Class 10 - Relativistic electrodynamics - II

Class material

Exercise 10.1 - Radiation of a relativistically moving source

Let us assume that a source radiates in its rest frame K^* with the following distribution of power

$$\frac{dP^*}{d\Omega^*} = \frac{d^2\mathcal{E}^*}{dt^* d\Omega^*} = f(\cos\theta^*, \phi^*) \tag{1}$$

1. Using Lorentz transformation show that the power of electromagnetic radiation by the source in the far zone has the following form in the laboratory frame K

$$\frac{dP_{\text{source}}}{d\Omega} = \frac{d^2\mathcal{E}}{dt d\Omega} = \frac{1}{\gamma^4(1-\beta\cos\theta)^3} f(\frac{\cos\theta - \beta}{1-\beta\cos\theta}, \phi) \tag{2}$$

   where the radiating charge moves with velocity $\mathbf{v}$ along the polar $z = z^*$ axes of the spherical coordinate systems (r, θ, ϕ) and (r^*, θ^*, ϕ^*), $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$, with c being the speed of light in vacuum.

2. What is the the angular distribution of radiation as detected by distant, fixed observers in the lab frame?

3. Using the above result, compute the angular distribution of the radiation by a single accelerated charge. What is the result when the acceleration is

   (a) parallel

   (b) perpendicular

   to the velocity? Draw the resulting angular distribution and compute the total radiated power.

Exercise 10.2 - Radiation loss in circular accelerator

Assume that a relativistic particle of momentum p, charge q and rest mass m is moving on a circle of radius R.

(a) Show that the total radiated power is

$$P = \frac{Z_0 q^2 c^2}{6\pi R^2}\left(\frac{p}{mc}\right)^4 \tag{3}$$

(b) Assuming that the particle is ultrarelativistic ($\mathcal{E} \approx pc$) show that the energy loss per revolution is

$$\delta\mathcal{E} = \frac{Z_0 q^2 c^2}{3R}\left(\frac{\mathcal{E}}{mc^2}\right)^4 \tag{4}$$

The CERN LEP2 electron-positron collider had a ring of circumference 27 km (assumed to be a perfect circle), and accelerated the electrons to energies $\mathcal{E} \approx 60\,\text{GeV}$. Using that for the electron $mc^2 \approx 0.5\,\text{MeV}$, compute the energy lost by one electron per revolution.

(c) Assuming that the total beam current is $2 \times 3\,\text{mA}$, how much is the radiation power loss in MW?

(d) Now do the same computation for the LHC, where the beams are composed of protons, the beam energy is increased to 7 TeV, while the circulating beam current is $2 \times 0.54\,\text{A}$, while the circumference of the accelerator is kept the same. What is the radiated power in this case and how does it compare to the case of the LEP2?

(e) Compute and compare the magnetic field necessary to keep the electron and proton beams in the above examples on circular trajectory.

Exercise 10.3 - Spectrum of synchrotron radiation - (Jackson, Section 14.6)

Compute the frequency-angle distribution of a relativistic charged particle of charge q in circular motion with radius starting from the results

$$\frac{d^2P}{d\Omega d\omega} = \frac{|\mathbf{e}\cdot\mathbf{C}(\omega)|^2}{\pi Z_0} \tag{5}$$

where $\mathbf{e}$ is the polarisation observed and

$$\mathbf{C}(\omega) = \frac{Z_0 q}{4\pi}\int\limits_{-\infty}^{\infty} dt e^{i\omega(t-\hat{\mathbf{x}}\cdot\gamma(t)/c)}\frac{d}{dt}\left[\frac{\hat{\mathbf{x}}\times(\hat{\mathbf{x}}\times\dot{\beta})}{1-\hat{\mathbf{x}}\cdot\beta}\right] \tag{6}$$

Use the notations

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{7}$$

and assume that the particle is ultrarelativistic i.e. $\gamma \gg 1$, so the distribution is concentrated at small angles $\theta \ll 1$.

