Electrodynamics 2

Lecture notes

Week 1: Potential theory I. Laplace equation in rectangular domains. Spherical coordinates

Potential theory

curl $\vec{E} = 0 \Rightarrow \vec{E} = -\text{grad}\,\Phi$

div $\vec{E} = 0 \Rightarrow \Delta\Phi = -\dfrac{\rho}{\epsilon_0}$ Poisson equation

Boundary conditions:

- Metal: $\Phi = \text{const}$
- Dielectrics: $\vec{E}_{t1} = \vec{E}_{t2}$ $\quad D_{n2} - D_{n1} = \sigma$

$$\Rightarrow \Phi_1 = \Phi_2 \qquad \epsilon_1 \frac{\partial \Phi_1}{\partial n} - \epsilon_2 \frac{\partial \Phi_2}{\partial n} = \sigma$$

Solution in vacuum:

$$\Phi_v(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

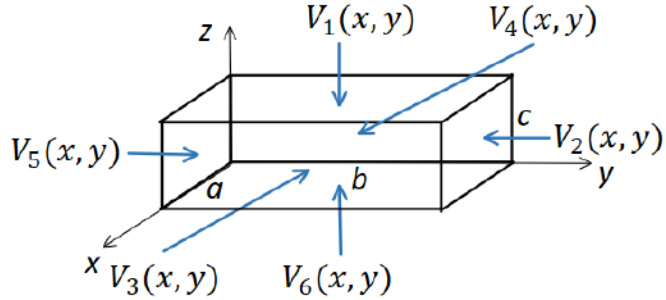
+ boundary condition: $\Phi(\vec{x}) = \Phi_v(\vec{x}) + K(\vec{x})$

$\Delta K = 0$ + appropriate boundary condition for K

Potential theory: the know-how of solving the Laplace equation

Example: rectangular box with metal walls

Divide et impera:



$$\Phi = \sum_{i=1}^{6} \Phi_i \qquad \Phi_i = 0 \text{ on all faces except the } i\text{th one}$$

Solution for top face $z = c$

$$\Phi(x=0,y,z) = \Phi(x=a,y,z) = \Phi(x,y=0,z) = \Phi(x,y=b,z) = \Phi(x,y,z=0)$$
$$\Phi(x,y,z=c) = V_1(x,y)$$

Separation of variables: $\Phi(x,y,z) = X(x)Y(y)Z(z)$

$$\Delta\Phi = X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0$$

$$\Rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = 0$$

$$X'' = -\alpha^2 X \quad Y'' = -\beta^2 Y \quad Z'' = \lambda^2 Z \quad \lambda^2 = \alpha^2 + \beta^2$$
$$X = e^{\pm i\alpha x} \quad Y = e^{\pm i\beta y} \quad Z = e^{\pm \lambda z}$$

Using the boundary conditions

$$\Phi(x=0,y,z) = 0 \Rightarrow X = \sin \alpha x \quad \Phi(x=a,y,z) = 0 \Rightarrow \alpha_n = \frac{n\pi}{a} \quad n = 1,2,\dots$$
$$\Phi(x,y=0,z) = 0 \Rightarrow Y = \sin \beta y \quad \Phi(x,y=b,z) = 0 \Rightarrow \beta_m = \frac{m\pi}{b} \quad m = 1,2,\dots$$
$$\Phi(x,y,z=0) = 0 \Rightarrow Z = \sinh \lambda_{nm} z \quad \lambda_{nm} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

Infinitely many factorized solutions

$$\Phi_{nm}(x,y,z) = \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \sinh \lambda_{nm} z$$

General solution:

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \sinh \lambda_{nm} z$$

Last boundary condition

$$\Phi(x,y,z=c) = \sum_{n,m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \sinh \lambda_{nm} c = V_1(x,y)$$

The functions

$$\sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right)$$

form a complete system in the rectangle $0 \le x \le a$, $0 \le y \le b$ and satisfy the orthogonality relations

$$\int_0^a dx \int_0^b dy \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \sin\left(\frac{n'\pi}{a}x\right) \sin\left(\frac{m'\pi}{b}y\right) = \frac{ab}{4}\delta_{nn'}\delta_{mm'}$$

So, we can write

$$\int_0^a dx \int_0^b dy \sum_{n,m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \sinh(\lambda_{nm} c) \sin\left(\frac{n'\pi}{a}x\right) \sin\left(\frac{m'\pi}{b}y\right)$$

$$= \frac{ab}{4} C_{n'm'} \sinh(\lambda_{n'm'} c) = \int_0^a dx \int_0^b dy\, V(x,y) \sin\left(\frac{n'\pi}{a} x\right) \sin\left(\frac{m'\pi}{b} y\right)$$

which determines

$$C_{nm} = \frac{4}{ab \sinh(\lambda_{nm} c)} \int_0^a dx \int_0^b dy\, V_1(x,y) \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right)$$

and fixes the solution

$$\Phi_1(x,y,z) = \sum_{n,m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right) \sinh \lambda_{nm} z$$

Strategy:

1. Find an adapted coordinate system, i.e. one such that the boundaries correspond to one of coordinates being constant
   -- depends on our abilities

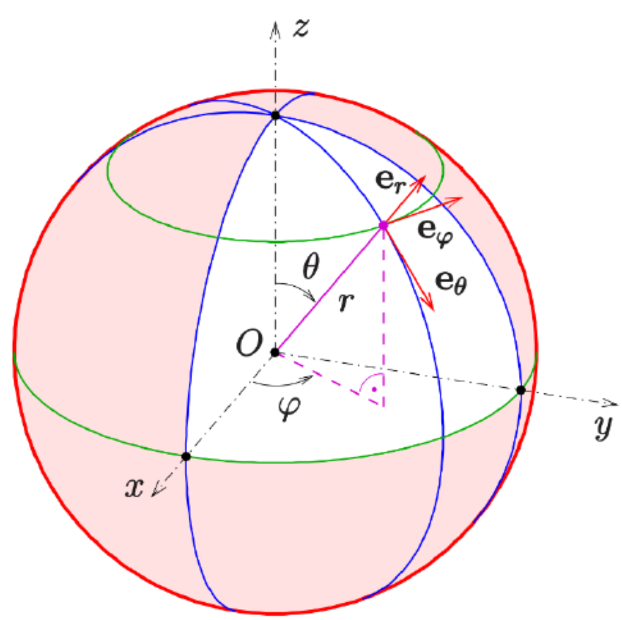
2. Solve the Laplace equation using separation of variables
   -- only works for certain coordinate systems

3. Fix boundary condition by using completeness and orthogonality of the factorized solutions
   -- guaranteed by the fact that the Laplacian with Dirichlet BC is a Hermitian operator and so its eigenfunctions form a complete orthogonal set in the space of all functions

Curvilinear coordinates

$\vec{x}(q^1, q^2, q^3)$

Tangent vectors to coordinate lines:



$$\vec{e}_i = \frac{\partial \vec{x}}{\partial q_i} \qquad h_i = |\vec{e}_i| \qquad \hat{e}_i = \frac{1}{h_i} \vec{e}_i$$

Orthogonal coordinates: $\vec{e}_i \cdot \vec{e}_j = 0 \quad i \neq j$
Jacobian determinant: $J = h_1 h_2 h_3$

Spherical coordinates

$$\vec{x} = \begin{pmatrix} r \sin\vartheta \cos\varphi \\ r \sin\vartheta \sin\varphi \\ r \cos\vartheta \end{pmatrix}$$

$$\vec{e}_r = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} \quad \vec{e}_\vartheta = \begin{pmatrix} r\cos\vartheta\cos\varphi \\ r\cos\vartheta\sin\varphi \\ -r\sin\vartheta \end{pmatrix} \quad \vec{e}_\varphi = \begin{pmatrix} -r\cos\vartheta\sin\varphi \\ r\cos\vartheta\cos\varphi \\ 0 \end{pmatrix}$$

$$h_r = 1 \qquad h_\vartheta = r \qquad h_\varphi = r\cos\vartheta$$

Basis vectors

$$\hat{e}_r = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} \quad \hat{e}_\vartheta = \begin{pmatrix} \cos\vartheta\cos\varphi \\ \cos\vartheta\sin\varphi \\ -\sin\vartheta \end{pmatrix} \qquad \hat{e}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

Cylindrical coordinates

$$\vec{x} = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ z \end{pmatrix} \qquad \vec{e}_\rho = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \quad \vec{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \vec{e}_\varphi = \begin{pmatrix} -\rho\sin\varphi \\ \rho\cos\varphi \\ 0 \end{pmatrix}$$

$$h_\rho = 1 \qquad h_z = 1 \qquad h_\varphi = \rho$$

Basis vectors: $\hat{e}_\rho = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \quad \hat{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \hat{e}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$

Gradient

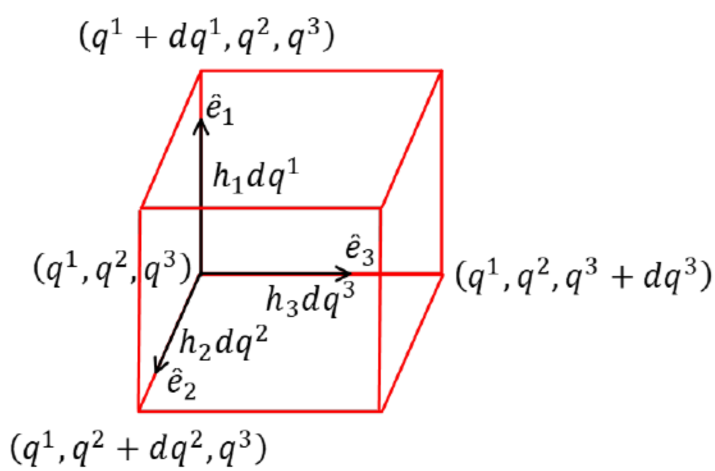
$$\frac{\partial\Phi}{\partial q_i} = \frac{\partial\vec{x}}{\partial q_i}\cdot \operatorname{grad}\Phi = h_i\,\hat{e}_i\cdot\operatorname{grad}\Phi \quad\Rightarrow\quad \operatorname{grad}\Phi = \sum_i \frac{1}{h_i}\frac{\partial\Phi}{\partial q_i}\hat{e}_i$$

Vector field in curvilinear coordinates

$$\vec{u}(\vec{x}) = \sum_i u_i(q^1,q^2,q^3)\hat{e}_i$$

Divergence

$$\iiint_V dV\,\operatorname{div}\vec{u} = \oiint_{\partial V} d\vec{f}\cdot\vec{u}$$

