Problem set for the course "Phase Transitions and Criticality", 2019

Rules: You can choose at wish from problems having the same main number (i.e. from a given section), but you can collect only 30 points at maximum from all other sections. You may submit more but the points per section will be limited to 30. There is one more constraint from problems 1.1 and 1.2 only one can be submitted.
You are supposed to work alone as much as possible but you are allowed to consult other students and discuss with them. While discussions among students are encouraged, solving a problem together as a team work is NOT ALLOWED. Of course, also feel free to contact me and ask questions; I can help and give you further hints if you are stuck.
Deadline: 18.06.2019. Delay penalty: 5points/day. (Days end at 4pm...)

Please scan the solutions and upload it to the moodle system: `http://newton.phy.bme.hu/moodle/`
Grading is as follows:

5: 71- points,
4: 56-70 points,
3: 41-55 points,
2: 31-40 points.

I. MEAN FIELD PROBLEMS

1.1 (20 p) Mean field theory of the 3-state Potts model.
In this problem, the mean field free energy of the 3-state Potts model is determined, and it is shown that the ferromagnetic phase transition in this model is of first order. The $N = 3$-state Potts model is defined by the following (dimensionful) Hamiltonian,

$$H = -J \sum_{<r,r'>} \delta_{S(r),S(r')} \ .$$

Here the index r runs over lattice sites, and the "Potts spin" $S(r)$ takes N different values, $S(r) = 1, \dots, N$. Neighboring Potts spins interact by a ferromagnetic interaction, and prefer to be aligned in the same direction.

a. **(10 p)** First compute the free energy within the mean field approximation as follows: Assuming that, independently of the other spins, every spin points with probability p_i in direction $S = i$ estimate the free energy density, by computing $E = \langle H \rangle$ and using $F = E - TS$. Express this quantity in terms of the magnetizations, $m_i \equiv p_i - 1/3$. Expanding the free energy density show that it is approximately given by

$$f \approx C(T) + \sum_{i=1}^{3} \Big[\big(\frac{3}{2}T - Jz/2\big) m_i^2 - \frac{3T}{2} m_i^3 + \frac{9T}{4} m_i^4 + \dots \Big] \ ,$$

with z the number of nearest neighbors.

b. **(5 p)** Introduce the variables, $\alpha = (m_2 - m_3)/\sqrt{2}$, and $\beta = (2m_1 - m_2 - m_3)/\sqrt{6}$, and express the above free energy expression in terms of these variables. Evaluate this approximate free energy numerically as a function of these two variables and show by producing contour plots that the free energy develops three symmetrical minima in the ordered phase, $T < T_C$, through a first order transition (5p).

c. **(5 p)** Determine numerically the mean field value of T_C. (Hint: consider the mean field free energy obtained in a. along the special direction, $m_1 = m$, $m_3 = m_2 = -m/2$. Plot $f(m, T)$ and determine T_C numerically, up to four digits.)

d. **(5 p)** (Bonus!) Introduce the following two variables,

$$m_\pm \equiv (m_1 + e^{\pm i2\pi/3} m_2 + e^{\mp i2\pi/3} m_3)/\sqrt{3} \ .$$

Using only symmetry arguments, construct the (Ginzburg-Landau) free energy as a function of these variables up to fourth order, and show that it takes on the form:

$$f(T) = a(T)m_+m_- + b(m_+^3 + m_-^3) + c\,(m_+m_-)^2 + \ldots\ .$$

Show that introducing the real variables, $(m_+ + m_-)/\sqrt{2}$ and $(m_+ - m_-)/(i\sqrt{2})$, this is precisely of the form one gets by expanding the mean field free energy. (Notice that now no symmetry forbids the third order term.)

1.2 (20p) Exercise 2.4 from Cardy's book: Mean field theory of antiferromagnetic Ising model.
Let us consider the following antiferromagnetic Ising Hamiltonian:

$$H = \frac{1}{2}\sum_{r,r'} J(r-r')\; S(r)S(r') - H\sum_r S(r)\ ,$$

where r and r' run over a d-dimensional cubic lattice, $J(r-r') = J > 0$ for nearest neighbor sites, and it vanishes otherwise. Notice that J and H have now dimension of temperature (energy). We shall construct the mean field phase diagram of this model as a function of T and H/J.

a. **(5 p)** First construct the $T = 0$ part of the phase diagram by comparing the energy of the ferromagnetic and antiferromagnetic states. What is the order of the phase transition found?

