Lecture 1: Potential theory I

1. Reproduce the expressions of gradient, divergence, curl and Laplacian in cylindrical and spherical coordinates.

2. Prove
$$\Delta\vec{v} = \text{grad div } \vec{v} - \text{curl curl } \vec{v}$$

Lecture 2: Potential theory II

1. Consider the solution of the Legendre equation
$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + l(l+1)P = 0$$
using the Frobenius Ansatz
$$P(x) = x^\alpha \sum_n a_n x^n$$
Compute α and derive the recurrence relation satisfied by the coefficients a_n.

2. In the calculation of the edge effect we substituted *y=(1-x)/2* in the Legendre equation. Apply the Frobenius Ansatz in the variable *y*
$$P(y) = y^\alpha \sum_n a_n y^n$$
and show that $\alpha = 0$.

Lecture 3: Potential theory III

1. Consider the associated Legendre equation
$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P = 0$$
Assume that $m \geq 0$ and expand the solution around $x = 1$ by introducing the new variable $y = 1 - x$ and writing
$$P(y) = y^\alpha \sum_n a_n y^n$$
Show that the appropriate solution for α is $\alpha = m/2$.

2. Using the properties of the associated Legendre functions, show that the spherical harmonics
$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}$$
satisfy the orthogonality relation
$$\int d\Omega\ Y_l^m(\theta,\phi)Y_{l'}^{m'}(\theta,\phi)^* = \delta_{ll'}\delta_{mm'}$$

3. The completeness property of the spherical harmonics states that any function $f(\theta,\phi)$ square integrable on the unit sphere can be expanded as

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}C_{lm}\,Y_l^m(\theta,\phi)$$

Show that

$$C_{lm}=\int d\Omega\,f(\theta,\phi)\,Y_l^m(\theta,\phi)^*$$

4. Express the Descartes components of the electric dipole moment with the spherical multipole moments

$$\begin{pmatrix}p_x\\p_y\\p_z\end{pmatrix}=\sqrt{\frac{2\pi}{3}}\begin{pmatrix}q_{1,-1}-q_{1,1}\\i(q_{1,-1}+q_{1,1})\\\sqrt{2}q_{1,0}\end{pmatrix}$$

Lecture 4: Surface effects in conductors. General theory of wave guides

1. Following the methods used in the lecture, derive the splitting of the equation

$$\vec{\nabla}\times\vec{B}=-i\omega\mu\epsilon\vec{E}$$

Into components parallel and perpendicular to the *z* direction.

2. Using the general wave guide equations verify that the following relation holds between the transverse fields

$$\vec{H}_T=\frac{1}{Z}\vec{e}_z\times\vec{E}_T$$

where the wave impedance is given by

$$Z=\begin{cases}\dfrac{k}{\epsilon\omega}=\dfrac{k}{k_0}\sqrt{\dfrac{\mu}{\epsilon}} & (\mathrm{TM})\\ \dfrac{\mu\omega}{k}=\dfrac{k_0}{k}\sqrt{\dfrac{\mu}{\epsilon}} & (\mathrm{TE})\end{cases}$$

Lecture 5: TEM, TE and TM modes in wave guides

1. Finish the computation of the energy per unit length U and check that the energy velocity

$$v_E=\frac{P}{U}$$

(computed as the ratio of the transmitted power *P* and the energy per unit length *U*) agrees with the group velocity

$$v_g=\frac{1}{\sqrt{\epsilon\mu}}\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}$$

both for the TE and TM modes.

Lecture 6: Resonant cavities

No homework assigned.

Lecture 7: Electromagnetic waves in matter, dispersion, Kramers-Kronig relations

1. Starting from the Lorentz-Drude formula

$$\epsilon_r(\omega) = 1 + \sum_j \frac{N_j^2 q_j^2}{\epsilon_0 m_j} \frac{1}{\omega_j^2 - \omega^2 - i\gamma_j \omega}$$

compute

$$\operatorname{Im} \epsilon_r(\omega) = \sum_j \frac{N_j^2 q_j^2}{\epsilon_0 m_j} \frac{\gamma_j \omega}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j^2 \omega^2}$$