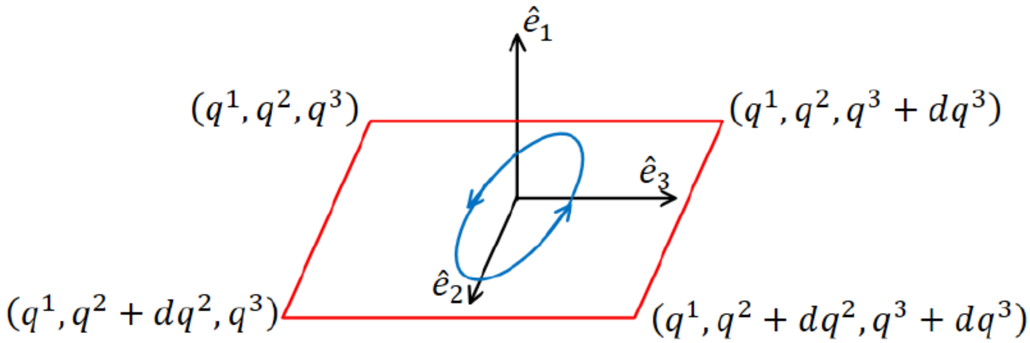


$$
\begin{aligned}
(\operatorname{div}\vec{u})\overbrace{h_1h_2h_3}^{J}\,dq^1dq^2dq^3 \\
= &\ (u_1h_2h_3)(q^1+dq^1,q^2,q^3)dq^2dq^3-(u_1h_2h_3)(q^1,q^2,q^3)dq^2dq^3 \\
+ &\ (u_2h_1h_3)(q^1,q^2+dq^2,q^3)dq^1dq^3-(u_2h_1h_3)(q^1,q^2,q^3)dq^1dq^3 \\
+ &\ (u_3h_1h_2)(q^1,q^2,q^3+dq^3)dq^1dq^2-(u_3h_1h_2)(q^1,q^2,q^3)dq^1dq^2
\end{aligned}
$$

$$
\operatorname{div}\vec{u}=\frac{1}{J}\sum_i\frac{\partial}{\partial q_i}\left(\frac{J}{h_i}u_i\right)
$$

Curl

(German: rot)



$$
\iint_A d\vec{f}\cdot\operatorname{curl}\vec{u}=\oint_{\partial A}d\vec{s}\cdot\vec{u}
$$

$$
\begin{aligned}
(\operatorname{curl}\vec{u})_1h_2h_3dq^2dq^3 &= \big(h_3u_3(q^1,q^2+dq^2,q^3)-h_3u_3(q^1,q^2,q^3)\big)dq^3 \\
&\quad-\big(h_2u_2(q^1,q^2,q^3+dq^3)-h_2u_2(q^1,q^2,q^3)\big)dq^2
\end{aligned}
$$

$$
(\operatorname{curl}\vec{u})_1=\frac{1}{h_2h_3}\left(\frac{\partial}{\partial q_2}(h_3u_3)-\frac{\partial}{\partial q_3}(h_2u_2)\right)
$$

$$
(\operatorname{curl}\vec{u})_2=\frac{1}{h_1h_3}\left(\frac{\partial}{\partial q_3}(h_1u_1)-\frac{\partial}{\partial q_1}(h_3u_3)\right)
$$

$$
(\operatorname{curl}\vec{u})_3=\frac{1}{h_1h_2}\left(\frac{\partial}{\partial q_1}(h_2u_2)-\frac{\partial}{\partial q_2}(h_1u_1)\right)
$$

Laplacian

$$
\Delta\Phi=\operatorname{div}\operatorname{grad}\Phi=\frac{1}{J}\sum_i\frac{\partial}{\partial q_i}\left(\frac{J}{h_i^2}\frac{\partial\Phi}{\partial q_i}\right)
$$

Homework: determine the expression of gradient, divergence, curl and Laplacian in spherical and cylindrical coordinate systems!

Vector Laplacian

$\Delta\vec{u}=\operatorname{grad}\operatorname{div}\vec{u}-\operatorname{curl}\operatorname{curl}\vec{u}$

Homework: prove the above identity using Cartesian coordinates!

For the dedicated: write $\Delta\vec{u}$ in spherical and cylindrical coordinates!

Results for grad/div/curl/Laplacian can be found at:
https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates

Laplace equation in spherical coordinates

$$\Delta\Phi = \frac{1}{r}\partial_r^2(r\Phi) + \frac{1}{r^2}\left[\frac{1}{\sin\vartheta}\partial_\vartheta(\sin\vartheta\,\partial_\vartheta\Phi) + \frac{1}{\sin^2\vartheta}\partial_\varphi^2\Phi\right] = 0$$

1. Separation of variables

$$\Phi = \frac{U(r)}{r}P(\vartheta)Q(\varphi)$$

$$\frac{1}{r}\partial_r^2 U(r)P(\vartheta)Q(\varphi) + \frac{1}{r^3}\left[\frac{1}{\sin\vartheta}\partial_\vartheta\big(\sin\vartheta\,\partial_\vartheta P(\vartheta)\big)U(r)Q(\varphi) + \frac{1}{\sin^2\vartheta}U(r)P(\vartheta)\partial_\varphi^2 Q(\varphi)\right] = 0$$

$$/\cdot\frac{r^3\sin^2\vartheta}{UPQ}$$

$$r^2\sin^2\vartheta\left[\frac{\partial_r^2 U(r)}{U(r)} + \frac{1}{P(\vartheta)r^2\sin\vartheta}\partial_\vartheta(\sin\vartheta\,\partial_\vartheta P(\vartheta))\right] + \frac{1}{Q(\varphi)}\partial_\varphi^2 Q(\varphi) = 0$$

2. Solving the azimuthal part

$$\frac{1}{Q(\varphi)}\partial_\varphi^2 Q(\varphi) = -m^2 \Rightarrow Q(\varphi) = e^{im\varphi} \qquad m \in \mathbf{Z}$$

3. Separating and solving the radial part

$$r^2\sin^2\vartheta\left[\frac{\partial_r^2 U(r)}{U(r)} + \frac{1}{P(\vartheta)r^2\sin\vartheta}\partial_\vartheta(\sin\vartheta\,\partial_\vartheta P(\vartheta))\right] - m^2 = 0 \qquad /\cdot\frac{1}{\sin^2\vartheta}$$

$$r^2\frac{\partial_r^2 U(r)}{U(r)} + \frac{1}{P(\vartheta)\sin\vartheta}\partial_\vartheta\big(\sin\vartheta\,\partial_\vartheta P(\vartheta)\big) - \frac{m^2}{\sin^2\vartheta} = 0$$

$$r^2\frac{\partial_r^2 U(r)}{U(r)} = l(l+1) \qquad l(l+1) \in \mathbf{R}$$

$$\Rightarrow U(r) = Ar^{l+1} + Br^{-l}$$

4. The polar part

$$\frac{1}{P(\vartheta)\sin\vartheta}\partial_\vartheta\big(\sin\vartheta\,\partial_\vartheta P(\vartheta)\big) - \frac{m^2}{\sin^2\vartheta} = -l(l+1)$$

$$\frac{1}{\sin\vartheta}\partial_\vartheta\big(\sin\vartheta\,\partial_\vartheta P(\vartheta)\big) + \left[l(l+1) - \frac{m^2}{\sin^2\vartheta}\right]P(\vartheta) = 0$$

Azimuthal symmetry and Legendre polynomials

Axially symmetric case: potential independent of φ

$$\Phi = \Phi(r,\vartheta) = \frac{U(r)}{r}P(\vartheta) \qquad U(r) = Ar^{l+1} + Br^{-l}$$

$$\frac{1}{\sin\vartheta}\partial_\vartheta\big(\sin\vartheta\,\partial_\vartheta P(\vartheta)\big) + l(l+1)P(\vartheta) = 0$$

Change variables to $x = \cos\vartheta$:

$$\frac{d}{dx} = -\frac{1}{\sin\vartheta}\partial_\vartheta \qquad \sin\vartheta = \sqrt{1-x^2}$$

$$\frac{d}{dx}\left((1-x^2)\frac{dP(x)}{dx}\right) + l(l+1)P(x) = 0 \qquad \text{Legendre equation}$$

Frobenius method

Seek solution in terms of a power series Ansatz

$$P(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j$$

$$\sum_{j=0}^{\infty}\left\{(j+\alpha)(j+\alpha-1)a_j x^{j+\alpha-2} - [(j+\alpha)(j+\alpha+1) - l(l+1)]a_j x^{j+\alpha}\right\} = 0$$

$x^{\alpha-2}:\ \alpha(\alpha-1)a_0 = 0$
$x^{\alpha-1}:\ (\alpha+1)\alpha a_1 = 0$
$x^{j+\alpha}:\ (j+\alpha+2)(j+\alpha+1)a_{j+2} - [(j+\alpha)(j+\alpha+1) - l(l+1)]a_j = 0 \qquad j = 0,1,2,\ldots$

$a_0 \neq 0 \Rightarrow \alpha = 0,1 \quad a_1 \neq 0 \Rightarrow \alpha = -1,0$: same set of solutions, we can choose $\alpha = 0$

$$a_{j+2} = \frac{j(j+1) - l(l+1)}{(j+2)(j+1)}a_j$$

j runs either over even or odd integers

Homework: verify the above computations!

For large j: $a_j \to \text{const} \Rightarrow$ series diverges at $x = \pm 1$ unless it terminates. Regularity of solution in the full interval $\vartheta \in [-\pi, \pi]$ requires $l \in \mathbf{N}$

Result: **Legendre polynomials** $P_l(x)$ Normalisation $P_l(1) = 1$

$$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \quad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$

Some properties of the Legendre polynomials

$$P_l(-x) = (-1)^l P_l(x)$$

Rodrigues formula:

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2 - 1)^l$$

Recursion formulas:

$$\frac{dP_{l+1}(x)}{dx} - \frac{dP_{l-1}(x)}{dx} - (2l+1)P_l(x) = 0$$

$$(l+2)P_{l+1}(x) - (2l+1)xP_l(x) + lP_{l-1}(x) = 0$$

Orthogonality:

$$\int_{-1}^{1} dx\, P_l(x)P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'}$$

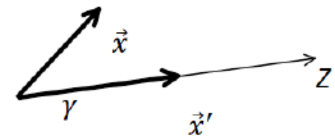
Completeness: for any continuous function $f: [-1,1] \to \mathbf{R}$

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x) \qquad A_l = \frac{2l+1}{2}\int_{-1}^{1} dx\, f(x)P_l(x)$$

Week 2: Potential theory II. Azimuthal symmetry. Spherical harmonics

Expansion of the Green's function with Legendre polynomials

Theorem



$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

$$r_< = \min(|\vec{x}|, |\vec{x}'|) \quad r_> = \max(|\vec{x}|, |\vec{x}'|) \quad \gamma = \angle(\vec{x}, \vec{x}')$$

Proof: choose the z axis in the direction of $\vec{x}'$ and denote $r = |\vec{x}| \quad r' = |\vec{x}'|$

$$\Delta_x \frac{1}{|\vec{x}-\vec{x}'|} = 0 \quad (\vec{x} \neq \vec{x}') \quad \Rightarrow \quad \frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-l-1})\, P_l(\cos\gamma)$$