(a) Show that the resulting frequency-angle distribution is

$$\frac{d^2P}{d\Omega d\omega} = \frac{Z_0 q^2}{12\pi^3}\left(\frac{\omega R}{c}\right)^2\left(\frac{1}{\gamma^2}+\theta^2\right)^2\left[K_{2/3}(\xi)^2 + \frac{\theta^2}{1/\gamma^2+\theta^2}K_{1/3}(\xi)^2\right] \tag{8}$$

where the first term corresponds to polarisation in the plane of orbit, while second to radiation polarised perpendicular to that plane, and

$$\xi = \frac{\omega R}{3c}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2} \tag{9}$$

The following formulae are useful:

$$\begin{aligned} K_{2/3}(\xi) &= \sqrt{3}\int\limits_0^\infty x\sin\left(\frac{3}{2}\xi(x+x^3/3)\right)dx \\ K_{1/3}(\xi) &= \sqrt{3}\int\limits_0^\infty \cos\left(\frac{3}{2}\xi(x+x^3/3)\right)dx \end{aligned} \tag{10}$$

(b) Show that the radiation drops steeply for ω above the cut-off frequency

$$\omega_c = \frac{3\gamma^3 c}{2R} \tag{11}$$

and the intensity behaves as

$$\left.\frac{d^2P}{d\Omega d\omega}\right|_{\theta=0} \sim \begin{cases} \left(\frac{\omega R}{c}\right)^{2/3} & \omega \ll \omega_c \\ \frac{\omega}{\omega_c}e^{-\omega/\omega_c} & \omega \gg \omega_c \end{cases} \tag{12}$$

(c) Show that for any fixed frequency most of the energy is confined to a narrow cone $\theta < \theta_c$. How does the angle θ_c depends for frequencies low/high compared to the cut-off frequency ω_c?

These problems are for further practice and to have some fun!

Exercise 10.4 - Radiation loss in linear accelerator

Assume that a relativistic particle of momentum p, charge q and rest mass m moves along a line.

(a) Show that the total radiated power is

$$P = \frac{Z_0 q^2}{6\pi m c^2}\left(\partial_t p\right)^2 \tag{13}$$

(b) Assuming that the particle is accelerated by a constant electric field E, show that the power radiated is the same order of magnitude as the power transmitted to the particle by the field when the electric field is

$$E \sim \frac{mc^2}{r_0 q} \tag{14}$$

where the classical charge radius is defined by

$$\frac{q^2}{4\pi\epsilon_0 r_0} = mc^2 \tag{15}$$

This puts an upper limitation on the strength of the accelerating field.

(c) What is the limiting field for (1) an electron (2) a proton? How does this compare to the critical field

$$E_{\text{crit}} = \alpha \frac{m_e c^2}{r_e q} \tag{16}$$

above which the electric field starts creating electron-positron pairs spontaneously from the vacuum?

Note that modern lasers can reach field strengths of up to $10^{12}\,\mathrm{V/m}$, *but only for short impulses. The results show that radiation loss can always be neglected for linear accelerators, in contrast to circular ones.*

Exercise 10.5 - Transition radiation (Jackson 13.12)

A relativistic particle of charge ze moves along the z axis with a constant speed βc. The half-space $z \leq 0$ is filled with a uniform isotropic dielectric medium with plasme frequency ω_1, and the space $z > 0$ with a similar medium whose plasma frequency is ω_2. Discuss the emission of transition radiation as the particle transverses the interface, using the approximation of Jackson Sec.13.7.

(a) Show that the radiation intensity per unit circular frequency interval and per unit solid angle is given approximately by

$$\frac{\mathrm{d}^2 I}{\mathrm{d}\omega \mathrm{d}\Omega} =\approx \frac{z^2 e^2 \theta^2}{\pi^2 c} \left| \frac{1}{\frac{1}{\omega^2} + \frac{\omega_1^2}{\omega^2} + \theta^2} - \frac{1}{\frac{1}{\omega^2} + \frac{\omega_2^2}{\omega^2} + \theta^2} \right|^2$$

where θ is the angle of emission relatice to the velocity of the particle and $\gamma = (1 - \beta^2)^{-1/2}$.

(b) Show that the total energy radiated is

$$I \approx \frac{z^2 e^2}{3c} \frac{(\omega_1 - \omega_2)^2}{(\omega_1 + \omega_2)^2} \gamma$$

Exercise 10.6 - Radiation by a charge in a headon collision (Jackson 14.5)

A *nonrelativistic* particle of charge ze, mass m, and kinetic energy E makes a *headon* collision with a fixed central force field of finite range. The interaction is repulsive and described by a potential $V(r)$, which becomes greater than E at close distances.