2. From

$$\epsilon_r(\omega + i\delta) = 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \, \frac{\epsilon_r(\omega') - 1}{\omega' - \omega - i\delta}$$

derive the relation

$$\epsilon_r(\omega) = 1 + \frac{1}{\pi i} P \int_{-\infty}^{\infty} d\omega' \, \frac{\epsilon_r(\omega') - 1}{\omega' - \omega}$$

using the distribution identity

$$\lim_{\delta \to +0} \frac{1}{x - i\delta} = P\frac{1}{x} + i\pi\delta(x)$$

Lecture 8: Multipole radiation

1. Quadrupole radiation - angular dependence
Assuming cylindrical symmetry around axis 3

$$Q_{33} = Q_0 \,, \quad Q_{11} = Q_{22} = -\frac{Q_0}{2} \,, \quad \text{while } \; Q_{jk} = 0 \; \text{ for } \; j \neq k$$

show that

$$\frac{dP}{d\Omega} = \frac{Z_0 \omega^6}{512\pi c^4} Q_0^2 \sin^2\theta \cos^2\theta$$

Lecture 9: Scattering of EM waves

1. Radiation part of electric field of oscillating electric and magnetic dipole

In the previous lecture we obtained the vector potential

$$\vec{A}_{rad}(t, \vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{-i\omega t + ikr}}{r} \vec{a}(k, \hat{x})$$

where up to the electric and magnetic dipole terms

$$\vec{a}(k,\hat{x}) = -i\omega\left(\vec{p} + \frac{1}{c}\vec{m} \times \hat{x}\right)$$

Compute the electric and magnetic field strength of the radiation emitted in direction $\hat{x} = \vec{n}$ and verify that

$$\vec{E}_{rad} = \frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r}\left[(\vec{n} \times \vec{p}) \times \vec{n} - \frac{1}{c}\vec{n} \times \vec{m}\right] \qquad \vec{H}_{rad} = \frac{1}{Z_0}\vec{n} \times \vec{E}_{rad}$$

2. Take an inhomogeneous medium with position dependent linear response coefficients:

$$\epsilon(\vec{x}) = \bar{\epsilon} + \delta\epsilon(\vec{x}) \qquad \mu(\vec{x}) = \bar{\mu} + \delta\mu(\vec{x})$$

Derive the wave equation for the magnetic field

$$\left(\Delta - \frac{1}{\bar{c}^2}\partial_t^2\right)\vec{B} = -\bar{\mu}\,\partial_t \text{curl}\left(\vec{D} - \bar{\epsilon}\vec{E}\right) - \text{curl curl}\left(\vec{B} - \bar{\mu}\vec{H}\right)$$

Lecture 10: EM field of a moving charge

1. Trajectory of charge: $\vec{\xi}(t)$
Derive

$$\vec{A}(\vec{x},t) = \frac{\mu_0 q}{4\pi}\frac{\vec{v}(\bar{t})}{R - \vec{\beta}(\bar{t}) \cdot \vec{R}}$$

2. Magnetic field strength

$$H_i = \frac{1}{\mu_0}\epsilon_{ijk}\partial_j A_k = \epsilon_{ijk}\partial_j\left(\frac{q}{4\pi}\frac{\vec{v}(\bar{t})}{R - \vec{\beta}(\bar{t}) \cdot \vec{R}}\right)$$

Do the differentiation using

$$\frac{\partial\bar{t}}{\partial t} = \frac{1}{1 - \hat{R} \cdot \vec{\beta}} \qquad \frac{\partial R}{\partial t} = -\frac{\hat{R} \cdot \vec{\beta}c}{1 - \hat{R} \cdot \vec{\beta}} \qquad \frac{\partial R_i}{\partial t} = -\frac{\beta_i c}{1 - \hat{R} \cdot \vec{\beta}}$$

$$\frac{\partial\bar{t}}{\partial x_i} = -\frac{1}{c}\frac{\hat{R}_i}{1 - \hat{R} \cdot \vec{\beta}} \qquad \frac{\partial R}{\partial x_i} = \frac{\hat{R}_i}{1 - \hat{R} \cdot \vec{\beta}} \qquad \frac{\partial R_j}{\partial x_i} = \delta_{ij} + \frac{\hat{R}_i\beta_j}{1 - \hat{R} \cdot \vec{\beta}}$$

and show that

$$\vec{H} = \frac{1}{Z_0}\hat{R} \times \vec{E}$$

where

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{R} - R\vec{\beta}}{\left(R - \vec{\beta} \cdot \vec{R}\right)^3}(1 - \beta^2) + \frac{\mu_0 q}{4\pi}\frac{\vec{R} \times \left(\left(\vec{R} - R\vec{\beta}\right) \times \vec{a}\right)}{\left(R - \vec{\beta} \cdot \vec{R}\right)^3}$$