(i) $r > r'$ and $\gamma = 0$: $P_l(\cos\gamma) = P_l(1) = 1$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r}\frac{1}{1-r'/r} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-l-1}) \Rightarrow A_l = 0,\ B_l = r'^l$$

(ii) $r < r'$ and $\gamma = 0$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r'}\frac{1}{1-r/r'} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r}{r'}\right)^l = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-l-1}) \Rightarrow A_l = r'^{-l-1},\ B_l = 0$$

Corollary: generating function of the Legendre polynomials

Choose $\vec{x}' = \vec{e}_z$ and $|\vec{x}| = t < 1$, denote $z = \cos\gamma$

$$|\vec{x}-\vec{x}'| = \sqrt{(\vec{x}-\vec{x}')^2} = \sqrt{1+t^2-2tz}$$

$$\frac{1}{\sqrt{1+t^2-2tz}} = \sum_{l=0}^{\infty} t^l P_l(z)$$

Generating function of the Legendre polynomials

Orthogonality and normalisation

(i) Orthogonality follows from the differential equation

$$0=\int_{-1}^{1}dx\left(P_{l'}(x)\left[\frac{d}{dx}\left((1-x^2)\frac{dP_l(x)}{dx}\right)+l(l+1)P_l(x)\right]\right.$$
$$\left.-P_l(x)\left[\frac{d}{dx}\left((1-x^2)\frac{dP_{l'}(x)}{dx}\right)+l'(l'+1)P_{l'}(x)\right]\right)$$

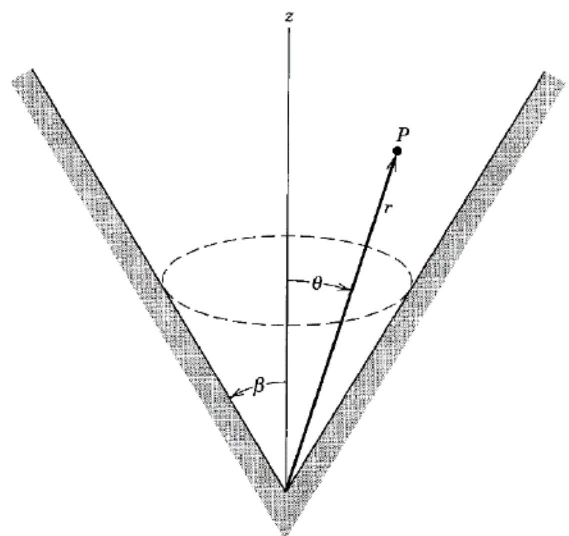
By a double partial integration, the differential terms cancel each other, and we are left with

$$0=\big(l(l+1)-l'(l'+1)\big)\int_{-1}^{1}dx\,P_l(x)P_{l'}(x)\ \Rightarrow\ \int_{-1}^{1}dx\,P_l(x)P_{l'}(x)=0\ \ l\neq l'$$

(ii) Normalisation: integrate the square of the generating function

$$\int_{-1}^{1}dz\left(\frac{1}{\sqrt{1+t^2-2tz}}\right)^2=\int_{-1}^{1}dz\sum_{l=0}^{\infty}t^lP_l(z)\sum_{l=0}^{\infty}t^{l'}P_{l'}(z)=\sum_{l=0}^{\infty}t^{2l}\int_{-1}^{1}dz\,\big(P_l(z)\big)^2$$
$$\int_{-1}^{1}dz\frac{1}{1+t^2-2tz}=\frac{1}{t}(\log(1+t)-\log(1-t))=\sum_{l=0}^{\infty}\frac{2}{2l+1}t^{2l}\quad\Rightarrow\quad\int_{-1}^{1}dz\,\big(P_l(z)\big)^2$$
$$=\frac{2}{2l+1}$$

Electrostatic field at sharp edges



Differences from what we had so far:
(i) Regularity at $\vartheta=\pi$ is not needed
(ii) Potential must vanish at $\vartheta=\beta$

We still have the polar equation
$$\frac{d}{dx}\left((1-x^2)\frac{dP(x)}{dx}\right)+\nu(\nu+1)P(x)=0$$
with the corresponding radial solution $A_\nu r^\nu+B_\nu r^{-\nu-1}$.

However, it is convenient to change variables to
$$y=\frac{1}{2}(1-x)=\sin^2\frac{\vartheta}{2}\qquad 1-x^2=4y(1-y)\qquad\frac{d}{dx}=-2\frac{d}{dy}$$
to get
$$\frac{d}{dy}\left(y(1-y)\frac{dP(y)}{dy}\right)+\nu(\nu+1)P(y)=0$$

Using Frobenius method (and also regularity at $y=0$ i. e. $\vartheta=0$):
$$P(y)=y^\alpha\sum_{n=0}^{\infty}a_ny^n\ \Rightarrow\ \alpha=0\quad a_{n+1}=\frac{(n-\nu)(n+\nu+1)}{(n+1)^2}a_n$$

$$a_0 = 1:\ \ P(y) = 1 + \frac{(-\nu)(\nu+1)}{(1)^2} y + \frac{(-\nu)(-\nu+1)(\nu+1)(\nu+2)}{(1 \cdot 2)^2} y^2 + \cdots$$

Gauss hypergeometric function

$$F(a,b;c|z) = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)} \frac{z^3}{3!} + \cdots$$

Hypergeometric equation:

$$z(1-z)\frac{d^2F}{dz^2} + [c - (a+b+1)z]\frac{dF}{dz} - abF = 0$$

Independent solutions: $F(a,b;c|z)$ and $z^{1-c}F(1+a-c, 1+b-c; 2-c|z)$, provided $c \notin \mathbf{Z}$

Many important differential equations can be mapped to the hypergeometric equation!

$$\tilde{P}_\nu(y) = F(-\nu, \nu+1, 1|y) \qquad P_l(x) = F\left(-l, l+1, 1\middle| \frac{1-x}{2}\right)$$

$\tilde{P}_\nu(y)$ is generally singular in $y = 1$ ($\vartheta = \pi$): it is only regular when $\nu = l \in \mathbf{N}$

Condition for ν:

$$\tilde{P}_\nu\left(\frac{1-\cos\ \beta}{2}\right) = 0 \ \Rightarrow\ 0 < \nu_1 < \nu_2 < \cdots$$

General solution for the potential regular at $r = 0$:

$$\Phi(r,\vartheta) = \sum_{i=0}^{\infty} A_i r^{\nu_i} \tilde{P}_{\nu_i}\left(\frac{1-\cos\vartheta}{2}\right)$$

Example: $\beta = \pi/2$

$$\tilde{P}_\nu(1/2) = 0 \ \Rightarrow\ \ \nu_1 = 1 \ \ \tilde{P}_1(y) = P_1(1-2y) = 1 - 2y = \cos\vartheta$$

$$\Phi(r,\vartheta) = A_1 r \cos\vartheta + \cdots = A_1 z + \cdots$$

First term gives homogeneous electric field!

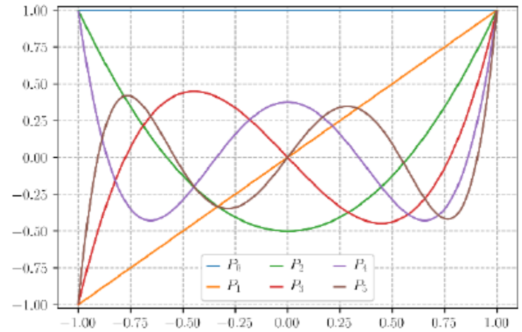
Case 1: $\beta \leq \pi/2$ (conical depression)

$\beta \leq \pi/2$ corresponds to $x = \cos\beta > 0$

$P_l(x)$ has largest zero at a position $x_1 \geq 0$ for $l = 1,2,\dots$

$x_1 = 0$ for $l = 1$ and it increases monotonically with l.

We conclude that



$$\beta \leq \frac{\pi}{2} \;\Rightarrow\; \nu_1 \geq 1$$

Electric field strength at the tip $r = 0$:

$$E_r = -\frac{\partial\Phi}{\partial r} = -A_1\nu_1 r^{\nu_1-1}\tilde{P}_{\nu_1}\left(\frac{1-\cos\vartheta}{2}\right) + \cdots \qquad E_\vartheta = -\frac{1}{r}\frac{\partial\Phi}{\partial\vartheta}$$
$$= -A_1 r^{\nu_1-1}\frac{\partial}{\partial\vartheta}\tilde{P}_{\nu_1}\left(\frac{1-\cos\vartheta}{2}\right) + \cdots$$

which eventually goes to zero as $r \to 0$!

For large ν:

$$\tilde{P}_\nu(y) = F(-\nu, \nu+1, 1|y) \approx J_0\left((2\nu+1)\sqrt{y}\right) = J_0\left((2\nu+1)\sin\frac{\vartheta}{2}\right)$$

First zero of Bessel function:

$$(2\nu_1+1)\sin\frac{\beta}{2} \approx 2.405 \Rightarrow \nu_1 \approx \frac{1}{2}\left(\frac{2.405}{\sin\frac{\beta}{2}} - 1\right)$$

For a needle-like depression (small β):

$$\nu_1 \approx \frac{2.405}{\beta} - \frac{1}{2} \gg 1$$

i.e., the field is extremely small near the tip!

Case 2: $\beta > \pi/2$ (sharp tip)

$\nu_1 < 1$: field blows up a tip!

For a very sharp needle, $\pi - \beta$ small:

$$\nu_1 \approx \left[2\log\left(\frac{2}{\pi-\beta}\right)\right]^{-1} \to 0 \quad \text{for} \quad \beta \to \pi$$

Electric field strength becomes singular near sharp edges (lightning rod!).

Spherical harmonics

Laplace equation in spherical coordinates

$$\Delta\Phi = \frac{1}{r}\partial_r^2(r\Phi) + \frac{1}{r^2}\left[\frac{1}{\sin\vartheta}\partial_\vartheta(\sin\vartheta\,\partial_\vartheta\Phi) + \frac{1}{\sin^2\vartheta}\partial_\varphi^2\Phi\right] = 0$$

Reminder: factorized Ansatz (separation of variables)

$$\Phi = \frac{U(r)}{r}P(\vartheta)Q(\varphi)$$

- Radial part: $U(r) = Ar^{l+1} + Br^{-l} \quad l(l+1) \in \mathbf{R}$
- Azimuthal part: $Q(\varphi) = e^{im\varphi} \quad m \in \mathbf{Z}$

Polar part satisfies

$$\frac{1}{\sin\vartheta}\partial_\vartheta\big(\sin\vartheta\,\partial_\vartheta P(\vartheta)\big) + \left[l(l+1) - \frac{m^2}{\sin^2\vartheta}\right]P(\vartheta) = 0$$

Change variables to $x = \cos\vartheta$:

$$\frac{d}{dx} = -\frac{1}{\sin\vartheta}\partial_\vartheta \qquad \sin\vartheta = \sqrt{1-x^2}$$

$$\frac{d}{dx}\left((1-x^2)\frac{dP(x)}{dx}\right) + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P(x) = 0 \qquad \text{associated Legendre equation}$$

Generally: two independent solutions

- $P_l^m(x)$: diverges at $x = 1$
- $Q_l^m(x)$: diverges at $x = -1$

For $l = 0,1,2,\ldots$ and $m = -l, -l+1, \ldots, l$ there is a solution $P_l^m(x)$ (unique up to normalisation) which is regular everywhere for $-1 \le x \le 1$.