b. **(5 p)** To construct a more complete picture, divide the lattice onto two sub-lattices, A and B. Construct the mean field effective Hamiltonian by assuming that fluctuations are small, and thus approximate the product $S(r)S(r') \approx S(r)\langle S(r')\rangle + S(r')\langle S(r)\rangle - \langle S(r)\rangle\langle S(r')\rangle$, as usual. However, allow for two different values for the magnetizations on the two sub-lattices, $\langle S(r)\rangle = m_{A,B}$. Then evaluate the partition function with this mean field Hamiltonian and show that the dimensionless mean field free energy density ($f = F/(T\ N)$) is given by:

$$f_{MF} = -\frac{1}{2}\left(\tilde{J}\ m_A m_B + \ln \mathrm{ch}(h_A - \tilde{J}m_B) + \ln \mathrm{ch}(h_B - \tilde{J}m_A)\right)\ ,$$

where we allowed for a field that is different on the two sub-lattices, $h_{A,B} = H_{A,B}/T$, and $\tilde{J} = zJ/T$, with z the coordination number. Derive the self-consistency equations for $m_{A,B}$ from this expression by differentiating with respect to h_A and h_B. [Be careful: $m_A = -\frac{1}{N_A}\frac{\partial F}{\partial H_A} = -2\frac{\partial f}{\partial h_A}$.]

c. **(5 p)** Introduce the ferromagnetic and antiferromagnetic order parameters, $m = (m_A + m_B)/2$ and $n = (m_A - m_B)/2$, and rewrite the self-consistency equations obtained in terms of these. Assume that in the high temperature phase $n = 0$, and compute the value of m approximately as a function of $h \equiv h_A \equiv h_B$ from the self-consistency equations. Then show that the $n = 0$ solution gets unstable when:

$$1 = \tilde{J}(1 - m^2) \approx \tilde{J}\left(1 - \frac{h^2}{(1+\tilde{J})^2}\right)\ ,$$

thus the magnetic field suppresses T_C quadratically. What is the order of the phase transition in a small but finite magnetic field?

d. **(5 p)** Finally, draw the phase diagram, and interpret it in terms of fixed points. Argue that there must be a tricritical fixed point, where a first order phase transition line and a second order phase transition line meet. (There are six fixed points for $h \geq 0$: two of them are stable, one is a discontinuity fixed point, one is a regular critical point, and the fifth one is the tricritical point described above, and for $h = 0$ there is a high temperature fixed point, too. There are three more fixed points for $h < 0$.)

1.3 (20 p) Perform the Hubbard-Stratonovich transformation for the following d-dimensional $O(3)$ Heisenberg model,

$$\mathcal{H} = -\frac{J}{2}\sum_{r,r'} \mathbf{n}_r \cdot \mathbf{n}_{r'}\ ,$$

and derive the corresponding continuum field theory. Here the $\mathbf{n}$-s denote vector spins of unit length, $|\mathbf{n}| = 1$, and only nearest neighbor sites give a contribution.

a. **(5 p)** Following the procedure at class, decompose the interaction term of the Hamiltonian by introducing the new vector fields ϕ_i^a at each site ($a = x, y, z$) using the identity (prove it!)

$$\exp\{\frac{1}{2}\sum_{r,r',a} x_r^a A_{r,r'} x_{r'}^a\} \sim \int \prod_{r,a} dy_r^a \ \exp\{-\frac{1}{2}\sum_{r,r',a} y_r^a [A^{-1}]_{r,r'} y_{r'}^a + \sum_{r,a} y_r^a x_r^a\} \ .$$

b. **(5 p)** Next, carry out the integrals over the $\mathbf{n}$'s in the terms $\sim \exp\{\mathbf{n}\cdot\phi\}$. (Hint: One can use spherical coordinates and align the z axis parallel to the field ϕ.) Show that the result depends only on $|\phi|$. Re-exponentiate and expand the result up to second order in ϕ^2 to determine the coefficient of the ϕ^2 and $|\phi|^4$ terms in the effective action.

c. **(5 p)** Treat the first term $\sim J^{-1}$ as at class, and then write the full effective action for the field ϕ_r. (Go over to Fourier space, invert $J_{r,r'}$ there, expand it in q, and then go back to real space.) What is the temperature at which the phase transition takes place (the coefficient of the ϕ^2 term changes sign)?

d. **(5 p)** For $T < T_c$ determine the ground state (of the quartic terms in the effective action (Hamiltonian) and look for the Goldstone modes. Assume that ϕ is parallel to the z axis, and then expand the effective action (Hamiltonian) up to second order in the small fluctuations:

$$\phi = (\delta\phi_x, \delta\phi_y, \phi_0 + \delta\phi_z) \ . \tag{1}$$

Show that the energy of transverse fluctuations goes as q^2 in Fourier space. Show also that the energy of longitudinal fluctuations (Higgs modes) remains finite at $q = 0$.