(a) Show that the total energy radiated is given by

$$\Delta W = \frac{4}{3} \frac{z^2 e^2}{m^2 c^3} \sqrt{\frac{m}{2}} \int_{r_{\min}}^{\infty} \left| \frac{\mathrm{d}V}{\mathrm{d}r} \right|^2 \frac{\mathrm{d}r}{\sqrt{V(r_{\min}) - V(r)}}$$

where $r_{\min}$ is the closest distance of approach in the collision.

(b) If the interaction is a Columb potential $V(r) = zZe^2/r$, show that the total energy radiated is

$$\Delta W = \frac{8}{45} \frac{zmv_0^5}{Zc^3}$$

where v_0 is the velocity of the charge at infinity.

Hint: use the Larmor formula to compute the radiated power in term of the acceleration and turn the time integral into a radial one using the equation of motion.

Exercise 10.7 - Radiation of particle accelerated by external electric and magnetic fields (Jackson 14.11)

A particle of charge ze and mass m moves in external electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$.

(a) Show that the classical relativistic result for the instantaneous energy radiated per unit time can be written

$$P = \frac{2z^4e^4}{3m^2c^2}\gamma^2\left[(\mathbf{E}+\beta\times\mathbf{B})^2 - (\beta\cdot\mathbf{E})^2\right]$$

where $\mathbf{E}$ and $\mathbf{B}$ are evaluated at the position of the particle.

(b) Show that the expression in part a) can be put into the manifestly Lorentzinvariant form

$$P = \frac{2z^4r_0^2}{3m^2c}F^{\mu\nu}p_\nu p^\lambda F_{\lambda\mu}$$

where $r_0 = e^2/mc^2$ is the classical charged particle radius.

Exercise 10.8 - Radiation of charge in harmonic motion (Jackson 14.12)

A charge e moves in simple harmonic motion along the z axis: $z(t') = a\cos\omega_0 t'$.

(a) Show that the instantaneous power radiated per unit solid angle is

$$\frac{dP(t')}{d\Omega} = \frac{e^2c\beta^4}{4\pi a^2}\frac{\sin^2\theta\cos^2\omega_0 t'}{(1+\beta\cos\theta\sin\omega_0 t')^5}$$

(b) By performing a time averaging, show that the average power per unit solid angle

$$\frac{dP}{d\Omega} = \frac{e^2c\beta^4}{32\pi a^2}\frac{4+\beta^2\cos^2\theta}{(1-\beta^2\cos^2\theta)^{7/2}}$$

(c) Make rough sketches of the angular distribution for nonrelativistic and relativistic motion.

Exercise 10.9 - Field of a relativistic particle moving in a medium (Jackson 13.10)

A particle of charge ze moves along the z axis with constant speed v, passing $z = 0$ at $t = 0$. the medium through which the particle moves is described by dielectric constant $\epsilon(\omega)$ and magnetic permeability $\mu_r = 1$.

(a) Defining the Fourier representation by

$$F(\mathbf{x},t) = \frac{1}{s(2\pi)^2}\int d^3k\int d\omega F(\mathbf{k},\omega)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \tag{17}$$

follow our derivation of Cherenkov radiation to solve the wave equation in the medium to obtain the following result for the potentials

$$\begin{aligned}\Phi(\mathbf{k},\omega) &= \frac{ze}{2\pi\epsilon(\omega)}\frac{\delta(\omega-\mathbf{k}\cdot\mathbf{v})}{k^2-\omega^2\epsilon(\omega)/c^2} \\ \mathbf{A}(\mathbf{k},\omega) &= \epsilon(\omega)\mu\mathbf{v}\Phi(\mathbf{k},\omega) \end{aligned} \tag{18}$$

(b) Beginning with the potential $\Phi(\mathbf{k},\omega)$ above, show that the potential of frequency ω is given as a function of spatial coordinate $\mathbf{x}$ by

$$\Phi(\mathbf{x},\omega) = \frac{ze}{4\pi v\epsilon(\omega)}\sqrt{\frac{2}{\pi}}K_0\left(\frac{|\omega|\,\rho}{v}\sqrt{1-\beta^2\epsilon_r}\right)\mathrm{e}^{\imath\omega z/v}$$

where z and $\rho = \sqrt{x^2+y^2}$ are the cylindrical coordinates of the observation point.

(c) Assuming that ϵ is independent of frequency and that $\beta^2\epsilon < 1$, take the Fourier transformation with respect to ω of the expression in part (a) and obtain $\Phi(\mathbf{x},t)$. Calculate the electric and magnetic fields and compare them to fields of a charge moving with constant speed v in the vacuum. Show that, among other things, the vacuum factor γ is replaced by $\Gamma = (1-\beta^2\epsilon)^{-1/2}$.