Assume $m \ge 0$. Behaviour at the endpoints:

- $x \sim 1$: $1 - x^2 \to 2(1-x)$

$$2\frac{d}{dx}\left((1-x)\frac{dP(x)}{dx}\right) - \frac{m^2}{2(1-x)}P(x) = 0$$

$$P(x) \to (1-x)^{\alpha} \to \ \alpha = \pm\frac{m}{2}$$

Only + sign is acceptable

- $x \sim -1$: similar.

Regular solution: $P_l^m(x) = (1-x^2)^{m/2} \cdot$ polynomial

Spherical harmonics

$$Y_{l,m}(\vartheta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l+m)!}{(l-m)!}}P_l^m(\cos\vartheta)e^{im\varphi} \qquad Y_{l,-m}(\vartheta,\varphi) = (-1)^m Y_{l,m}^*(\vartheta,\varphi)$$

Associated Legendre function

$$P_l^m(x) = (-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_l(x) \quad m \geq 0$$

$$P_l^{-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x) \qquad \text{convenient extension to negative } m$$

Homework for the dedicated: verify that $P_l^m(x)$ is indeed a solution!
Hint:

- Substitute first $P(x) = (1-x^2)^{m/2}p(x)$ and derive the equation for $p(x)$.
- Using the Frobenius method show that there is only one regular solution for $p(x)$ and it exists only for $l = 0,1,2,\dots$ and $m = -l,-l+1,\dots,l$ when the power series terminates.
- Taking m derivatives of the Legendre equation show that
$$\frac{d^m}{dx^m}P_l(x)$$
solves the same equation as $p(x)$.

Orthogonality relation

$$\int_{-1}^{1}dx\, P_l^m(x)P_{l'}^m(x) = \frac{2(l+m)!}{(2l+1)(l-m)!}\delta_{ll'}$$

Note: no simple relation between solutions corresponding to different m!

Orthogonality

$$\int d\Omega\, Y_{l,m}(\vartheta,\varphi)Y_{l',m'}^*(\vartheta,\varphi) = \delta_{ll'}\delta_{mm'}$$

Homework: prove orthogonality of the spherical harmonics!
Note the following: $d\Omega = \sin\vartheta\, d\vartheta d\varphi = d(\cos\vartheta)d\varphi = dxd\varphi$

Completeness of spherical harmonics

Let $f: S^2 \to \mathbf{C}$ be a square integrable function on the sphere, i.e.
$\int d\Omega\, |f(\vartheta,\varphi)|^2 < \infty$

Then the following expansion converges to f almost everywhere

$$f(\vartheta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{m} f_{lm} Y_{l,m}(\vartheta,\varphi) \qquad f_{lm} = \int d\Omega\, f(\vartheta,\varphi) Y^*_{l,m}(\vartheta,\varphi)$$

General solution of Laplace equation in spherical coordinates

$$\Phi(r,\vartheta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{m} (A_{lm} r^l + B_{lm} r^{-l-1}) Y_{l,m}(\vartheta,\varphi)$$

Homework:

1. Given the potential on a sphere of radius R: $\Phi(R,\vartheta,\varphi) = V(\vartheta,\varphi)$ expand the solution in spherical harmonics
    a. inside i.e., for $r < R$
    b. outside i.e., for $r > R$

2. Given the potential on two concentric spheres of radiuses R_1 and R_2:

    $\Phi(R_1,\vartheta,\varphi) = V_1(\vartheta,\varphi)$ and $\Phi(R_2,\vartheta,\varphi) = V_2(\vartheta,\varphi)$

    expand the solution between the spheres in spherical harmonics!

Group of rotations and spherical harmonics

Orthogonal transformations: $\vec{x}' = O\vec{x} \quad \vec{x}' \cdot \vec{y}' = \vec{x} \cdot \vec{y} \Rightarrow O^{-1} = O^T$

They form a group:
$O(3)$: orthogonal matrices
$SO(3)$: rotations (orthogonal matrices with $\det O = +1$

$\Delta = \vec{\nabla} \cdot \vec{\nabla}$ rotationally invariant

$$\Delta f = \frac{1}{r}\partial_r^2(rf) + \frac{1}{r^2}\Delta_\Omega f \qquad \Delta_\Omega f = \frac{1}{\sin\vartheta}\partial_\vartheta(\sin\vartheta\, \partial_\vartheta f) + \frac{1}{\sin^2\vartheta}\partial_\varphi^2 f$$

$\Delta_\Omega f$: Laplacian on the sphere - it is rotationally invariant as well!

Notation: $Y_{l,m}(\vartheta,\varphi) =: Y_{l,m}(\vec{n}) \quad \vec{n} = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}$

We then have

$$\Delta_\Omega Y_{l,m}(\vec{n}) = l(l+1)Y_{l,m}(\vec{n})$$

$$\Delta_\Omega Y_{l,m}(O\vec{n}) = l(l+1)Y_{l,m}(O\vec{n})$$

Therefore, there exists a matrix $D^{(l)}(O)$ dependent on O such that

$$Y_{l,m}(O\vec{n}) = \sum_{m'=-l}^{l} D^{(l)}_{mm'}(O)Y_{l,m'}(\vec{n})$$

Properties of $D^{(l)}(O)$:

(1) $D^{(l)}(O_1)D^{(l)}(O_2) = D^{(l)}(O_1O_2)$ - true due to completeness of the $Y_{l,m}(\vec{n})$.

This means that the matrices $D^{(l)}(O)$ form a **representation** of the group $SO(3)$.

(2) $D^{(l)}(O)D^{(l)}(O)^\dagger = \mathbf{1}$

i.e., the representation $D^{(l)}(O)$ is **unitary**. The proof uses the rotation invariance of $d\Omega$ and the orthogonality of the spherical harmonics:

$$\begin{aligned}
\delta_{mm'} &= \int d\Omega\, Y_{l,m}(O\vec{n})Y^*_{l,m'}(O\vec{n}) = \int d\Omega \sum_{m_1,m_2} D^{(l)}_{mm_1}(O)D^{(l)*}_{m'm_2}(O)Y_{l,m_1}(\vec{n})Y^*_{l,m_2}(\vec{n}) \\
&= \sum_{m_1,m_2} D^{(l)}_{mm_1}(O)D^{(l)*}_{m'm_2}(O)\delta_{m_1m_2} = \sum_{m_1} D^{(l)}_{mm_1}(O)D^{(l)*}_{m'm_1}(O) \\
&= \sum_{m_1} D^{(l)}_{mm_1}(O)D^{(l)}_{m_1m'}(O)^\dagger \\
&= \left(D^{(l)}(O)D^{(l)}(O)^\dagger\right)_{mm'}
\end{aligned}$$

Summary: the lth spherical harmonics transform as a $2l+1$-dimensional unitary representation under rotations. It can also be shown that this representation is irreducible.

Week 3: Potential theory III. Spherical multipole expansion. Surface effects in metals

Addition theorem for spherical harmonics

Theorem

$$P_l(\vec{n}' \cdot \vec{n}) = \frac{4\pi}{2l+1} \sum_{l=-m}^{m} Y_{l,m}^*(\vartheta', \varphi') Y_{l,m}(\vartheta, \varphi)$$

$$\vec{n}' \cdot \vec{n} = \cos\gamma = \cos\vartheta \cos\vartheta' + \sin\vartheta \sin\vartheta' \cos(\varphi - \varphi')$$

Proof: recall that

$$P_l^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_l(x) \quad P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) \qquad m \geq 0$$

$$Y_{l,m}(\vartheta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l+m)!}{(l-m)!}} P_l^m(\cos\vartheta) e^{im\varphi}$$

and also $P_l(1) = 1$. Therefore

$$Y_{l,m}(\vec{e}_z) = \sqrt{\frac{2l+1}{4\pi}} \delta_{m0} \quad \vec{e}_z\text{: } \vartheta = 0$$

which leads to

$$Y_{l,m}(O^{-1}\vec{e}_z) = \sum_{m'=-l}^{l} D_{mm'}^{(l)}(O^{-1}) Y_{l,m'}(\vec{e}_z) = \sum_{m'=-l}^{l} D_{mm'}^{(l)}(O^{-1}) \sqrt{\frac{2l+1}{4\pi}} \delta_{m'0}$$

$$= \sqrt{\frac{2l+1}{4\pi}} D_{m0}^{(l)}(O^{-1})$$

i.e.

$$(1)\ Y_{l,m}(O^{-1}\vec{e}_z) = \sqrt{\frac{2l+1}{4\pi}} D_{0m}^{(l)}(O)^*$$

We also have

$$(2)\ Y_{l,0}(\vec{n}) = \sqrt{\frac{2l+1}{4\pi}} P_l^0(\cos\vartheta) = \sqrt{\frac{2l+1}{4\pi}} P_l(\vec{e}_z \cdot \vec{n})$$

Now we can reason as follows: there is always a rotation O such that $\vec{e}_z = O\vec{n}'$ i.e. $\vec{n}' = O^{-1}\vec{e}_z$

$$P_l(\vec{n}' \cdot \vec{n}) = P_l(O^{-1}\vec{e}_z \cdot \vec{n}) = P_l(O^T\vec{e}_z \cdot \vec{n}) = P_l(\vec{e}_z \cdot O\vec{n}) = \sqrt{\frac{4\pi}{2l+1}} Y_{l,0}(O\vec{n})$$

$$= \sqrt{\frac{4\pi}{2l+1}} \sum_{m=-l}^{l} D_{0m}^{(l)}(O)\, Y_{l,m}(\vec{n}) = \sqrt{\frac{4\pi}{2l+1}} \sum_{m=-l}^{l} D_{m0}^{(l)}(O^{-1})^*\, Y_{l,m}(\vec{n})$$

$$= \sqrt{\frac{4\pi}{2l+1}} \sum_{m=-l}^{l} \sqrt{\frac{4\pi}{2l+1}} Y_{l,m}(O^{-1}\vec{e}_z)^* Y_{l,m}(\vec{n})$$

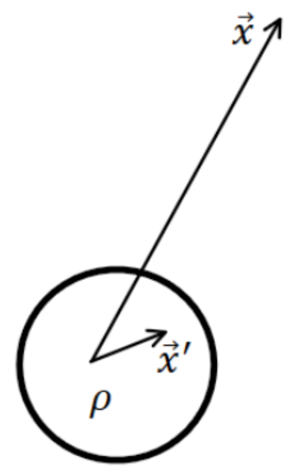
$$= \frac{4\pi}{2l+1} \sum_{l=-m}^{m} Y_{l,m}^*(\vec{n}')Y_{l,m}(\vec{n})$$

Q.e.d.