II. LATTICE RG AND UNIVERSAL SCALING

2.1 (10 p) Exercise 3.2 from Cardy's book. Scaling for the 1D Potts model.
Let us consider the classical one-dimensional Q-state Potts model:

$$\mathcal{H} = -J\sum_i \delta_{\sigma_i,\sigma_{i+1}} \ ,$$

where the 'spins' σ_i can take Q different values. Construct the renormalization transformation for $b = 2$.

a. **(3 p)** Rewrite $\exp\{-\mathcal{H}\}$ as a product of terms using the identity:

$$T_{\sigma_i\sigma_{i+1}} \equiv e^{J\delta_{\sigma_i\sigma_{i+1}}} = 1 + \delta_{\sigma_i\sigma_{i+1}}(e^J - 1) \ .$$

Then do the decimation by summing over every second spin.

b. **(3 p)** Give the renormalization group transformation for the free energy density. Use the variable $x \equiv e^J - 1$. Show that in the large x limit the transformation simplifies to:

$$x' \approx \frac{x}{2} - \frac{Q}{4} \ .$$

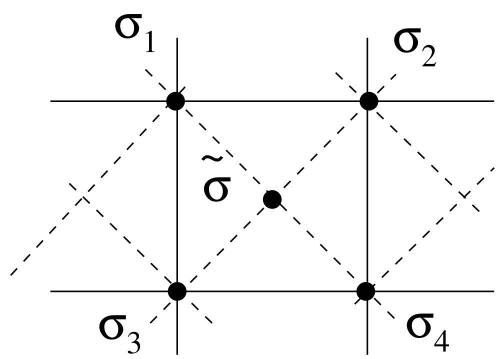
c. **(4 p)** Determine the correlation length: Proceed as at class. Start from a very large value of J, and use an approximate form of the relation $x \to \tilde{x}$ appropriate in this limit to determine the number of decimations after which the effective coupling becomes of the order of unity, $\tilde{x} \sim 1$. Determine the correlation length from the number of iterations needed. Keep also the subleading term in the approximate relation above, and take the large Q limit. How can you interpret the expression you get? Explain the Q-dependence of it? (Hint:Think about domain walls.)

2.2 (25 p) Construction of the RG transformation for the two-dimensional Ising model. Consider the two-dimensional Ising model on a square lattice with nearest neighbor interaction:

$$H = -K_1 \sum_{(i,j)} \sigma_i\sigma_j \ ,$$

Consider the plaquet shown in the figure. The contribution of this plaquet to the partition function is

$$Z = \sum_{\dots,\sigma_1,\sigma_2,\sigma_3,\sigma_4,\tilde{\sigma}\dots} \dots \times \exp(K_1(\sigma_1 + \dots + \sigma_4)\tilde{\sigma}) \times \dots$$



a. **(8 p)** Integrate out the spin $\tilde{\sigma}$ and show that the summation over $\tilde{\sigma}$ produces a factor

$$\sim cst. \times \ \exp\Big\{ A\sigma_1\sigma_2\sigma_3\sigma_4 + B \sum_{\substack{i,j=1 \\ i<j}}^{4} \sigma_i\sigma_j \Big\} ,$$

where $B = \frac{1}{8}\ln[\mathrm{ch}\ 4K_1]$. What is the value of the four-spin interaction A ?

b. **(7 p)** Now neglect all generated interaction beyond second nearest neighbor interaction. What is the RG transformation $K_1 \to K_1'$, $K_2 \to K_2'$ like, if you assume that the spins that are integrated out are independent? (I.e., you neglect the interaction K_2 between these spins.) Show that the approximate transformation obtained this way has a non-trivial fixed point determined by the equation:

$$K_1^* = \frac{3}{8}\ln[\mathrm{ch}(4K_1^*)] , \qquad K_2^* = K_1^*/3 .$$

[Be careful: One new bond has two neighboring plaquets...]

c. **(10 p)** Linearize the transformation around this point and show that it becomes:

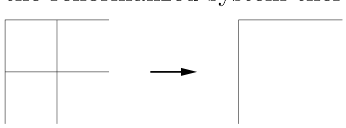
$$\delta K_1' = \mathrm{th}(4K_1^*)\ \delta K_1 + \delta K_2 , \tag{2}$$

$$\delta K_2' = \frac{1}{2}\mathrm{th}(4K_1^*)\ \delta K_1 . \tag{3}$$

Determine the corresponding eigenvalues and the value of the exponent ν. [Compute numerically the value of K_1^*.]

