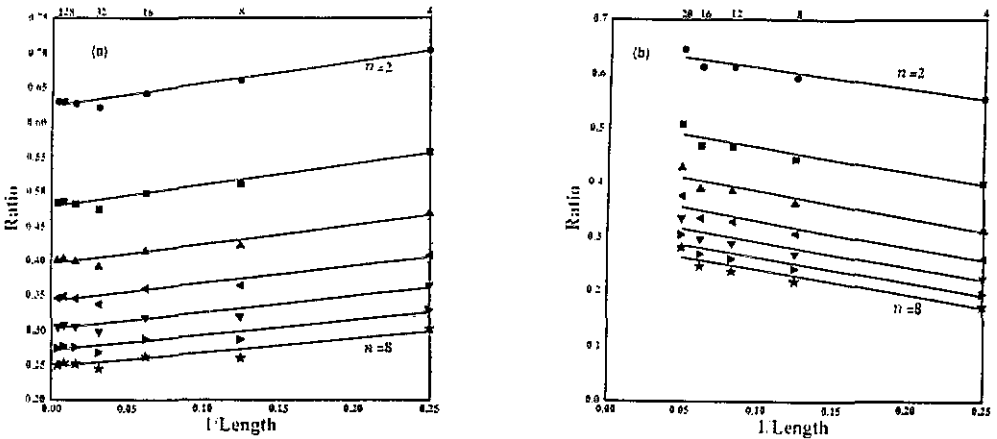
Figure 52. (*a*) $\langle \mathrm{Tr}(tt^\dagger)^n\rangle/\langle\mathrm{Tr}(tt^\dagger)\rangle$ plotted against inverse length for squares of size $4 \leq L \leq 256$ and $E = 1.0$ and $W = 3.0$ using the Anderson Model (16) averaged over 128 samples. (*b*) As before but for cubes of size $4 \leq L \leq 20$ and $E = 1.0$ and $W = 10.0$ (Pendry *et al* 1992).

11. Summary and conclusions

In this review we have tried to summarize the present status of the field of localization of quantum states in random systems from the experimental as well as the theoretical point of view. We have attempted not only to review the results of the literature, but also to treat some of the key concepts like hopping transport, asymptotic behaviour of the density of states, 1D localization, and the bounds for the critical exponents in a self-contained way.

After a brief review of the history of the field in the second section, which provides insight into the way the fundamental concepts have developed in the course of time, a concise description of the main experimental evidence is given in the third section. Here we have again concentrated on the key experiments, which are characteristic of the different regimes of interest. In most cases we have reported the first experimental evidence, such as the logarithmic weak localization correction to the low-temperature behaviour of the resistance of 2D metallic films, and the Aharonov–Bohm like quantum interference oscillations of the magnetoresistance of thin metallic cylinders. We have summarized the status of the experimental results concerning the critical behaviour at the disorder-induced metal–insulator transition, while emphasizing the fact that the agreement between theory and experiment is far from satisfactory. Needless to say, we consider this topic to be one of the most important problems for future research, especially in the experimental area. The connection between quantum localization and other areas of physics such as classical wave phenomena has been made using the two examples of water waves subject to scattering from a random assembly of obstacles, and enhanced backscattering of coherent optical radiation from a random assembly of glass spheres. An important consequence of quantum coherence and localization is the occurrence of reproducible stochastic fluctuations of the transport properties upon variation of external parameters of a system, such as an applied magnetic field, and/or a gate voltage, at very low temperature. We have discussed a few experimental observations in this rapidly developing field of current research at the end of the third section.

The basic concepts and models have been introduced in detail in the fourth section. The quantities of interest, and the concepts of self-averaging and the configurational average, have also been explained here. The vast number of partially differing definitions of localization including the asymptotic behaviour of quantum mechanical wavefunctions and Green's functions and multifractality of quantum states, and the connection between these and the transport properties were summarized in the fifth section. As a key example, localization in 1D disordered systems was covered in the sixth section in a self-contained way, including the theory of the averages of the resistance, the conductance, and the localization length. Here, the first explicit indication of the non-self-averaging property of transport was obtained.