Spherical multipole expansion

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\vec{n}' \cdot \vec{n}) = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} \sum_{l=-m}^{m} Y_{l,m}^*(\vec{n}')Y_{l,m}(\vec{n}) \qquad \vec{x} = r\vec{n} \quad \vec{x}' = r'\vec{n}'$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$$



Assuming $r \gg r'$ (field far away from localised charge distribution)

$$\Phi(\vec{x}) = \frac{1}{\epsilon_0} \sum_{l=0}^{\infty} \sum_{l=-m}^{m} \frac{1}{2l+1} \int d^3x'\, \rho(\vec{x}')\, \frac{r'^l}{r^{l+1}} Y_{l,m}^*(\vartheta', \varphi')Y_{l,m}(\vartheta, \varphi)$$

$$= \frac{1}{\epsilon_0} \sum_{l=0}^{\infty} \sum_{l=-m}^{m} \frac{q_{lm}}{2l+1} \frac{1}{r^{l+1}} Y_{l,m}(\vartheta, \varphi)$$

Spherical multipole moments

$$q_{l,m} = \int d^3x'\, \rho(\vec{x}') r'^l Y_{l,m}^*(\vartheta', \varphi')$$

Examples:

i. Monopole
$$q_{0,0} = \int d^3x'\, \rho(\vec{x}') r'^0 Y_{0,0}^*(\vartheta', \varphi') = \frac{1}{\sqrt{4\pi}} \int d^3x'\, \rho(\vec{x}')$$

ii. Dipole
$$\vec{p} = \int d^3x'\, \rho(\vec{x}')\vec{x}'$$

$$q_{1,+1} = \int d^3x'\, \rho(\vec{x}') r' Y_{1,1}^*(\vartheta', \varphi') = \int d^3x'\, \rho(\vec{x}') r' \sqrt{\frac{3}{8\pi}} \sin\vartheta'\, e^{-i\varphi'}$$

$$= \int d^3x' \, \rho(\vec{x}') r' \sqrt{\frac{3}{8\pi}} \sin\vartheta' \, (\cos\varphi' - i \sin\varphi') = \sqrt{\frac{3}{8\pi}} \int d^3x' \, \rho(\vec{x}')(x' - iy')$$

$$= \sqrt{\frac{3}{8\pi}} (p_x - ip_y)$$

$$q_{1,0} = \int d^3x' \, \rho(\vec{x}') r' Y_{1,0}^*(\vartheta', \varphi') = \int d^3x' \, \rho(\vec{x}') r' \sqrt{\frac{3}{4\pi}} \cos\vartheta' = \sqrt{\frac{3}{4\pi}} \int d^3x' \, \rho(\vec{x}') z'$$

$$= \sqrt{\frac{3}{4\pi}} p_z$$

$$q_{1,-1} = \int d^3x' \, \rho(\vec{x}') r' Y_{1,-1}^*(\vartheta', \varphi') = \int d^3x' \, \rho(\vec{x}') r' \left(-\sqrt{\frac{3}{8\pi}} \sin\vartheta' \, e^{i\varphi'} \right)$$

$$= \int d^3x' \, \rho(\vec{x}') r' \sqrt{\frac{3}{8\pi}} \sin\vartheta' \, (-\cos\varphi' - i \sin\varphi') = \sqrt{\frac{3}{8\pi}} \int d^3x' \, \rho(\vec{x}')(-x' - iy')$$

$$= \sqrt{\frac{3}{8\pi}} (-p_x - ip_y)$$

Inverse relation

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \sqrt{\frac{2\pi}{3}} \begin{pmatrix} q_{1,1} - q_{1,-1} \\ i(q_{1,1} + q_{1,-1}) \\ \sqrt{2} q_{1,0} \end{pmatrix}$$

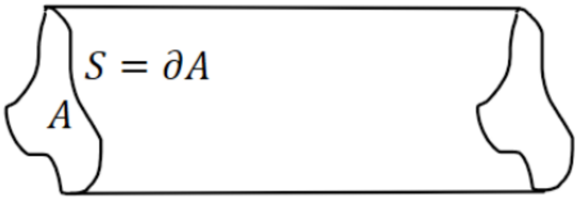
Homework: find the relation between the quadrupole moment

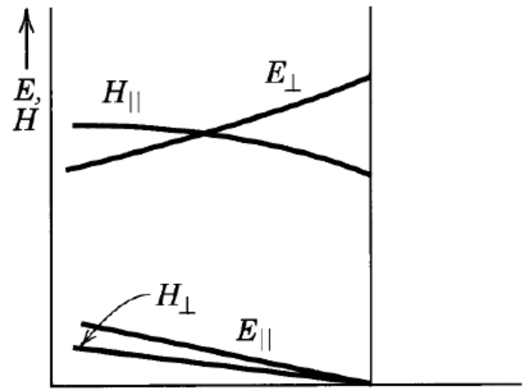
$$Q_{rs} = \int d^3x' \, \rho(\vec{x}')(3x'_r x'_s - r'^2 \delta_{rs})$$

and the spherical moments q_{2m}! Are the five independent spherical moments sufficient to describe the quadrupole? Why? (Hint: examine the Q_{rs} matrix!)

Note that by deriving the spherical form of the multipole expansion we have just shown that in general, the lth multipole (2^l-pole) has exactly $2l + 1$ independent components!

Wave guide: a long tube with conducting walls





Wall: good, but not ideal conductor ⇒ surface effects

Surface effects

Boundary conditions for ideal conductor: fields vanish in conductor.
From the inside of the tube:

$$\vec{n}\cdot\vec{D} = \Sigma \text{ (surface charge density)} \qquad \vec{n}\times\vec{E} = 0$$
$$\vec{n}\times\vec{H} = \vec{K} \text{ (surface current density)} \qquad \vec{n}\cdot\vec{B} = 0$$

Real conductor: there is a transient layer - skin effect with some thickness δ.
There are fields $\vec{E}_c$ and $\vec{H}_c$ inside the conductor. We assume that the conductor is a material with linear response to fields: $\vec{B}_c = \mu_c\vec{H}_c$

Inside conductor: $\vec{J} = \sigma\vec{E}_c$ On the wall: $\vec{n}\times\vec{H} = \vec{n}\times\vec{H}_c$

We also assume that frequency of field is low enough to apply quasi-stationary approximation inside the conducting wall:

$$|\vec{J}| \gg \left|\frac{\partial\vec{D}_c}{\partial t}\right| \quad \text{i.e.} \quad \sigma \gg \epsilon\omega$$
$$\vec{\nabla}\times\vec{H}_c = \sigma\vec{E}_c \quad \Rightarrow \quad \vec{E}_c = \frac{1}{\sigma}\vec{\nabla}\times\vec{H}_c$$
$$\vec{\nabla}\times\vec{E}_c = i\omega\mu_c\vec{H}_c \;\Rightarrow\; \vec{H}_c = -\frac{i}{\omega\mu_c}\vec{\nabla}\times\vec{E}_c$$

We know that these equations lead to a skin depth

$$\delta = \sqrt{\frac{2}{\omega\mu_c\sigma}}$$

Rewriting

$$\sigma \gg \epsilon\omega = \omega\frac{1}{c_n^2\mu_c} \Rightarrow \frac{1}{\omega\mu_c\sigma} \ll \frac{c_n^2}{\omega^2} = \frac{\lambda^2}{4\pi^2 n^2} \quad \text{where} \quad c_n = \frac{c}{n} \text{ is speed of light in wall}$$

Typically, $n \sim O(1)$ so quasi-stationary approximation means that the skin depth is much smaller than the electromagnetic wavelength:

$$\delta \ll \lambda$$

We also assume that the curvature radius of the wall satisfies $R \gg \delta, \lambda \Rightarrow$ the wall can be assumed to be flat on the scales where the surface effect happens. These are typically satisfied for the wave guides in use (frequencies up to several 100 MHz and metal walls).

Choice of coordinates: *x, y* along wall surface, *z* perpendicular to wall, *z*>0: conductor. Then

$$\frac{\partial}{\partial z} \sim \delta^{-1} \gg \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \sim R^{-1}, \lambda^{-1} \;\Rightarrow\; \vec{\nabla} = \vec{e}_z \frac{\partial}{\partial z} = -\vec{n}\,\frac{\partial}{\partial z}$$

$$\vec{E}_c = \frac{1}{\sigma}\vec{\nabla} \times \vec{H}_c \approx -\frac{1}{\sigma}\vec{n} \times \frac{\partial \vec{H}_c}{\partial z} \qquad \vec{H}_c = -\frac{i}{\omega\mu_c}\vec{\nabla} \times \vec{E}_c = \frac{i}{\omega\mu_c}\vec{n} \times \frac{\partial \vec{E}_c}{\partial z}$$

From the first equation

$$\frac{\partial^2(\vec{n} \times \vec{H}_c)}{\partial z^2} = -\sigma \frac{\partial \vec{E}_c}{\partial z}$$

Note that $\vec{n} \cdot \vec{E}_c = 0 = \vec{n} \cdot \vec{H}_c$ so $\vec{n} \times (\vec{n} \times \vec{E}_c) = -\vec{E}_c$ and $\vec{n} \times (\vec{n} \times \vec{H}_c) = -\vec{H}_c$ so

$$\frac{\partial \vec{E}_c}{\partial z} = -i\omega\mu_c \vec{n} \times \vec{H}_c$$

from which we can derive

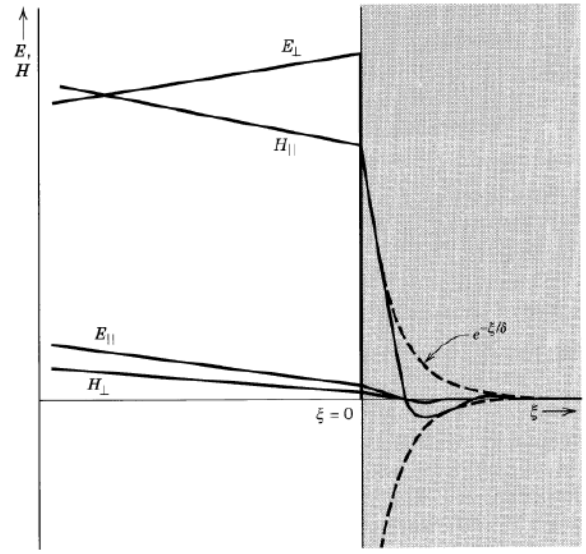
$$\frac{\partial^2(\vec{n} \times \vec{H}_c)}{\partial z^2} = -\frac{2i}{\delta^2}\vec{n} \times \vec{H}_c = \left(\frac{1-i}{\delta}\right)^2 \vec{n} \times \vec{H}_c$$

From the boundary condition $\vec{n} \times \vec{H} = \vec{n} \times \vec{H}_c$

$$\vec{H}_c(z=0) = \vec{H}_{\parallel}$$

where $\vec{H}_\parallel$ is the component of the magnetic field inside the wave guide parallel with the wall. The solution bounded for $z>0$ is then

$$\vec{H}_c(z) = \vec{H}_\parallel e^{-z/\delta} e^{iz/\delta}$$



Note that the electric field inside the conductor is

$$\vec{E}_c \approx -\frac{1}{\sigma}\vec{n} \times \frac{\partial \vec{H}_c}{\partial z} = \sqrt{\frac{\mu_c \omega}{2\sigma}}(1-i)\vec{n} \times \vec{H}_\parallel e^{-z/\delta} e^{iz/\delta}$$

So at the surface it develops a small parallel component, which is equal to the one inside of the wave guide on the wall:

$$\vec{E}_\parallel = \sqrt{\frac{\mu_c \omega}{2\sigma}}(1-i)\vec{n} \times \vec{H}_\parallel$$

Therefore, the boundary condition deviates slightly from that of the ideal metal, which can be taken into account at the next order.