2.3 (15 p) Construction of the RG transformation for the two-dimensional bond percolation.
Consider a two-dimensional square lattice. Links are present with probability p. In percolation theory we call the system percolating if there exists an infinite connected cluster. In the thermodynamic limit there is a critical point p_c above which there is an infinite cluster with probability 1 under which there is no infinite connected cluster.

a. **(5 p)** Perform the renormalization of the two-dimensional bond percolation. Consider a 2×2 system with its 8 bonds and do the renormalization with $b = 2$. Write down the renormalization group equation. Note that in the renormalized system there are only two bonds, see figure.



b. **(2 p)** Solve the renormalization equation for p_c.

c. **(3 p)** determine the exponent ν.

d. **(5 p)** Do the same exercise for the two dimensional site percolation. In this model the sites of the two-dimensional square lattice is filled with probability p. Do the renormalization with $b = 2$. Consider the configurations connecting only one opposing walls of the 2×2 cluster with weight 1/2 for the RG equation.

III. QUANTUM CRITICALITY

3.1 (20 p) Consider the 2-dimensional transverse field Ising model. Using simple scaling arguments determine the scaling form of the susceptibility along the z-direction, close to the quantum phase transition. Use the finite size scaling property

$$f_s(B_z, B_x - B_x^{(c)}, \beta) = b^{-(d+1)}\ f_s(b^{y_h} B_z, b^{y_t}\ (B_x - B_x^{(c)}), \beta/b)$$

and show (5p) that

$$\chi_z(T, B) = T^{-x}\ Q_\pm(T/|B - B_C|^y)\ ,$$

with Q a universal scaling function, and B the field in the x direction. What are the precise (numerical) values of the exponents (5p)? (Use the corresponding table in Cardy's book!) What are the asymptotical properties of the function $Q_\pm$ (5p)? Sketch the behavior of $\chi(T, B)$ as a function of T while B crosses the critical value, $B = B_C$ (5p). (Remember that in this system one has a gap away from the critical point and that for $B > B_C$ one has a paramagnet, while for $B < B_C$ a ferromagnet is found.)

3.2 (15 +1 p) Diagonalization of the transverse field Ising model
Consider the transverse field Ising model,

$$H = -J \sum_j \hat\sigma_j^z \hat\sigma_{j+1}^z - B \sum_j \hat\sigma_j^x\ .$$

a. **(5 p)** First use a special form of the Jordan-Wigner transformation

$$\hat\sigma_j^x = 2c_j^\dagger c_j - 1\ , \quad \hat\sigma_j^z = i\ (-1)^{\sum_{k<j}(c_k^\dagger c_k - 1)}\ (c_j - c_j^\dagger)\ ,$$

with the c_j denoting spinless fermions. Show that these operators satisfy the relations $(\hat\sigma_j^x)^2 = (\hat\sigma_j^z)^2 = 1$ and $[\hat\sigma_j^x, \hat\sigma_k^x] = [\hat\sigma_j^z, \hat\sigma_k^z] = 0$ (3p). Show that, in this fermionic language, the Hamiltonian is quadratic (2p):

$$H = \sum_j \left[J(c_j + c_j^\dagger)(c_{j+1} - c_{j+1}^\dagger) - 2B\ c_j^\dagger c_j \right] + cst\ .$$

b. **(5 p)** Now introduce the Fourier transform of the operators c_j and show that

$$H = \frac{1}{2} \sum_j (c_q^\dagger, c_{-q}) \begin{pmatrix} \omega_q & -2iJ\sin(q) \\ 2iJ\sin(q) & -\omega_q \end{pmatrix} \begin{pmatrix} c_q \\ c_{-q}^\dagger \end{pmatrix} + cst\ , \tag{4}$$

with $\omega_q = 2J\cos(q) - 2B$.

