Laplacian in spherical coordinates

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In this note we express the Laplacian in spherical coordinates. The Laplacian is defined as

$$\Delta = \text{div grad}. \tag{1}$$

We will tread div and grad separately, and then put the two together.

The spherical coordinates r, θ, ϕ are connected to the Cartesian coordinates x, y, z by

$$z = r\cos\theta; \qquad x + iy = r\sin\theta e^{i\phi}. \tag{2}$$

To each point we associate a local Cartesian coordinate system with axis aligned to the spherical coordinates. For this we define unit vectors : $\mathbf{r}$, $\boldsymbol{\theta}$, $\boldsymbol{\phi}$. The differential displacements along these vectors read,

$$d\mathbf{r} = \mathbf{r}dr; \qquad d\boldsymbol{\theta} = \boldsymbol{\theta}rd\theta; \qquad d\boldsymbol{\phi} = \boldsymbol{\phi}r\sin\theta d\phi. \tag{3}$$

The the components of the gradient in these directions are given by partial derivatives,

$$\boldsymbol{\nabla} f = \mathbf{r}\partial_r f + \boldsymbol{\theta}\frac{1}{r}\partial_\theta f + \boldsymbol{\phi}\frac{1}{r\sin\theta}\partial_\phi f. \tag{4}$$

The divergence of a vector function is obtained by taking an infinitesimal cube, and taking the differences of the fluxes on the opposite faces, and dividing by the volume. If we align the cube with the local system of coordinates, we get for the faces dA_j, with $j \in \{r, \theta, \phi\}$, and the volume dV,

$$dA_r = r^2\sin\theta d\phi d\theta; \qquad dA_\theta = r\sin\theta d\phi dr; \qquad dA_\phi = rd\theta dr; \qquad dV = r^2\sin\theta d\theta d\phi dr. \tag{5}$$

We have to plug these into the formula for the divergence,

$$\text{div } \mathbf{a} = \sum_{j\in\{r,\theta,\phi\}} \frac{(a_j dA_j)|_+ - (a_j dA_j)}{dV}. \tag{6}$$

We can evaluate all three terms in the sum. For the first term, we have

$$\frac{a_r|_+(r+dr)^2\sin\theta - a_r r^2\sin\theta}{r^2\sin(\theta)dr} = \frac{1}{r^2}\partial_r(a_r r^2). \tag{7}$$

We could simplify by $\sin\theta$ because it does not change with r. Repeating this calculation for the other two terms, we find

$$\text{div } \mathbf{a} = \frac{1}{r^2}\partial_r(r^2 a_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin(\theta)a_\theta) + \frac{1}{r\sin\theta}\partial_\phi a_\phi. \tag{8}$$

All that is left is to plug Eq. (4) into Eq. (8):

$$\Delta f = \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin^2(\theta)\partial_\theta f) + \frac{1}{r^2\sin\theta}\partial_\phi^2 f. \tag{9}$$