Dissipation and effective surface current

Power carried away on unit surface of the wall:

$$\frac{dP}{da} = \vec{n} \cdot \vec{S} = \vec{n} \cdot \frac{1}{2}\mathrm{Re}\left(\vec{E} \times \vec{H}^*\right)\Big|_{z=0} = -\frac{1}{2}\sqrt{\frac{\mu_c \omega}{2\sigma}}\left|\vec{H}_\parallel\right|^2 = -\frac{1}{2\sigma\delta}\left|\vec{H}_\parallel\right|^2$$

Interpretation

The current density inside the metal wall is

$$\vec{J}_c = \sigma \vec{E}_c = \frac{1}{\delta}(1-i)\vec{n} \times \vec{H}_\parallel e^{-z/\delta} e^{iz/\delta}$$

Ohmic dissipation per unit area

$$\int_0^\infty dz \frac{1}{2}\vec{J}_c \cdot \vec{E}_c^* = \int_0^\infty dz \frac{1}{2\sigma}\left|\vec{J}_c\right|^2 = \int_0^\infty dz \frac{1}{2\sigma}\frac{2}{\delta^2}\vec{H}_\parallel^2 e^{-2z/\delta} = \frac{1}{2\sigma\delta}\left|\vec{H}_\parallel\right|^2$$

Effective surface current density

$$\vec{K}_{eff} = \int_0^\infty dz\ \vec{J}_c = \vec{n} \times \vec{H}_\parallel$$

Note that

1. The field satisfies the ideal boundary condition with $\vec{K}_{eff}$.
2. The dissipated power per unit wall area is nothing else than

$$\frac{dP}{da} = \frac{1}{2\sigma\delta}\left|\vec{H}_{\parallel}\right|^2 = \frac{1}{2\sigma\delta}\left|\vec{K}_{eff}\right|^2$$

Strategy of solving the wave guide

1. Solve the ideal case first.

2. Compute the dissipation using the formula

$$\frac{dP}{da} = \frac{1}{2\sigma\delta}\left|\vec{H}_{\parallel}\right|^2$$

Week 4: Wave guides

Strategy of solving the wave guide

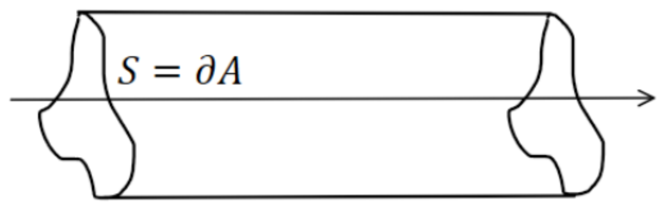
1. Solve the ideal case first.

2. Compute the dissipation using the formula

$$\frac{dP}{da} = \frac{1}{2\sigma\delta}\left|\vec{H}_{\parallel}\right|^2$$

General theory of ideal wave guide

Let us now choose the z direction along the axis of the wave guide.



The Maxwell equations

$$\vec{\nabla} \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t} \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \cdot \vec{E} = 0$$

imply the wave equations

$$(\Delta - \mu\epsilon\partial_t^2)\begin{Bmatrix}\vec{E}\\ \vec{B}\end{Bmatrix} = 0$$

Wave along the guide

$$\begin{Bmatrix}\vec{E}\\ \vec{B}\end{Bmatrix} = \begin{Bmatrix}\vec{E}(x,y)\\ \vec{B}(x,y)\end{Bmatrix} e^{ikz-i\omega t} \;\Rightarrow\; (\Delta_T - k^2 + \mu\epsilon\omega^2)\begin{Bmatrix}\vec{E}(x,y)\\ \vec{B}(x,y)\end{Bmatrix} = 0 \qquad \Delta_T = \partial_x^2 + \partial_y^2$$

Decomposing Maxwell's equations

$$\vec{E} = E_z\vec{e}_z + \vec{E}_T \quad \vec{B} = B_z\vec{e}_z + \vec{B}_T \quad \vec{\nabla} = \vec{e}_z\partial_z + \vec{\nabla}_T$$

Note: $\vec{E}_T = \left(\vec{e}_z \times \vec{E}\right) \times \vec{e}_z \quad \vec{B}_T = \left(\vec{e}_z \times \vec{B}\right) \times \vec{e}_z$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \left(\vec{e}_z\partial_z + \vec{\nabla}_T\right) \times \left(E_z\vec{e}_z + \vec{E}_T\right) = i\omega\left(B_z\vec{e}_z + \vec{B}_T\right)$$

so

$$-\vec{e}_z \times \vec{\nabla}_T E_z + \vec{e}_z \times \partial_z\vec{E}_T + \vec{\nabla}_T \times \vec{E}_T = i\omega\left(B_z\vec{e}_z + \vec{B}_T\right)$$

Transverse component: $\vec{e}_z \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla}_T E_z - \partial_z\vec{E}_T = i\omega\vec{e}_z \times \vec{B}_T$

Longitudinal component: $\vec{e}_z \cdot \left(\vec{\nabla} \times \vec{E}\right) = \vec{e}_z \cdot \left(\vec{\nabla}_T \times \vec{E}_T\right) = i\omega B_z$

$$\partial_z\vec{E}_T + i\omega\vec{e}_z \times \vec{B}_T = \vec{\nabla}_T E_z \qquad \vec{e}_z \cdot \left(\vec{\nabla} \times \vec{E}\right) = i\omega B_z$$

HW: apply a similar decomposition to the equation

$$\vec{\nabla} \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t}$$

to derive $\quad \partial_z \vec{B}_T - i\omega\mu\epsilon \vec{e}_z \times \vec{E}_T = \vec{\nabla}_T B_z \qquad \vec{e}_z \cdot (\vec{\nabla} \times \vec{B}) = -i\omega\mu\epsilon E_z$

HW: decompose the equations $\vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \cdot \vec{E} = 0$ to obtain

$$\vec{\nabla}_T \cdot \vec{E}_T = -\partial_z E_z \qquad \vec{\nabla}_T \cdot \vec{B}_T = -\partial_z B_z$$

Note that ∂_z can be substituted with ik to give

$$ik\vec{E}_T + i\omega\vec{e}_z \times \vec{B}_T = \vec{\nabla}_T E_z$$
$$ik\vec{B}_T - i\omega\mu\epsilon\vec{e}_z \times \vec{E}_T = \vec{\nabla}_T B_z$$

We can also write for the longitudinal components

$$(\Delta_T + \gamma^2)\begin{Bmatrix} E_z \\ B_z \end{Bmatrix} = 0 \qquad \gamma^2 = k^2 - \mu\epsilon\omega^2$$

Summary: decomposition of Maxwell's equations

$$\vec{E}, \vec{B} \propto e^{ikz - i\omega t}$$
$$\vec{E} = E_z\vec{e}_z + \vec{E}_T \qquad \vec{B} = B_z\vec{e}_z + \vec{B}_T$$

$$\frac{\partial \vec{E}_T}{\partial z} + i\omega\vec{e}_z \times \vec{B}_T = \vec{\nabla}_T E_z \qquad \vec{e}_z \cdot (\vec{\nabla}_T \times \vec{E}_T) = i\omega B_z$$
$$\frac{\partial \vec{B}_T}{\partial z} - i\omega\mu\epsilon\vec{e}_z \times \vec{E}_T = \vec{\nabla}_T B_z \qquad \vec{e}_z \cdot (\vec{\nabla}_T \times \vec{B}_T) = -i\omega\mu\epsilon E_z$$
$$\vec{\nabla}_T \cdot \vec{E}_T = -\frac{\partial E_z}{\partial z} \qquad \vec{\nabla}_T \cdot \vec{B}_T = -\frac{\partial B_z}{\partial z}$$

Solving for the transverse components gives

$$\vec{E}_T = \frac{i}{\gamma^2}\left(k\vec{\nabla}_T E_z - \omega\vec{e}_z \times \vec{\nabla}_T B_z\right) \quad \vec{B}_T = \frac{i}{\gamma^2}\left(k\vec{\nabla}_T B_z + \mu\epsilon\omega\vec{e}_z \times \vec{\nabla}_T E_z\right)$$

Equation for longitudinal components

$$(\Delta_T + \gamma^2)B_z = 0 = (\Delta_T + \gamma^2)E_z \qquad \gamma^2 = \epsilon\mu\omega^2 - k^2$$

TEM, TM and TE modes

(1) <u>TEM</u>: $E_z = B_z = 0$

We then get a 2D electrostatic problem

$$\vec{\nabla}_T \times \vec{E}_T = 0 \quad \vec{\nabla}_T \cdot \vec{E}_T = 0$$

$$\frac{\partial \vec{E}_T}{\partial z} + i\omega \vec{e}_z \times \vec{B}_T = 0 \qquad \frac{\partial \vec{B}_T}{\partial z} - i\omega\mu\epsilon \vec{e}_z \times \vec{E}_T = 0$$
$$ik\vec{E}_T + i\omega \vec{e}_z \times \vec{B}_T = 0 \quad ik\vec{B}_T - i\omega\mu\epsilon \vec{e}_z \times \vec{E}_T = 0$$

$$\Rightarrow \ \vec{e}_z \times (ik\vec{E}_T + i\omega \vec{e}_z \times \vec{B}_T) = \ ik\vec{e}_z \times \vec{E}_T - i\omega \vec{B}_T = 0$$

Consistency requires
$$\frac{ik}{-i\omega} = \frac{-i\omega\mu\epsilon}{ik} \ \Rightarrow \ \omega = \frac{1}{\sqrt{\epsilon\mu}} k$$

Linear dispersion (identical to the one without boundaries)!