(d) Repeat the calculations of parts (a) and (b) with $\beta^2\epsilon > 1$. Show that now

$$\Phi(\mathbf{x},\omega) = \frac{ze}{4\pi v\epsilon(\omega)}\sqrt{\frac{\pi}{2}}\,\mathrm{e}^{\imath\omega z/v}\left[-N_0\left(\frac{|\omega|\,\rho}{v}\sqrt{\beta^2\epsilon_r-1}\right)\pm\imath J_0\left(\frac{|\omega|\,\rho}{v}\sqrt{\beta^2\epsilon_r-1}\right)\right]$$

for $\omega \gtrless 0$. Calculate the remaining Fourier transform to obtain $\Phi(\mathbf{x},t)$.

Exercise 10.10 - Radiation by a charge in a collision with a nonzero impact parameter (Jackson 14.6)

(a) Generalize the circumstances of the collision of Exercise 10.6 to nonzero angular momentum (impact parameter) and show that the total energy radiated is given by

$$\Delta W = \frac{4}{3}\frac{z^2e^2}{m^2c^3}\sqrt{\frac{m}{2}}\int_{r_{\min}}^{\infty}\left|\frac{\mathrm{d}V}{\mathrm{d}r}\right|^2\frac{\mathrm{d}r}{\sqrt{E-V(r)-\dfrac{L^2}{2mr^2}}}$$

where $r_{\min}$ is the closest distance of approach (root of $E - V - L32/2mr^2$), $L = mbv_0$, where b is the impact parameter, and v_0 is the incident speed ($E = mv_0^2/2$).

(b) Specialize to the repulsive Coulomb potential $V(r) = zZe^2/r$. Show that ΔW can be written in terms of the impact parameter as

$$\Delta W = \frac{2zmv_0^5}{Zc^3}\left[-t^{-4}+t^{-5}\left(1+\frac{t^2}{3}\right)\tan^{-1}t\right]$$

where $t = bmv_0^2/zZe^2$ is the ratio of twice the impact parameter to the distance of closest approach in a head-on collision.

Show that in the limit of t going to zero the result of Exercise 10.6 (b) is recovered, while for $t >> 1$ one obtains the approximate result of Exercise 10.11 (a).

(c) Using the relation between the scattering angle θ and $t(=\cot\theta/2)$, show that ΔW can be expressed as

$$\Delta W = \frac{2zmv_0^5}{Zc^3}\tan^3\frac{\theta}{2}\left[\frac{1}{6}(\pi-\theta)\left(1+3\tan^2\frac{\theta}{2}\right)-\tan\frac{\theta}{2}\right]$$

(d) What charges occur for an *attracite* Couloomb potential?

Exercise 10.11 - Radiation by a non-relativistic charge passing through a fixed Coulomb potential (Jackson 14.7)

A nonrelativistic particle of charge ze, mass m and initial speed v_0 is incident on a fixed charge Ze at an mpact parameter b that is large enough to ensure that the particle's deflection in the course of the collision is very small.

(a) Using the Larmor power formula and Newton's second law, calculate the total energy radiated, assuming (after you have computed the acceleration) that the particle's trajectory can be approximated by a straight line at constant speed:

$$\Delta W = \frac{\pi z^4 Z^2 e^6}{3m^2 c^3 v_0} \frac{1}{b^3}$$

(b) The expression found in (a) is an approximation which fails at small enough impact parameter. For a repulsive potential the closest distance of approach at zero impact parameter, $r_c = 2zZe^2/mv_0^2$ serves as a length against which to measure b. The approximation will be valid for $b \gg r_c$. Compare the result of replacing b by r_c in part (a) with the answer of Problem 10.6 for head-on collision.

Exercise 10.12 - Radiation by a relativistic charge passing through a fixed Coulomb potential (Jackson 14.8)

A swiftly moving particle of charge ze and mass m passes a fixed point charge Ze in an approximately straight-line path at impact parameter b and nearly constant speed v. Show that the total energy radiated in the encounter is

$$\Delta W = \frac{\pi z^4 Z^2 e^6}{4m^2 c^4 \beta} \left(\gamma^2 + \frac{1}{3}\right) \frac{1}{b^3}$$

This is the relativistic generalization of the result of Problem 10.11.