c. **(5+1 p)** Find the eigenvalues $\pm\Omega_q$ of the 2×2 matrix H_q in (4), and denote the corresponding eigenvectors by $u_{q,\pm}$. Plot the dispersion Ω_q at and off the critical point (1p). Define the following annihilation operators,

$$a_q \equiv (u_{q,+})^+ \cdot \begin{pmatrix} c_q \\ c_{-q}^\dagger \end{pmatrix}\ , \quad b_q \equiv (u_{q,-})^+ \cdot \begin{pmatrix} c_q \\ c_{-q}^\dagger \end{pmatrix}\ ,$$

and show that the structure of the 2×2 matrix H_q implies (2p)

$$a_q^\dagger = b_{-q}\ .$$

Using this property and the spectral representation $H_q = \Omega_q (u_{q,+})\ (u_{q,+})^+ - \Omega_q (u_{q,-})\ (u_{q,-})^+$, show that (2p)

$$H = \sum_q \Omega_q\ a_q^\dagger a_q + cst.$$

Thus Ω_q just gives the energy of (fermionic) quasiparticles, and the spectrum becomes gapless at the quantum critical point, $B = J$. How does the gap scale with $B - B_C$? What does that imply for the critical exponents of the Ising model? (+1p)

3.3 (20 p) Dynamical correlations of an Ising spin and mapping to the 1D classical Ising model. Consider the following Hamiltonian describing a spin in a magnetic field:

$$\hat{H}_Q \equiv -B\,\hat{\sigma}_x\,,$$

with $\hat{\sigma}_x$ the Pauli Matrix.

a. **(5 p)** First, as a warm-up, repeat what we did at class: Compute the partition function:

$$Z \equiv \mathrm{Tr}\{e^{-\beta\,\hat{H}_Q}\} \tag{5}$$

using the Trotter formula: divide β in the exponential into N pieces, $\Delta\tau \equiv \beta/N$, and insert a complete set $|\sigma\rangle$ at every time $\tau_i = i \cdot \Delta\tau$ $(i = 0, .., N-1)$ using the identity $1 = \sum_\sigma |\sigma\rangle\langle\sigma|$, and

$$(e^{B\Delta\tau})_{\sigma\sigma'} = C(J)\begin{pmatrix} e^{J} & e^{-J} \\ e^{-J} & e^{J} \end{pmatrix}\,. \tag{6}$$

(Show that $\tanh(B\Delta\tau) = e^{-2J}$. What is the expression for $C(J)$?) Show that the partition function of the quantum system then reads,

$$Z = C^N \sum_{\{\sigma_i\}} \exp\{J\sum_i \sigma_i\sigma_{i+1}\}\,,$$

i.e., it is just the classical partition function of the Ising model with an appropriate $J = -\frac{1}{2}\ln\tanh(B\Delta\tau)$.

b. **(5 p)** Now introduce the imaginary time Heisenberg operators,

$$\hat{\sigma}_z(t) = e^{i\hat{H}_Q t}\hat{\sigma}_z e^{-i\hat{H}_Q t} \to \hat{\sigma}_z(\tau) = e^{\hat{H}_Q\tau}\hat{\sigma}_z e^{-\hat{H}_Q\tau}$$

and their correlation functions

$$C(\tau_1-\tau_2) \equiv \langle\hat{\sigma}_z(\tau_1)\hat{\sigma}_z(\tau_2)\rangle_{\hat{H}_Q} \equiv \mathrm{Tr}\{e^{-\beta\,\hat{H}_Q}\hat{\sigma}_z(\tau_1)\hat{\sigma}_z(\tau_2)\}/Z\,,$$

with $\tau_1 > \tau_2$. Show that this correlation function depends indeed only on the difference $\tau_1 - \tau_2$. Now repeat the previous procedure, by choosing $\tau_1 = i\times\Delta\tau$ and $\tau_2 = j\times\Delta\tau$, and using the identity $\hat{\sigma}_z\sum_\sigma|\sigma\rangle\langle\sigma| = \sum_\sigma|\sigma\rangle\sigma\langle\sigma|$. Show that

$$C(\tau_1-\tau_2) = \langle\sigma_i\sigma_j\rangle_{\mathcal{H}} = \frac{\sum_{\{\sigma_k\}}\sigma_i\sigma_j\exp\{J\sum_k\sigma_k\sigma_{k+1}\}}{\sum_{\{\sigma_k\}}\exp\{J\sum_k\sigma_k\sigma_{k+1}\}},$$

where the average is taken with the classical Hamiltonian:

$$\mathcal{H} = -J\sum_k \sigma_k\sigma_{k+1}.$$

This means that the imaginary time correlation function of a quantum spin in a transverse magnetic field is identical to the spatial correlation function of a one-dimensional Ising chain.