If we have $\vec{E}_T$ then the magnetic field can be computed

$$\vec{B}_T = \frac{\omega\mu\epsilon}{k} \vec{e}_z \times \vec{E}_T = \sqrt{\epsilon\mu}\, \vec{e}_z \times \vec{E}_T$$

We can write
$$\vec{E}_T = -\vec{\nabla}_T \Phi \quad \Delta_T \Phi = 0$$

Boundary condition on wall: $\vec{E}_T$ must only have normal component $\Rightarrow \Phi|_S = \text{const.}$

Problem to solve:

$$\Delta_T \Phi = 0 \quad \Phi|_S = \text{const.}$$

This only has trivial solutions if there is just a single wall (simply connected domain).

However, TEM can propagate e.g. in coax cables, since

$$\Delta_T \Phi = 0 \quad \Phi|_{S_1} = \phi_1 \quad \Phi|_{S_2} = \phi_2$$

has nontrivial solution!

- No dispersion (only if there are losses i.e., wall is not ideal), therefore no signal distortion
- No lower cut-off frequency

(2) TM mode: $B_z = 0$

$$\vec{E}_T = \frac{ik}{\gamma^2} \vec{\nabla}_T \Psi \quad \vec{B}_T = \frac{i\mu\epsilon\omega}{\gamma^2} \vec{e}_z \times \vec{\nabla}_T \Psi$$

Master function: $\Psi = E_z$ on the ideal conductor boundary: $E_z = 0$

$$(\Delta_T + \gamma^2)\Psi = 0 \quad \Psi|_S = 0 \quad \gamma^2 = \epsilon\mu\omega^2 - k^2$$

(3) TE mode: $E_z = 0$

$$\vec{E}_T = -\frac{i\omega}{\gamma^2}\vec{e}_z \times \vec{\nabla}_T\Psi \quad \vec{B}_T = \frac{ik}{\gamma^2}\vec{\nabla}_T\Psi$$

Master function: $\Psi = B_z$ on the ideal conductor boundary: $\vec{B}_T$ is parallel to the wall

$$(\Delta_T + \gamma^2)\Psi = 0 \quad \left.\frac{\partial\Psi}{\partial n}\right|_S = 0 \quad \gamma^2 = \epsilon\mu\omega^2 - k^2$$

Dispersion relation

Determining modes

$$(\Delta_T + \gamma^2)\Psi = 0 \quad \Psi|_S = 0 \ \text{(TM)} \ \text{or} \quad \left.\frac{\partial\Psi}{\partial n}\right|_S = 0 \ \text{(TE)}$$

Eigenvalues with given BC: γ_α

- TM: eigenproblem is the same as membrane with fixed edge
- TE: eigenproblem is the same as membrane with free edge

$$k = \sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_\alpha^2} \qquad \omega_\alpha = \frac{\gamma_\alpha}{\sqrt{\mu\epsilon}} \quad \text{cut-off frequency}$$

$$\omega > \omega_\alpha : k \ \text{real} \qquad \Rightarrow \ \text{propagating wave}$$

$$\omega < \omega_\alpha : k \ \text{imaginary} \ \Rightarrow \text{evanescent field}$$

Phase velocity

$$v_{\text{ph}} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}\frac{1}{\sqrt{1 - \frac{\omega_\alpha^2}{\omega^2}}} > \frac{1}{\sqrt{\mu\epsilon}}$$

Group velocity

$$v_g = \frac{d\omega}{dk} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{1 - \frac{\omega_\alpha^2}{\omega^2}} < \frac{1}{\sqrt{\mu\epsilon}}$$

Note that

$$v_{\text{ph}}v_g = \frac{1}{\mu\epsilon}$$

Energy carried in a mode

Time-averaged energy current density (Poynting vector)

$$\vec{S} = \frac{1}{2}\mathrm{Re}\left(\vec{E} \times \vec{H}^*\right)$$

Transmitted power

$$P = \iint_A d^2a\, \vec{e}_z \cdot \vec{S} = \iint_A d^2a \frac{1}{2}\mathrm{Re}\, \vec{e}_z \cdot \left(\vec{E}_T \times \vec{H}_T^*\right)$$

$$\text{TM: } \vec{E}_T = \frac{ik}{\gamma^2}\vec{\nabla}_T\Psi \quad \vec{B}_T = \frac{i\mu\epsilon\omega}{\gamma^2}\vec{e}_z \times \vec{\nabla}_T\Psi$$
$$\text{TE: } \vec{E}_T = -\frac{i\omega}{\gamma^2}\vec{e}_z \times \vec{\nabla}_T\Psi \quad \vec{B}_T = \frac{ik}{\gamma^2}\vec{\nabla}_T\Psi$$

$$P = \frac{\omega k}{2\gamma^4}\begin{Bmatrix}\epsilon \\ 1/\mu\end{Bmatrix} \iint_A d^2a\, \vec{\nabla}_T\Psi^* \cdot \vec{\nabla}_T\Psi \quad \begin{Bmatrix}\text{TM} \\ \text{TE}\end{Bmatrix}$$

Green's theorem (in the transverse 2 dimensions)

$$\iint_A d^2a\left[\Psi^*\Delta_T\Psi + \vec{\nabla}_T\Psi^* \cdot \vec{\nabla}_T\Psi\right] = \oint_S dl\, \underbrace{\Psi^*\frac{\partial\Psi}{\partial n}}_{0 \text{ on } S} = 0$$
$$\iint_A d^2a\, \vec{\nabla}_T\Psi^* \cdot \vec{\nabla}_T\Psi = -\iint_A d^2a\, \Psi^*\Delta_T\Psi = \gamma^2 \iint_A d^2a|\Psi|^2$$

So

$$P = \frac{\omega k}{2\gamma_\alpha^2}\begin{Bmatrix}\epsilon \\ 1/\mu\end{Bmatrix} \iint_A d^2a\, |\Psi|^2 \quad \begin{Bmatrix}\text{TM} \\ \text{TE}\end{Bmatrix}$$
$$= \frac{1}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\alpha}\right)^2 \sqrt{1 - \frac{\omega_\alpha^2}{\omega^2}}\begin{Bmatrix}\epsilon \\ 1/\mu\end{Bmatrix} \iint_A d^2a\, |\Psi|^2 \quad \begin{Bmatrix}\text{TM} \\ \text{TE}\end{Bmatrix}$$

Energy in a mode per unit length

Energy density (time averaged)
$$w = \frac{1}{2}\epsilon\left(\frac{1}{2}\mathrm{Re}\,\vec{E} \cdot \vec{E}^*\right) + \frac{1}{2\mu}\left(\frac{1}{2}\mathrm{Re}\,\vec{B} \cdot \vec{B}^*\right)$$
$$\text{TM: } \vec{E}_T = \frac{ik}{\gamma^2}\vec{\nabla}_T\Psi \quad \vec{B}_T = \frac{i\mu\epsilon\omega}{\gamma^2}\vec{e}_z \times \vec{\nabla}_T\Psi \quad \Psi = E_z$$

Energy per unit length
$$U_{TM} = \frac{1}{4}\iint_A d^2a\left\{\epsilon\left(\frac{k}{\gamma^2}\vec{\nabla}_T\Psi\right)^2 + \epsilon|\Psi|^2 + \frac{1}{\mu}\left(\frac{\mu\epsilon\omega}{\gamma^2}\vec{\nabla}_T\Psi\right)^2\right\}$$

TE: $\vec{E}_T = -\frac{i\omega}{\gamma^2}\vec{e}_z \times \vec{\nabla}_T \Psi \quad \vec{B}_T = \frac{ik}{\gamma^2}\vec{\nabla}_T \Psi \quad \Psi = B_z$

$$U_{TE} = \frac{1}{4}\iint_A d^2a \left\{ \epsilon \left(\frac{\omega}{\gamma^2}\vec{\nabla}_T \Psi\right)^2 + \frac{1}{\mu}|\Psi|^2 + \frac{1}{\mu}\left(\frac{k}{\gamma^2}\vec{\nabla}_T \Psi\right)^2 \right\}$$

HW: finish the calculation and derive

$$U = \frac{1}{2}\left(\frac{\omega}{\omega_\alpha}\right)^2 \begin{Bmatrix} \epsilon \\ 1/\mu \end{Bmatrix} \iint_A d^2a\, |\Psi|^2 \quad \begin{Bmatrix} \text{TM} \\ \text{TE} \end{Bmatrix}$$

Velocity of energy propagation: identical to the group velocity!

$$v_E = \frac{P}{U} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{1 - \frac{\omega_\alpha^2}{\omega^2}} = v_{gr}$$

Attenuation

Supposing that wall is almost ideal, to leading order

$$\frac{dP}{dz} = \frac{1}{2\sigma\delta}\oint_S dl\, \left|\vec{n} \times \vec{H}\right|^2$$

A detailed computation can be found in Jackson's book. It turns out that for any given mode

$$\frac{dP}{dz} = -2\beta_\alpha P \quad \Rightarrow \quad P(z) = P(z=0)e^{-2\beta_\alpha z}$$

where β_α is a mode dependent dissipation constant of unit 1/length.

Another way: use first order perturbation theory (analogous to QM) to account for deviation of boundary condition from ideal. Result:

$$k_\alpha(\omega) = k_\alpha^0 + \Delta k_\alpha + i\beta_\alpha$$

where β_α is the same for the previous method, but there is also a change in the dispersion relation Δk_α (it happens to be the case that $\Delta k_\alpha = \beta_\alpha$).

HW: wave impedance

Show for a given mode that it is always true that

$$\vec{H}_T = \frac{1}{Z}\vec{e}_z \times \vec{E}_T$$

where

$$Z = \begin{Bmatrix} \dfrac{k}{\epsilon\omega} = \dfrac{k}{k_0}\sqrt{\dfrac{\mu}{\epsilon}} & \text{(TM)} \\ \dfrac{\mu\omega}{k} = \dfrac{k_0}{k}\sqrt{\dfrac{\mu}{\epsilon}} & \text{(TE)} \end{Bmatrix} \quad k_0 = \omega\sqrt{\epsilon\mu}$$

is the wave impedance.

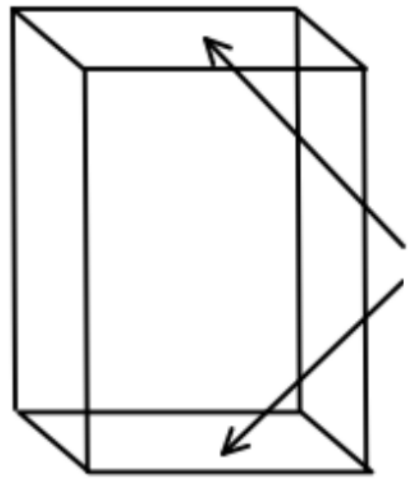
This is not related to any dissipation (which rather comes from attenuation) however it is important for impedance matching at the ends of the power line.

