Short questions for Electrodynamics 2 exam

1. Curvilinear coordinates

Cylindrical coordinates: $\boldsymbol{r} = (\rho\cos\varphi\,,\ \rho\sin\varphi\,, z)$

Compute the expression of the gradient, divergence and the scalar Laplacian in cylindrical coordinates.

2. Laplace-equation with azimuthal symmetry

The Laplace equations has the following form in spherical coordinates:

$$\frac{1}{r}\partial_r{}^2(r\Phi) + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta\Phi) + \frac{1}{\sin^2\theta}\partial_\phi^2\Phi\right) = 0$$

a) Separate the variables for azimuthal symmetry (no dependence on ϕ).

b) Solve the radial part and write down the general solution, denoting the angular part with $P_l(\cos\theta)$.

c) Formulate the statement that the Legendre polynomials $P_l(x)$form a complete orthogonal system in the space of square integrable functions on the interval $[-1,1]$.

d) Prove that for $r' = |\vec{x}'| > r = |\vec{x}|$ we have

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \frac{r^l}{r'^{l+1}} P_l(\cos\gamma)$$

where γ is the angle between the vectors $\vec{x}$ and $\vec{x}'$, provided we normalise $P_l(1) = 1$!

e) Formulate and prove the corresponding statement for in the case $r' = |\vec{x}'| < r = |\vec{x}|$.

3. Potentials in planar polar coordinates

Polar coordinates in the plane: $x = r\cos\varphi \quad y = r\sin\varphi$

a) The Laplace equation in planar polar coordinates is $\frac{1}{r}\partial_r(r\partial_r\Phi) + \frac{1}{r^2}\partial_\varphi{}^2\Phi = 0$. Separate the variables.

b) What is the solution for the rotation invariant (φ-independent) case?

c) Write down the general solution for the potential.

d) Assume that the potential is given on a circle of radius R:

$$\Phi(r = R, \varphi) = V(\varphi) = \sum_n V_n e^{in\varphi}$$

Write the solution inside and outside the circle.

4. Spherical harmonics: eigenfunctions of the spherical Laplace operator

$$\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta)+\frac{1}{\sin^2\theta}\partial_\phi^2\right]Y_{lm}(\theta,\phi)=l(l+1)Y_{lm}(\theta,\phi)$$

a) Write the orthogonality relation for spherical harmonics.

b) State the completeness of the system of spherical harmonics.

c) Denoting the unit vector in direction (θ,φ) by $\boldsymbol{n}$, the transformation of spherical harmonics under a rotation O is

$$Y_{lm}(O\boldsymbol{n})=\sum_{m'}D^{(l)}_{mm'}(O)Y_{lm'}(\boldsymbol{n})$$

Complete the following statements of the properties of the matrix $D^{(l)}$:

$D^{(l)}(O)^\dagger D^{(l)}(O)=$ $\qquad\qquad$ $D^{(l)}(O_1O_2)=$

5. Spherical multipoles

a) Using the result of 2 d) and e), plus the addition theorem $P_l(n\cdot n')=\frac{4\pi}{2l+1}\sum_m Y_{lm}(n)^*Y_{lm}(n')$, expand the Green's function of the Poisson equation $\frac{1}{|\vec{x}-\vec{x}'|}$ in spherical harmonics.

b) Use the above relation to write the spherical multipole expansion of the potential of a localised charge distribution. What is the number of independent components of the lth multipole?

c) Express the usual (Descartes) dipole vector and quadrupole tensor with the spherical ones using

$$Y_0^0(\theta,\phi)=\sqrt{\frac{1}{4\pi}}$$

$$Y_1^0(\theta,\phi)=\sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_1^1(\theta,\phi)=-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi} \qquad Y_1^{-1}(\theta,\phi)=\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}$$

$$Y_2^0(\theta,\phi)=\sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1) \quad Y_2^1(\theta,\phi)=-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi} \quad Y_2^{-1}(\theta,\phi)=\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{-i\phi}$$

$$Y_2^2(\theta,\phi)=\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{2i\phi} \qquad Y_2^{-2}(\theta,\phi)=\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{-2i\phi}$$

6. Quasi-stationary approximation

a) What are the boundary conditions satisfied by the electric and magnetic fields on the surface of an ideal conductor? Express the surface current density in terms of the fields on the surface!

b) What is the skin effect?

c) Write down the condition for the validity of the quasi-stationary approximation in terms of material constants σ and ε? How can this condition be expressed in terms of the wave-length and the skin depth?

d) Express the effective energy current density (of the real fields) assuming the complex time dependences $\boldsymbol{E}\sim e^{-i\omega t}$ and $\boldsymbol{H}\sim e^{-i(\omega t+\varphi)}$?

e) Express the energy loss in the wall in terms of the magnetic field at the surface.