Exercise 10.13 - Radiation by a charge moving perpendicularly to a uniform static magnetic field (Jackson 14.9)

A particle of mass m and charge q moves in a plane perpendicular to a uniform static magnetic field $\mathbf{B}$.

(a) Calculate the total energy radiated per unit time, expressing it in terms of the constants already defined and the ratio γ of the particle's total energy to its rest energy.

(b) If at time $t = 0$ the particle has total energy $E_0 = \gamma_0 mc^2$, show that it will have energy $E = \gamma mc^2 < E_0$ at time t, where

$$t = \frac{3m^3 c^5}{2q^4 B^2} \left(\frac{1}{\gamma} - \frac{1}{\gamma_0}\right)$$

provided $\gamma \gg 1$.

(c) If the particle is initially nonrelativistic and has a kinetic energy T_0 at $t = 0$, what is its kinetic energy at time t?

Exercise 10.14 - Transition radiation through a dielectric foil (Jackson 13.13)

Consider the transition radiation emitted by a relativistic particle trabersing a dielectric foil of thickness a perpendicular to its path. Assuming that reflections can be ignored because

$$\frac{n(\omega) - 1}{n(\omega) + 1}$$

is very small, show that the differential angular and frequency spectrum is given by the single-interface result (13.84) times the factor:

$$\mathcal{F} = 4 \sin^2 \Theta \,, \qquad \text{with} \qquad \Theta = \nu(1 + \frac{1}{\nu^2} + \eta) \frac{a}{4D} \,.$$

Here $D = \gamma c/\omega_p$ is the formation length, $\nu = \omega/\gamma\omega_o$, and $\eta = (\gamma\theta)^2$. Provided $a \gg D$, the fator $\mathcal{F}$ oscillates extremely rapidly in angle of frequency, averaging to $\langle \mathcal{F} \rangle = 2$. For such foils the smoothed intensity distribution is just twice that for a single interface. Frequency distributions for different values of $\Gamma = 2D/a$ are displayed in Fig.1 of G.B.Yodh, X.Artru and R.Ramaty, *Astrophys.J.* **181**, 725 (1973)

Exercise 10.15 - Transition radiation through a dielectric foil stack (Jackson 13.14)

Transition radiation is emitted by a relativistic particle traversing normally a uniform array of N dielectric foils, each of thickness a, separated by air gaps (effectively vacuum), each of length b. Assume that multiple reflections can be neglected for the whole stack. That requires

$$\left|\frac{n(\omega)-1}{n(\omega)+1}\right| \approx \frac{\omega_p^2}{4\omega^2} \ll \frac{1}{N}$$

(a) Show that if the dielectric constant of the medium varies in the z direction as $\epsilon(\omega, z) = 1 - (\omega_p^2/\omega^2)\rho(z)$, the differential spectrum of transition radiation is given approximately by the single-interface result

$$\frac{d^2I}{d\nu d\eta} \simeq \frac{z^2e^2\gamma\omega_p}{\pi c}\left[\frac{\eta}{\nu^4(1+\nu^{-2}+\eta)^2(1+\eta)^2}\right]$$

(Jackson 13.84), where $\eta = (\gamma\theta)^2$ is an appropriate angular variable and

$$\nu = \frac{\omega}{\gamma\omega_p},$$

multipled by the factor

$$\mathcal{F} = \left|\mu \int \mathrm{d}z \rho(z)\, \mathrm{e}^{\imath\omega z/v} \exp\left(-\imath \cos\theta \int^z k(z')\mathrm{d}z'\right)\right|^2$$

where $\rho(0) = 1$ by convention, $\mu = \omega/v - k(0)\cos\theta$, and $k(z) = (\omega/c)\sqrt{\epsilon(\omega, z)}$.

(b) Show that for the stack of N foils

$$\mathcal{F} = 4\sin^2\Theta \frac{\sin^2[N(\Theta+\Psi)]}{\sin^2[\Theta+\Psi]}$$

where Θ is defined in Exercise 10.14 and $\Psi = \nu(1+\eta)(b/4D)$. Compare G.M.Garibyan, *Zh. Eksp. Teor. Fiz.* **60**, 39 (1970) [transl. *Sov.Phys.Rev.D* **12**,1289] (1975).
The principal theory of multilayered transition radiation detectors is treated in great details by X.Artru, G.B.Yodh and G.Mennessier, *Phys.Rev.D.* **12** 1289 (1975).