3. Derive the potentials for a charge in uniform motion using Lorentz transformation!

$$\text{In rest frame: } \Phi(t,\vec{x}) = \frac{q}{4\pi\epsilon_0|\vec{x}|} \qquad \vec{A}(t,\vec{x}) = 0$$

Transformation to coordinate system moving with u in direction z:

$$t' = \gamma\left(t - \frac{uz}{c^2}\right) \qquad x' = x \qquad y' = y \qquad z' = \gamma(z - ut)$$

$$\Phi' = \gamma(\Phi - uA_z) \quad A'_x = A_x \quad A'_y = A_y \quad A'_z = \gamma\left(A_z - \frac{u\Phi}{c^2}\right)$$

Hint: we need $\Phi'(t', x')$ and $\vec{A}'(t', \vec{x}')$!

4. Derive the fields using Lorentz transformation!

$$\text{In rest frame: } \vec{E}(t, \vec{x}) = \frac{q}{4\pi\epsilon_0}\frac{\vec{x}}{|\vec{x}|^3} \qquad \vec{B}(t, \vec{x}) = 0$$

Transformation to coordinate system moving with u in direction z:

$$t' = \gamma\left(t - \frac{uz}{c^2}\right) \qquad x' = x \qquad y' = y \qquad z' = \gamma(z - ut)$$

$$E'_x = \gamma(E_x - uB_y) \qquad E'_y = \gamma(E_y + uB_x) \qquad E'_z = E_z$$

$$B'_x = \gamma\left(B_x + \frac{uE_y}{c^2}\right) \qquad B'_y = \gamma\left(B_y - \frac{uE_x}{c^2}\right) \qquad B'_z = B_z$$

Hint: we need $\vec{E}'(t', x')$ and $\vec{B}'(t', \vec{x}')$!

Lecture 11: Radiation field of accelerated charge

1. Using

$$\frac{dP}{d\Omega} = \frac{Z_0 q^2}{16\pi^2}\frac{\left|\hat{x} \times \left((\hat{x} - \vec{\beta}) \times \dot{\vec{\beta}}\right)\right|^2}{\left(1 - \vec{\beta} \cdot \hat{x}\right)^5}$$

derive the angular distribution for the radiation of a charge in circular motion:

$$\frac{dP}{d\Omega} = \frac{Z_0 q^2 a^2}{16\pi^2 c^2}\frac{1}{(1 - \beta\cos\theta)^3}\left\{1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1 - \beta\cos\theta)^2}\right\}$$

$$\propto \frac{1}{(1 + \gamma^2\theta^2)^3}\left\{1 - \frac{4\,\gamma^2\theta^2\cos^2\phi}{(1 + \gamma^2\theta^2)^2}\right\} \text{ for ultrarelativistic motion}$$

using the following choice of coordinates

$$\vec{\beta} = \beta\vec{e}_z \qquad \dot{\vec{\beta}} = \frac{a}{c}\vec{e}_x \qquad \hat{x} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\vec{\beta} \cdot \hat{x} = \beta\cos\theta \qquad \dot{\vec{\beta}} \cdot \hat{x} = \frac{a}{c}\sin\theta\cos\phi$$

2. Compute the energy loss for ultra-relativistic particles in a circular accelerator starting from the relativistic generalisation of Larmor formula:

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{Z_0 q^2}{6\pi c^2}\gamma^6\left(\vec{a}^2 - \left(\vec{\beta} \times \vec{a}\right)^2\right)$$

Show that

$$P \approx \frac{Z_0 q^2 c^2}{6\pi r^2}\left(\frac{W}{mc^2}\right)^4 \qquad T \approx \frac{2\pi r}{c}$$

which becomes

$$\delta W = \frac{Z_0 q^2 c}{3r}\left(\frac{W}{mc^2}\right)^4$$

under one revolution.

