1 HARRIS CRITERION: The result of Chayes et al.

Consider a disordered system with a term, in the exponent of the Boltzmann weight, of the form

$$\lambda \sum_x \epsilon_x A_x$$

with A_x some "local observables" and ϵ_x a collection of IID random variables.

For simplicity, consider the case where the ϵ_x are Gaussians $\mathcal{N}(m, \sigma^2)$ of mean m and variance σ^2, and to avoid that the disorder disappears we assume that $\sigma^2 > 0$.

Note that we can replace $\lambda \sum_x \epsilon_x A_x$ by $\sum_x \epsilon_x A_x$ if we say that the IID variables ϵ_x have probability density

$$\rho_\lambda(x) := \frac{1}{\lambda}\rho(\frac{x}{\lambda}),$$

with $\rho(x)$ the probability density of the $\mathcal{N}(m, \sigma^2)$ variables. In the following, we let $\mathbb{P}_\lambda$ denote the law of such IID variables with density $\rho_\lambda(x)$.

Now we assume that the system under consideration has a second-order phase transition when λ is varied. Call the critical point λ_c and assume that $\lambda_c \neq 0$ (at criticality disorder is not vanishing). In particular, for the finite-volume system (enclosed in a cube of side L) we assume that there exists some event B_L such that

- if $\lambda < \lambda_c$ then $\mathbb{P}_\lambda(B_L) \to 0$ when $L \to \infty$;
- if $\lambda = \lambda_c$ then $\mathbb{P}_{\lambda_c}(B_L) \geq c > 0$ for every L.

Theorem 1.1 (Chayes et al., '86)**.** *For $\lambda < \lambda_c$ define the "finite-size-scaling correlation length" $\xi(\lambda)$ as*

$$\xi(\lambda) = \max\left\{L : \mathbb{P}_\lambda(B_L) \geq \frac{1}{2}c\right\}.$$

Then, one has

$$\xi(\lambda) \geq C(\lambda_c - \lambda)^{-\frac{2}{d}}.$$

In other words, when $\lambda \to \lambda_c$ the correlation length explodes with an exponent ν that is at least $2/d$.

Example: ferromagnet with random couplings.

Consider a nearest-neighbor ferromagnet such that the coupling constant $J > 0$ between neighboring spins σ_x, σ_y is replaced by ϵ_{xy}, a Gaussian random variable $\mathcal{N}(J, \Sigma^2)$. In the Boltzmann weight, the term $\epsilon_{xy}\sigma_x\sigma_y$ is multiplied by β, the inverse temperature. Then, in this example one has simply $\lambda = \beta$.

Assume that at some critical inverse temperature β_c there is a ferromagnetic phase transition. It is natural to imagine that

- for $\beta < \beta_c$ spins are essentially independent, so that the magnetization

$$M_L = \frac{1}{L^d} \sum_x \sigma_x$$

  is of order $L^{-d/2}$, while

- for $\beta > \beta_c$ the magnetization is of order 1.

Then, typically it will happen that at the critical point

$$M_L \sim L^{-\gamma}$$

for some critical exponent $\gamma < d/2$. If this holds, then in this case a natural candidate for the event B_L is

$$B_L = \{\langle M_L^2 \rangle \geq L^{-2\gamma}\},$$

with $\langle \cdot \rangle$ the thermal average. Clearly, B_L has probability almost zero for $\beta < \beta_c$ and L large, has probability almost 1 for $\beta < \beta_c$ and L large and should have positive probability at the critical point.

The finite size scaling correlation length $\xi(\beta)$ has then the following meaning. If $\beta_c - \beta$ is small, then a small enough system will look like critical. $\xi(\beta)$ is the first system size at which the system "realizes that it is sub-critical".

Proof of the Theorem

The theorem is based on the following simple fact: let F be a function of the random variables $\epsilon_1, \ldots, \epsilon_N$. Then,

$$|\frac{d}{d\lambda}\mathbb{E}_\lambda F(\epsilon_1, \ldots, \epsilon_N)| \leq C\sqrt{N} \max |F| \tag{1}$$

where $\max|F|$ is the maximal value of F (in absolute value) and $C(\lambda)$ is some λ-dependent constant that does not diverge when $\lambda \to \lambda_c$.

For the system in the cube Q_L of side L, let F be the indicator function of the event B_L, that clearly depends on the L^d random variables $\epsilon_x, x \in Q_L$. Note that

$$\mathbb{P}_\lambda(B_L) = \mathbb{E}_\lambda(F).$$

By assumption,

$$\mathbb{P}_{\lambda_c}(B_L) \geq c$$

for every L. Integrating (1), we see that for $\lambda < \lambda_c$

$$\mathbb{P}_\lambda(B_L) = \mathbb{E}_\lambda(F) \geq c - C\sqrt{L^d}(\lambda_c - \lambda)$$

since $\max|F| = 1$ for an indicator function. One sees immediately that the right-hand side is larger than $c/2$ if

$$L \leq \left(\frac{c}{C(\lambda_c - \lambda)}\right)^{2/d}$$

so that we obtain

$$\xi(\lambda) \geq C'(\lambda_c - \lambda)^{-2/d}$$

as we wished.

Proof of (1).