7. Wave guides

a) What are the variables ("master functions") to which the determination of the electric and magnetic fields can be reduced in general? What is the equation satisfied by these variables?

b) What is the TEM mode and what are the conditions for its existence? Write the dispersion relation for it.

c) What are the TE and TM modes? Write their dispersion relation and state the condition for their propagation.

d) Determine the group and phase velocities of the propagating waves.

8. Resonant cavity: consider a wave guide bounded by two transverse conducting planes.

a) Derive the boundary condition satisfied by master functions of the TE and TM modes at the two bounding planes.

b) Starting from the dispersion relation of the wave guide write down the frequencies of the modes in the resonant cavity.

c) Define the quality factor. How does the spectral function look for a finite quality factor Q, and what is the half-width?

d) Select the properties that the quality factor depends on:

- geometry of the cavity;
- which mode is excited;
- material of the cavity walls;
- amplitude of the mode.

9. Linear response theory and dielectric coefficient

a) Write the relation between the electric field strength $\boldsymbol{E}$ and electric polarisation $\boldsymbol{P}$ in real time, and in Fourier components, assuming linear response. Express the dielectric coefficient with the response function (susceptibility).

b) What is the fundamental property implying the Kramers-Kronig relation? What are the physical quantities related?

c) What is the physical effect described the imaginary part of the refraction coefficient? What is the definition of opacity?

10. Dielectric coefficient of free electron gas

a) Write down the dielectric coefficient of free electrons in terms of the plasma frequency ω_p ?

b) What is the physical phenomenon corresponding to the plasma frequency ω_p?

c) How do the electromagnetic waves propagate if (1) $\omega > \omega_p$ (2) $\omega < \omega_p$?

11. Conductivity and dielectric coefficient

a) Express the polarisation current density with $\boldsymbol{P}$ and justify the result.

b) What is the relation between the dielectric coefficient and conductivity of metals from Ohm's law?

12. Retarded potential and radiation

a) Write down the retarded solution for the vector potential $\boldsymbol{A}$ assuming a time dependence $e^{-i\omega t}$.

b) What is the radiation zone and how does the radiation part of $\boldsymbol{A}$ look?

c) What expansion gives the multipole components of radiation and what is the condition of its validity?

13. Description of the radiation

a) What simplifications can be used when computing the radiation fields from the vector potential? Express the radiation parts of the magnetic and electric fields.

b) How can one obtain the effective energy current density **S** in terms of the magnetic field strength?

c) Using **S** express the power radiated in a unit solid angle.

d) Draw a qualitative picture of the direction dependence for (i) electric dipole (ii) magnetic dipole and (iii) electric quadrupole radiation.

14. Scattering cross section and form factor

a) Define the differential cross section $\frac{d\sigma}{d\Omega}$.

b) What is the form factor and how to compute it? Calculate the form factor of a random medium.

c) Derive the form factor of a cubic crystal.

d) What is the condition of amplification (Bragg condition)? How does a crystal scatter light with a wave-length much longer than the lattice constant?

15. Rayleigh scattering

a) How does the (electric) dipole scattering cross section depend on the wave number k and the incoming and outgoing polarisations $\boldsymbol{e}_0$ and $\boldsymbol{e}$? Compute the total scattering cross section and outgoing polarisation of unpolarised light.

c) What is the total cross section in a gas of volume V and density ρ? Compute the attenuation coefficient.

d) Why is the sky blue and why are the rising and setting suns red?

e) Which part of the sky emits completely polarised light and what is its direction of polarisation?

f) How does a medium close to the vapour-liquid critical point affect the propagation of light? Give a short qualitative explanation.

16. Retardation

a) A point charge moves on the trajectory $\boldsymbol{\xi}(t)$. The fields are observed at time t and position $\boldsymbol{x}$. Write down the equations determining the the retarded time *t* and the retarded distance *R***.**

b) What is the relative factor $\frac{\partial t}{\partial \bar{t}}$ between observation time and retarded time?

c) Write down the scalar and vector potentials.

17. Radiation of an ultra-relativistic charged particle

a) Characterise the direction dependence of the radiation of an ultra-relativistic charge particle. Compare it with the result obtained in dipole approximation and draw both qualitatively.

b) Draw the frequency spectrum of synchrotron radiation. Estimate the characteristic frequency by a simple argument.

c) What are the main features of radiation loss in a circular and a linear accelerator?

18. Cherenkov and transition radiation

a) What is the condition determining the allowed frequencies of Cherenkov radiation?

b) What determines the frequency dominating the Cherenkov radiation?

c) Determine the angle of Cherenkov radiation to the particle trajectory.

d) What is the characteristic angular dependence of transition radiation? What are the dominant frequencies for a material with plasma frequency ω_p?

19. Radiation backreaction

a) Derive the Abraham-Lorentz force using a simple argument.

b) What is the origin of the radiation backreaction?

c) List the main problems with the simple Abraham-Lorentz result and briefly describe the solutions.