Class 6 - Resonant cavities

Class material

Exercise 6.1 - TE modes in a rectangular cavity resonator

Investigation of the electric and magnetic field of the TE modes in a rectangular cavity with sides a, b and c.

(a) Write down the Maxwell equations for the TE modes inside the cavities assuming ideal conductor boundaries.

(b) Solve the equations and give the frequency of the possible modes.

Exercise 6.2 - Eigenfrequencies of a spherical cavity resonator

Determine the eigenfrequencies of a spherical cavity resonator with radius R and metal wall.

(a) Write down the Maxwell equations for $\mathbf{rE}$ and $\mathbf{rB}$.

(b) Specify the boundary conditions for the case of TE and TM modes.

(c) Solve the equations in spherical coordinates (hint: use spherical Bessel functions).

(d) Determine the eigenfrequencies of the cavity from the boundary conditions.

Exercise 6.3 - Cylindrical resonator cavity: TM modes

Consider a cylindrical resonator cavity with radius a and height b. Determine the lowest frequency for TM modes.

Homework

The following problems (marked with an asterisk) form the basis of the short test at the beginning of the next class.

Exercise 6.4 - TM modes in a rectangular cavity resonator*

Investigation of the electric and magnetic field of the TM modes in a rectangular cavity with sides a, b and c.

(a) Write down the Maxwell equations for the TE modes inside the cavities assuming ideal conductor boundaries.

(b) Solve the equations and give the frequency of the possible modes.

Exercise 6.5 - Dissipation in the TE modes in a rectangular cavity resonator*

Using the results of Exercise 6.1 what is the energy loss in TE modes in a rectangular resonator if the walls have a finite but large σ conductivity?

Exercise 6.6 - Cylindrical resonator cavity: TE modes*

Consider a cylindrical resonator cavity with radius a and height b. Determine the lowest frequency for TE modes.

These problems are for further practice and to have some fun!

Exercise 6.7 - Dissipation from a metal plane

Consider an infinite metal plane with thickness b. The plane is parallel with the $z = 0$ plane, such that its surfaces are the planes $z = \pm b/2$, and its conductivity is σ. A plane wave propagates perpendicularly to the plane in the $+z$ direction. The polarisation (direction of the electric field $\vec{E}(\vec{r}, t)$) of the wave points in the x direction.

(a) Determine the electric field $\vec{E}(\vec{r}, t)$ and the magnetic field strength $\vec{H}(\vec{r}, t)$ vector inside the metal plane.

(b) Compute the $\vec{S}(\vec{r}, t)$ Poynting vector inside the metal plane.

(c) Determine the dissipated energy in a square-shaped region of the plane with side length a.

Exercise 6.8 - Insulator/vacuum transition in a coaxial cable

Consider a cylindrical coaxial wire along the z axis with core radius a and shield radius $b = 2a$. In the part $z > 0$ there is vacuum between the cylinders, while in the part $z < 0$ the space between the cylinders is filled with an insulator with dielectric coefficient ϵ. We excite a TEM mode in the waveguide which propagates in the $+z$ direction.

(a) Specify the boundary conditions on the two sides of the $z = 0$ plane.

(b) What portion of the energy of the incoming wave transmits through the $z = 0$ plane?

Exercise 6.9 - Metal/insulator/vacuum stacking

Consider a thick, infinite metal plate. The upper surface of the plate is the plane $x = 0$, and the metal plate is covered with a homogenous insulator of thickness b and electric/magnetic properties (ϵ_I, μ_I). There is vacuum over the insulator, and inside the insulator a TM mode electromagnetic plane wave propagates in the $+z$ direction.

(a) Write down the general form of the $E_z(x,t)$ component of the electromagnetic wave in the insulator and the vacuum for this case.

(b) Specify the boundary conditions and determine the $E_z(x,t)$ function.

(c) Determine the $E_x(x,t)$ and $H_y(x,t)$ components of the wave.

Exercise 6.10 - Coaxial resonator cavity

Consider two coaxial metal cylinder along the z axis with inner radius a and outer radius b. The height of the cylinders are h, and the top and bottom opening is covered with an annulus shaped metal to form a cavity resonator. Determine the lowest frequency

(a) TEM_{ijk} mode;

(b) TE_{ijk} mode; and

(c) TM_{ijk} mode.

Exercise 6.11 - Copper cylindrical resonant cavity (Jackson 8.6)

A resonation cavity of copper consists of a hollow, right circular cylinder of inner radius R and length L, with flat end faces.

(a) Determine The resonant frequencies of the cavity for all types of waves. With $(1/\sqrt{\mu\epsilon}R)$ as a unit of frequency, plot the lowest four resonant frequencies of each type as a function of R/L for $0 < R/L < 2$. Does the same mode have the lowest frequency for all R/L?

(b) If $R = 2$ cm, $L = 3$ cm, and the cavity is made of pure copper, what is the numerical value of Q for the lowest resonant mode?

Exercise 6.12 - Sphere shell shaped resonant cavity (Jackson 8.7)

A resonant cavity consists of the empty space between two perfecty conducting, concentric spherical shells, the smaller having an outer radius a and the larger an inner radius b. The azimuthal magnetic field has a radial dependence given by the Bessel functions, $j_l(kr)$ and $n_l(kr)$, where $k = \omega/c$.

(a) Write down the transcendental equation for the characteristic frequencies of the cavity for arbitrary l.

(b) For $l = 1$ use the explicit forms of the spherical Bessel functions to show that the characteristic frequencies are given by

$$\frac{\tan kh}{kh} = \frac{\left(k^2 + 1/ab\right)}{k^2 + ab\left(k^2 - 1/a^2\right)\left(k^2 - 1/b^2\right)}$$

where $h = b - a$.

(c) For $h/a \ll 1$, varify that the result of part (b) yields the frequency

$$\omega_l = \simeq \sqrt{l(l+1)}\frac{c}{a}$$

found for the Schumann resonances and find the first order correction in h/a.

The result of part (b) seems to have been derived by J.J. Thomson and published in his book *Recent Research in Electricity and Magnetism*, Oxford Clarendon Press, 1893, pp.373 ff.

Exercise 6.13 - Quality factor of a spherical cavity (Jackson 9.23)

Assume that a spherical resonant cavity has permeable walls of large, but finite conductivity. In the approximation that the skin depth δ is small compared to the cavity radius a, show that the quality factor Q of the cavity is given by

$$Q = \begin{cases} \frac{a}{\delta} & \text{for all TE modes} \\ \frac{a}{\delta}\left(1 - \frac{l(l+1)}{x_{lm}^2}\right) & \text{for all TM modes} \end{cases} \tag{1}$$

where

$$x_{lm} = \frac{a}{c}\omega_{lm} \tag{2}$$

for TM modes.

Exercise 6.14 - Normal modes of a perfectly conducting solid sphere (Jackson 9.24)

Discuss the normal modes of oscillation of a perfectly conducting solid sphere of radius a in free space.

(a) Determine the characteristic equations for teh eigenfrequencies for TE and TM modes of oscillation. Show that the roots for ω always have a negative imaginary part assuming a time dependence of $e^{-i\omega t}$.

(b) Calculate the eigenfrecquencies for the $l = 1$ and $l = 2$ TE and TM modes. Tabulate the wave-length (determined from the real part of the frequency) in units of the radius a and the lifetime (a.k.a. decay time) in units of the transit time a/c for each of the modes.