We have

$$\mathbb{E}_\lambda(F) = \int dx_1 \dots dx_N F(x_1, \dots, x_N) \prod_j \frac{1}{\lambda}\rho(x_j/\lambda) \tag{2}$$

so that

$$\left|\frac{d}{d\lambda}\mathbb{E}_\lambda(F)\right| = \left|\int dx_1 \dots dx_N F(x_1, \dots, x_N) \prod_{j=1}^N \frac{1}{\lambda}\rho(x_j/\lambda) \sum_{i=1}^N \left(-\frac{1}{\lambda} - \frac{x_i}{\lambda^2}\frac{\rho'(x_i/\lambda)}{\rho(x_i/\lambda)}\right)\right| \tag{3}$$

$$\leq \frac{\max|F|}{|\lambda|} \int dx_1 \dots dx_N \prod_{j=1}^N \rho(x_j) \left|\sum_{i=1}^N \left(1 + x_i\frac{\rho'(x_i)}{\rho(x_i)}\right)\right| \tag{4}$$

where we did a change of variables $x_i/\lambda \to x_i$. Now we use the Cauchy-Schwartz inequality:

$$(\mathbb{E}(f))^2 \leq \mathbb{E}(f^2)$$

to obtain

$$\left|\frac{d}{d\lambda}\mathbb{E}_\lambda(F)\right|^2 \leq \frac{(\max|F|)^2}{\lambda^2} \int dx_1 \dots dx_N \prod_{j=1}^N \rho(x_j) \sum_{i,j} \left(1 + x_i\frac{\rho'(x_i)}{\rho(x_i)}\right)\left(1 + x_j\frac{\rho'(x_j)}{\rho(x_j)}\right). \tag{5}$$

The diagonal terms ($i = j$) give altogether a contribution of order

$$N\frac{(\max|F|)^2}{\lambda^2}.$$

The non-diagonal terms instead vanish: indeed, one has

$$\int dx_i dx_j (\rho(x_i) + x_i\rho'(x_i))(\rho(x_j) + x_j\rho'(x_j)) = \left[\int dx(\rho(x) + x\rho'(x))\right]^2 \tag{6}$$

$$= \left[\int \frac{d}{dx}(x\rho(x))dx\right]^2 = 0. \tag{7}$$

Altogether,

$$\left|\frac{d}{d\lambda}\mathbb{E}_\lambda(F)\right|^2 \le CN\frac{(\max|F|)^2}{\lambda^2} \tag{8}$$

and since $\lambda_c \neq 0$ by assumption, the constant C/λ^2 is not divergent at criticality.

2 The Imry-Ma argument

Imry, Ma, PRL35 (1975), 1399-1401.

We have seen that the Harris (and the Weinrib-Halperin) argument gives a criterion for stability of a second-order phase transition with respect to the introduction of a small amount of disorder.

The Imry-Ma argument in a sense is in the same spirit, but concerns the stability of spontaneous symmetry breaking (an order parameter being non-zero at low temperature and breaking a symmetry of the Hamiltonian).

Consider to be concrete a nearest-neighbor ferromagnet on $\mathbb{Z}^d$, with n-dimensional spins (spins are vectors on the sphere S^{n-1}; the Ising case corresponds to $n = 1$). The spin-spin interaction is $J\sigma_x \cdot \sigma_y$ (scalar product). Since J is positive, at low temperature spins tend to align. Is there a spontaneous magnetization at low temperature? This depends on d and n.

Suppose we have all spins aligned (upwards) and we want to create a defect: a cube of side L of spin aligned downwards. The minimal cost (i.e. increase in energy) to produce this defect is of order L^{d-1} if $n = 1$ (Ising model) and of order L^{d-2} if $n \ge 2$. For $n \ge 2$, due to the continuous character of the spins, the domain wall is energetically less costly because spins can move from "up" to "down" in a smooth way, over a distance of order L.

Then, one guesses that at low enough temperature large defects are very unfavorable as soon as $d > 1$ (if $n = 1$) and as soon as $d > 2$ (if $n \ge 2$), so that the "up-magnetized phase" is stable. This is correct. Indeed, in these situations there is spontaneous symmetry breaking (non-zero magnetization) at low enough temperature.

Now one can ask: What happens if we introduce quenched randomness in the Hamiltonian? in particular, a random magnetic field. So that the Hamiltonian becomes

$$H = -J\sum_{<xy>} \sigma_x \cdot \sigma_y - \epsilon \sum_x h_x \cdot \sigma_x$$

with h_x IID random variables with zero average (say, Gaussian variables for instance). Here ϵ is a positive (small) constant that tunes the disorder intensity.

At first sight one might think that, since the varibles are centered, their effect on a large volume will be negligible because of cancellations. However, this is not true.

Let us repeat carefully the argument above. Assume that at low temperature there is spontaneous magnetization and spins are (mostly) aligned upwards. Let us check if this situation is stable. If we create a large defect of cubic shape Q_L of linear size L of downward aligned spins, we have seen that this induces a boundary energy increase of order L^{d-1} if $n = 1$ and L^{d-2} if $n \geq 2$. However, it may happen that the sum of the magnetic fields h_x inside Q_L is negative, which favors spins being downward. In fact, the energetic decrease due to the fields when spins in Q_L are flipped is $2\epsilon \sum_{x \in Q_L} h_x$ (if this sum is negative, otherwise the energy decrease is actually an increase, that is unfavorable). Since the magnetic fields are IID, the sum

$$\sum_{x \in Q_L} h_x$$

is approximately a Gaussian with variance of order L^d, i.e. its typical fluctuations are of order $L^{d/2}$. Then, if $d/2 \geq d-1$ (for $n = 1$) or if $d/2 \geq d-2$ (for $n \geq 2$) it is intuitive that we will find arbitrarily large regions where the bulk energy decrease $2\epsilon \sum_{x \in Q_L} h_x$ due to the random field fluctuations beats the boundary energy increase. This suggests that, whatever the value of $\epsilon > 0$, the symmetry breaking is unstable when $d \leq 2$ (if $n = 1$) or if $d \leq 4$ (if $n \geq 2$).

This is the Imry-Ma prediction and can be proven rigorously (This was done by Aizenman-Wehr, PRL 62 (1989), 2503–2506)).