In section seven we have treated the weak localization approach in some detail, again to some extent self-contained, explaining the essential steps and the approximations involved in such a way that they can be followed and controlled without resorting too heavily to the original literature. It is consistent that in the weak localization approach, which is a perturbative approximation method for the quantum corrections to the conductivity, non-interacting free electrons are used as a starting point for the mathematical treatment. In addition we have given a comprehensive picture in position space which serves to illustrate the basic physical content of the approach, and which can even be traced by experiment.

Section eight contained a concise treatment of the scaling approach for the Anderson transition. Though definitely not aiming to be self-contained, we have tried to summarize the recent status of the results of this method which, as it is now becoming more and

more apparent, cannot yet be considered as the 'solution of the problem of the disorder-induced MIT'. Besides outlining the concept of scaling from a general point of view at the beginning of the section, we have provided a short summary and critical discussion of the field theoretical formulation. We have also placed considerable emphasis on the numerical results obtained by investigating the scaling properties of the transmission properties of quasi-1D disordered systems of finite cross section. This includes results for the critical exponents as well as explicit demonstration of the *existence of a universal scaling function* in the critical regime, and a short discussion with respect to experimental data.

As an illustrative example, we have discussed localization in the magnetic field in the ninth section. The reason for this is threefold. Firstly, 2D disordered systems in the presence of a magnetic field are of considerable interest in connection with the quantum Hall effect. Secondly, universality has been explicitly demonstrated for this limit, the critical exponent has been determined with extraordinary precision, and has been confirmed in a direct experiment. Thirdly, there are many experiments concerning the MIT in 3D systems subject to a magnetic field with different and partially contradictory results. Astonishingly enough, recent numerical work suggests that the critical behaviour is not changed when a magnetic field is applied. Further experimental and theoretical work is definitely needed in order to substantiate this first indication that the conventional scheme of universality classes for the Anderson transition is incomplete.

Section ten has been devoted to a summary of the field of reproducible conductance fluctuations. As this is presently an open and still growing area, the discussion is necessarily incomplete and exemplifies only specific points, such as the universal fluctuations in the metallic regime, which can be considered to be more or less understood, at least in its basic aspects, and the fluctuations in the regime of hopping transport, which is the subject of thorough and extensive current investigations. The validity of the one-parameter scaling approach, in particular, relies heavily on the choice of quantity to use as the scaling variable, which should be self-averaging. It is one of the main objectives of the theory of the fluctuations in the transport properties to contribute to the clarification of this question, i.e. to clarify which quantity is measured in a transport experiment done on a disordered system at very low temperature in the regime where quantum effects dominate.

Theory and experiments in the field of localization have matured considerably since the early days when the problem was formulated by proposing the *absence of diffusion in certain random lattices*. However, in contrast to the common folklore in the past decade most of the questions are far from being solved. Despite all of the efforts, the nature of most of the experimentally observed metal–insulator transitions must be considered as not understood. The determination of the critical behaviour turned out to be a forbiddingly complicated enterprise. The situation in the theoretical section is not much better. It is presently not clear whether or not the widely celebrated scaling idea can be used for the conductance. It seems that it is the logarithm of the conductance that must be used as a scaling variable—numerical works and theoretical considerations have yielded many hints in this direction. If this is the case, then all of the moments of the conductance must be evaluated in order to understand the nature of the disorder-induced metal–insulator transition.

It is, however, completely open how this is affected when interactions have to be taken into account.

Although the problems are not yet solved, the situation is nevertheless encouraging for the following reasons. Firstly, in the metallic limit a number of quantitative theoretical results are available that allow for experimental tests of the main ideas, such as quantum interference. Secondly, powerful methods, analytical as well as numerical, have been devised that should, at least in principle, be capable of treating some of the open questions,

such as the critical behaviour. Thirdly, experimental techniques have improved considerably, especially in connection with low temperatures, sample preparation and characterization, and measuring methods. Finally, the field of localization, originally restricted to solid state physics, has widened by adopting new ideas from other fields, such as optics and classical waves.

Altogether there are good reasons to believe that the field will be alive and extremely active in the coming years, and that eventually the long-standing problem of the metal–insulator transition in condensed matter can be solved.

Acknowledgments

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