(a) Estimate the energy loss per particle per revolution in the LEP2 (Large Electron-Positron Collider) of circumference 27 km for electron/positron beams of energy $W = 100$ GeV. What is the power requirement to keep the beam energy constant, if the total beam current is 2x5 mA, and the efficiency of the accelerating radio-frequency cavities is 50%?

(b) Do the same calculations for the LHC (Large Hadron Collider), which accelerates two proton beams in the same tunnel as the LEP2, with beam energies of $W = 7$ TeV, with a total beam current of 2x5 A.

(c) What is the magnetic field required for keeping the beams on the appropriate circular trajectories for the LEP2 machine? How does this compare to the magnetic fields required for the LHC?

3. Compute the energy loss for ultra-relativistic particles in a linear accelerator starting from the relativistic generalisation of Larmor formula:

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{Z_0 q^2}{6\pi c^2}\gamma^6\left(\vec{a}^2 - \left(\vec{\beta} \times \vec{a}\right)^2\right)$$

Show that

$$P = \frac{Z_0 q^2}{6\pi m^2 c^2}\left(\frac{dp}{dt}\right)^2$$

and

$$\frac{P}{\partial_t W} = \frac{Z_0 q^2}{6\pi m^2 c^3}\frac{dW}{dx}$$

Assuming that the accelerated particle is an electron and using

$$\frac{dW}{dx} = qE$$

show that the critical field above which no further acceleration is possible (losses are larger than the energy gain) is

$$E \approx 2.7 \cdot 10^{20}\,\frac{V}{m}$$

Note that this is much larger than the theoretically achievable maximum static field strength in vacuum:

$$E_{max} \approx 10^{18}\,\frac{V}{m}$$

According to quantum electrodynamics, fields larger than this value lead to a run-away pair production of electrons and positrons, known as the Schwinger effect and similar to the breakdown of insulators over a critical field strength.

Lecture 12: Radiation of a moving charge: distribution in frequency and angle

No homework assigned.

Lecture 13: Cherenkov and transition radiation

1. Compute the electric and magnetic fields of a moving charge in variables $\vec{x}, \omega$. Starting from the solution in variables $\vec{k}, \omega$
$$\vec{E}(\omega, \vec{k}) = \frac{2\pi i q}{\epsilon}(\omega\mu\epsilon\vec{v} - \vec{k})\frac{\delta(\omega - \vec{k}\cdot\vec{v})}{k^2 - \mu\epsilon\omega^2} \qquad \vec{B}(\omega, \vec{k}) = \mu\epsilon\vec{v}\times\vec{E}(\omega, \vec{k})$$
use the inverse Fourier transform to derive
$$E_z(\omega, \vec{x}) = -\frac{iq\lambda^2}{4\pi\epsilon\omega v}\int_{-\infty}^{\infty} ds \frac{e^{is\lambda r}}{\sqrt{s^2+1}} \quad E_x(\omega, \vec{x}) == -\frac{iq\lambda}{4\pi\epsilon v}\int_{-\infty}^{\infty} ds \frac{s e^{is\lambda r}}{\sqrt{s^2+1}}$$
with
$$\lambda = \frac{\omega}{v}\sqrt{1-\mu\epsilon v^2} = \frac{\omega}{v}\sqrt{1-\frac{v^2}{c_n^2}}$$
in a coordinate system where $\vec{v} = (0,0,v) \quad \vec{x} = (r,0,0)$. Using the formulae
$$2K_0(z) = \int_{-\infty}^{\infty} ds \frac{e^{isz}}{\sqrt{s^2+1}} \quad \text{and} \quad K_1(z) = -\frac{d}{dz}K_0(z)$$
establish that
$$E_x(\omega, \vec{x}) = \frac{q\lambda}{4\pi\epsilon v}2K_1(\lambda r) \quad E_y(\omega, \vec{x}) = 0 \qquad E_z(\omega, \vec{x}) = -\frac{iq\lambda^2}{4\pi\epsilon\omega}2K_0(\lambda r)$$
$$B_x(\omega, \vec{x}) = 0 \qquad B_y(\omega, \vec{x}) = \mu\epsilon v E_x(\omega, \vec{x}) \quad B_z(\omega, \vec{x}) = 0$$