c. **(5 p)** Now compute the correlation function III by simply diagonalizing $\hat{H}_Q$. (Hint: Construct the eigenvectors $|\pm\rangle$ of $\hat{H}_Q$ and the corresponding eigenvalues, and use these to evaluate the trace. You will have to compute the matrix elements of $\hat{\sigma}_z$ between them to evaluate the correlation function.) Show that the correlation length in time direction is simply given by:

$$\xi_\tau = \frac{1}{\Delta}, \tag{7}$$

with $\Delta = 2B$ the "gap", i.e., the energy difference between the ground state and the excited state. Express also this correlation length in terms of J and $\Delta\tau$ in the limit $\Delta\tau \to 0$ using the connection found above, and show that this corresponds indeed to the result we obtained at class for the 1D classical Ising model in the limit $J \gg 1$.

d. **(5 p)** To think: Can you generalize c. to a quantum system in higher dimensions with a gap in the excitation spectrum, and show that the relation $\xi_\tau = \frac{1}{\Delta}$ holds in general for ANY system with a gap?

IV. SURFACE ROUGHENING

4.1 (10 p) Calculate the scaling of the width of the surface in the Edwards-Wilkinson model
Start from the solution of the Edwards-Wilkinson model in the Fourier space:

$$h(\mathbf{q},t) = h(\mathbf{q},0)e^{-\nu q^2 t} + \int_0^t e^{-\nu q^2 (t-t')}\eta(\mathbf{q},t')dt' \tag{8}$$

a. **(4 p)** Derive the width

$$W^2(L,t) = \langle \overline{h^2(\mathbf{x},t)} - \overline{h(\mathbf{x},t)}^2 \rangle \tag{9}$$

using the correlator:

$$\langle \eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle = \Gamma\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \tag{10}$$

b. **(3 p)** Using the change of variable $\mathbf{s} = L\mathbf{q}$, determine the scaling function $w(u)$

$$W(L,t) \propto L^\alpha w\left(\frac{t}{L^z}\right) \tag{11}$$

c. **(3 p)** determine the exponents $\alpha,\ \beta,\ z$.

4.2 (15 p) Show that the Edwards-Wilkinson and the KPZ equations have the same long time characteristics (same exponent α) in one dimension
Let $\mathcal{N}$ be either the EW or the KPZ operator and $P(h,t)$ the probability of a certain height profile h at time t. The Fokker-Planck equation reads as:

$$\frac{\partial P}{\partial t} = \int \frac{\delta}{\delta h}\left[-\mathcal{N}P + \frac{\Gamma}{2}\frac{\delta P}{\delta h}\right] d\mathbf{x} \tag{12}$$

a. **(5 p)** Derive formally the stationary solution P_s^{EW} for the EW probability distribution.

b. **(8 p)** Show that in one dimensions the difference between the EW and KPZ operators does not contribute to the stationary solution of P_s, namely

$$\int \frac{\delta}{\delta h}\left[(\nabla h)^2 P_s^{EW}\right] d\mathbf{x} = 0 \tag{13}$$

c. **(2 p)** Show that Eq. (13) does not hold for two dimensions.

4.3 (15 p) Write a code which simulates surface roughening in one dimension:

a. **(5 p)** Write a code which simulates the Random deposition with relaxation (RDR) model.

A discrete one dimensional periodic surface is characterized by its height $h_i \in \mathbb{N}$, $i \in [0, 1, \dots L-1]$. New particles arrive at random position i. If $h_i \leq \min(h_{i+1}, h_{i-1})$ the particle stays where it landed. If $h_i \geq \max(h_{i+1}, h_{i-1})$ then the particle moves randomly to one of its neighboring sites. If only one neighbor is lower than h_i the particle moves there.

b. **(5 p)** Write a code which simulates the ballistic deposition (BD) model. A discrete one dimensional periodic surface is characterized by its height $h_i \in \mathbb{N}$, $i \in [0, 1, \dots L-1]$. New particles arrive at random position i. The new value of h_i is

$$h_i \to \max(h_{i+1}, h_i + 1, h_{i-1}) \tag{14}$$

c. **(5 p)** Measure $W(L,t)$ for different $L \in \{10, 20, 40\}$ and t and verify the Family-Vicsek (dynamic) scaling and determine the exponents. Compare them to the Edwards-Wilkinson and to the Kadar-Parisi-Zhang model.