2. Starting from the expression valid for large r
$$\frac{dW}{dz} = -2r\int_{c_n(\omega)<v} d\omega\left(-\frac{i\lambda\epsilon v^2}{\omega}\right)|E_x(\omega, \vec{x})|^2 \qquad E_x(\omega, \vec{x}) \approx \frac{q}{4\pi\epsilon v}\sqrt{\frac{2\pi\lambda}{r}}e^{-\lambda r}$$
compute in details the Frank-Tamm formula for energy lost per unit distance by Cherenkov radiation:
$$\frac{dW}{dz} = -\frac{q^2}{4\pi}\int_{v>\frac{c}{n(\omega)}} d\omega\, \omega\mu(\omega)\left|1 - \frac{c^2}{v^2 n(\omega)^2}\right| \quad \text{where} \quad n(\omega) = \sqrt{\epsilon_r(\omega)\mu_r(\omega)}$$

3. The depth $d(\omega)$ of the region emitting transition radiation satisfies
$$\frac{\omega}{c}\left(\frac{1}{\beta} - n(\omega)\cos\theta\right)d(\omega) \simeq 1$$
In the ultra-relativistic limit
$$\frac{1}{\beta} \approx 1 + \frac{1}{2\gamma^2}$$
using plasma approximation for the refraction index at high frequencies
$$n(\omega) \approx 1 - \frac{\omega_p^2}{2\omega^2}$$
derive the following result for the depth:
$$d(\omega) = \frac{2\gamma c}{\omega_p}\frac{1}{\nu + \nu^{-1}} \qquad \nu = \frac{\omega}{\gamma\omega_p}$$

Lecture 14: Radiation backreaction

1. Show that

$$\vec{\nabla}(R^{n+1}) = (n+1)R^{n-1}\vec{R}$$

$$\partial_i'\left(R^{n-1}R_j\right) = -R^{n-1}\left(\frac{(n-1)R_i}{R^2}R_j + \delta_{ij}\right)$$

where $\vec{R} = \vec{x} - \vec{x}'$.

2. Verify that indeed

$$\frac{1}{4\pi}\int d\Omega_\mathrm{R}\left(\frac{\vec{v}\cdot\vec{R}}{vR}\right)^2 = \frac{1}{3}$$

Hint: choose z axis in direction of $\vec{v}$ and use $\vec{R} = R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

3. Compute the self-energy of a charged sphere with radius r and charge q assuming that
(a) the charge is uniformly distributed over the surface of the sphere;
(b) the charge is uniformly distributed throughout the volume of the sphere.

4. Consider a radiatively damped oscillator obeying the equation of motion

$$m\frac{d^2x}{dt^2} - m\tau\frac{d^3x}{dt^3} + m\omega_0^2 x = 0$$

(a) Using the ansatz: $x \propto e^{-i\omega t}$ solve for ω assuming that $\omega_0\tau \ll 1$ and show that

$$\omega = \omega_0 - \frac{5}{8}\omega_0^3\tau^2 - \frac{1}{2}i\omega_0^2\tau + O(\tau^3)$$

Hint: derive the equation for ω. Substitute

$$\omega = \omega_0 + A\tau + B\tau^2 + O(\tau^3)$$

and solve for the coefficients of the $O(\tau)$ and $O(\tau^2)$ terms.

(b) Repeat the argument we used for resonant cavities to find the intensity spectrum

$$I(\omega) \propto \frac{1}{(\omega - \omega_0 - \Delta\omega)^2 + \left(\frac{\Gamma}{2}\right)^2}$$

with the frequency shift and half-width given by

$$\Delta\omega = -\frac{5}{8}\omega_0^3\tau^2 \quad \Gamma = \omega_0^2\tau$$

(c) Show that the half-width of the intensity curve in the wavelength λ is

$$\Delta\lambda = 2\pi\frac{c}{\omega_0^2}\Gamma = 2\pi c\tau$$

Hint: neglect $\Delta\omega$ and use $\lambda = 2\pi c/\omega$.