Note: these are different from wave impedance in vacuum

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 120\pi$$

Week 5: Resonant cavities. Dispersive media

Cylindrical cavity



Boundary condition at the ends

TM: $\vec{\nabla} \cdot \vec{E} = 0$

i.e. $\partial_z \Psi + \vec{\nabla}_T \cdot \vec{E}_T = 0$

At the endcap $\vec{E}_T = 0$ so

TM: $\partial_z \Psi|_{z=0,d} = 0 \;\Rightarrow\; \Psi \propto \cos k_p z \quad k_p = \frac{\pi p}{d}$

TE: $\Psi|_{z=0,d} = 0 \;\Rightarrow\; \Psi \propto \sin k_p z \quad k_p = \frac{\pi p}{d}$

Eigenfrequencies

$$\gamma^2 = \epsilon\mu\omega^2 - k^2 \Rightarrow \omega_{\lambda p} = \frac{1}{\sqrt{\epsilon\mu}}\sqrt{\gamma_\lambda^2 + \left(\frac{\pi p}{d}\right)^2}$$

TM: $p = 0,1,2, \ldots$ TE: $p = 1,2, \ldots$

Non-ideal cavity: quality factor

$$Q = \omega_0 \frac{\text{stored energy}}{\text{power loss}}$$

ω_0: ideal frequency

Stored energy

$$U = \int_V d^3x \left(\frac{1}{2}\epsilon(E_z^2 + \vec{E}_T^2) + \frac{1}{2}\mu(H_z^2 + \vec{H}_T^2)\right) \propto d\int_A |\Psi|^2$$

Power loss:

$$P = \frac{1}{2\sigma\delta}\oint_{\partial V} \left|\vec{n} \times \vec{H}\right|^2 \propto \cdots \int_A |\Psi|^2 + d\oint_{\partial A} |\Psi|^2$$

Amplitude drops out! $\Rightarrow Q$ depends only on mode and geometry via ratio

$$\frac{\oint_{\partial A}|\Psi|^2}{\int_A |\Psi|^2}$$

Jackson:

$$Q = \frac{\mu}{\mu_c}\frac{V}{S\delta} \cdot \text{(geometric factor)}$$

V: volume where energy is stored
$S\delta$: volume where dissipation happens (skin effect)

Dissipation

$$\frac{dU}{dt} = -\frac{\omega_0}{Q}U \Rightarrow U(t) = U(0)e^{-\omega_0 t/Q}$$

Lifetime: $\tau = \frac{Q}{\omega_0} = \frac{Q}{2\pi}T$

Amplitude: $\Psi(t) \propto e^{-\omega_0 t/2Q}e^{-i(\omega_0+\Delta\omega)t}$

$\Delta\omega$: frequency shift due to non-ideal BC

Fourier transform:

$$\Psi(\omega) \propto \int_0^\infty dt\, e^{-\omega_0 t/2Q}e^{-i(\omega_0+\Delta\omega)t}e^{i\omega \mathrm{t}} \propto \frac{1}{\frac{\omega_0}{2Q} - i(\omega - \omega_0 - \Delta\omega)}$$

Frequency dependence of intensity of radiation field:

$$I(\omega) \propto |\Psi(\omega)|^2 \propto \frac{1}{(\omega - \omega_0 - \Delta\omega)^2 + \left(\frac{\omega_0}{2Q}\right)^2}$$

Lorentzian curve

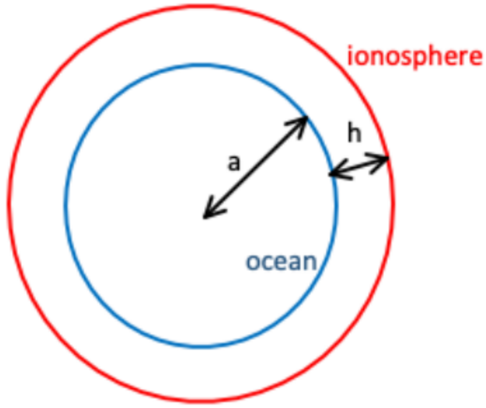
Half-width:

$$\Gamma = \frac{\omega_0}{Q} = \frac{1}{\tau} \quad \text{(analogy to QM)}$$

Earth as a spherical resonant cavity

(mini-project A6)

Ionosphere: $\sigma \sim 10^{-7} - 10^{-4} \frac{1}{\Omega m}$
Seawater: $\sigma \sim 4.4 \frac{1}{\Omega m}$
Not nearly perfect…
Estimating lowest frequencies:



TE mode: $H_r = 0$ at the boundaries
$\Rightarrow \omega_{TE} \sim \pi c/\mathrm{h}$

TM mode: E_r can be almost constant in r
$\Rightarrow \omega_{TM} \sim c/a$

$\mathrm{h} \ll a \Rightarrow \omega_{TE} \gg \omega_{TM}$
Jackson:

$$\nu_l = \frac{\omega_l}{2\pi} = \sqrt{l(l+1)}\frac{c}{2\pi a} \qquad \frac{c}{2\pi a}$$
$$= 7.46 \text{ Hz}$$

Schumann, 1952

Measured: 5.8 Hz (1960)

Quality factor: $Q = 4 \dots 10$ - very low!

Frequency dependent refractive index

Linear response theory for electric polarisation

$$P(t) = \int_{-\infty}^{\infty} dt' \underbrace{\theta(t-t')}_{\text{causality}} G(t,t')E(t')$$

Electric susceptibility

$$\epsilon_0 \chi(t-t') = \theta(t-t') \underbrace{G(t-t')}_{\substack{\text{no memory} \\ \text{effect}}} \quad \Rightarrow \quad P(t) = \epsilon_0 \int_{-\infty}^{\infty} dt' \, \chi(t-t')E(t')$$

With spatial dependence (for translationally invariant i.e., homogeneous system):

$$P(t,x) = \epsilon_0 \int_{-\infty}^{\infty} dt' \int d^3x' \, \chi(t-t', x-x')E(t',x')$$

Fourier transform: Reminder:

$$P(\omega) = \epsilon_0 \chi(\omega) E(\omega)$$

$$D(\omega) = \epsilon_0 \underbrace{(1 + \chi(\omega))}_{\epsilon_r(\omega)} E(\omega)$$

$$f(\omega) = \int_{-\infty}^{\infty} dt f(t) e^{+i\omega t}$$

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(\omega) e^{-i\omega t}$$

Reality: $E^*(\omega) = E(-\omega)$ and $P^*(\omega) = P(-\omega) \;\Rightarrow\; \chi^*(\omega) = \chi(-\omega)$

Lorentz-Drude model for dielectrics

Damped oscillators with charge q:

$$m\frac{d^2x}{dt^2} + \Gamma\frac{dx}{dt} + Dx = qE(t) \;\Rightarrow\; \frac{d^2p}{dt^2} + \gamma\frac{dp}{dt} + \omega_0^2 p = \frac{q^2}{m}E(t)$$

$$p = qx \quad \gamma = \frac{\Gamma}{m} \quad \omega_0^2 = \frac{D}{m}$$

$$E(t) = E(\omega)e^{-i\omega t} \;\Rightarrow\; p(t) = p(\omega)e^{-i\omega t}$$

$$p(\omega) = \frac{q^2}{m}\frac{1}{\omega_0^2 - i\gamma\omega - \omega^2} E(\omega) \Rightarrow P(\omega) = \underbrace{\frac{Nq^2}{m}\frac{1}{\omega_0^2 - i\gamma\omega - \omega^2}}_{\epsilon_0\chi} E(\omega)$$

N: volume density of charges

With more than one type of charge carriers

$$\epsilon_r = 1 + \chi = 1 + \sum_j \frac{N_j q_j^2}{\epsilon_0 m_j}\frac{1}{\omega_j^2 - i\gamma_j\omega - \omega^2}$$

Imaginary part of refractive index

$$\text{Im}\,\epsilon_r(\omega) = \frac{1}{2i}(\epsilon_r(\omega) - \epsilon_r(\omega)^*) = \sum_j \frac{N_j q_j^2}{\epsilon_0 m_j}\frac{\gamma_j\omega}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j^2\omega^2} > 0 \;\;!!!$$

Refractive index: usually $\mu_r \approx 1$, so

$$n(\omega) = \sqrt{\epsilon_r(\omega)} = n_r(\omega) + i n_i(\omega) \quad \text{with} \quad n_i(\omega) > 0$$

Wave propagation with complex refractive index

$$\text{curl}\,\vec{H} = \frac{\partial\vec{D}}{\partial t} \;\Rightarrow\; \text{curl}\,\vec{B} = \mu_0\frac{\partial\vec{D}}{\partial t} = -i\omega\mu_0\vec{D} = -i\omega\mu_0\epsilon_0 \underbrace{\epsilon_r(\omega)}_{n(\omega)^2}\vec{E}$$

$$\Rightarrow \text{curl}\,\vec{B} = -i\frac{\omega n(\omega)^2}{c^2}\vec{E}$$

$\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$
$\Rightarrow \text{curl}\,\vec{E} = -i\omega\vec{B}$

Using $\text{div}\,\vec{E} = 0 = \text{div}\,\vec{B}$ we get the wave equation

$$\left(\Delta + \frac{\omega^2 n(\omega)^2}{c^2}\right)\binom{\vec{E}}{\vec{B}} = 0$$

Plane wave solution:

$\vec{E}, \vec{B} \propto e^{i\vec{k}\cdot\vec{x} - i\omega t}$ with $\vec{k}\cdot\vec{k} = \frac{\omega^2 n(\omega)^2}{c^2} \Rightarrow \vec{k} = \frac{\omega n(\omega)}{c}\hat{k}$ $\hat{k}$: direction of wave

Therefore

$$\vec{E}(t,\vec{x}) = \vec{E}_0\, e^{-i\omega t - i\,\hat{k}\cdot\vec{x}\frac{\omega}{c}n_r(\omega)}\, e^{-\hat{k}\cdot\vec{x}\frac{\omega}{c}n_i(\omega)}$$

Attenuation!

Attenuation of the EM wave intensity (given by magnitude of Poynting vector)

$S(x) \propto \left|\vec{E}(t,\vec{x})\right|^2 = S_0 e^{-\rho\kappa x}$

$x = \hat{k}\cdot\vec{x}$: distance travelled by the wave

$\kappa = \frac{2\omega n_i(\omega)}{c\rho}$ **opacity**

Opacity: the smaller it is, the more transparent the material is.

Remark: Earth's atmosphere is transparent for
$\lambda \sim 400 - 800$ nm: visible light (evolution!)
$\lambda \sim 1$ cm $- 1$ m: radio waves

Plasma frequency

Free electron gas/ideal metal: $\omega_j = \gamma_j = 0$

$$\epsilon_r = 1 + \sum_j \frac{N_j q_j^2}{\epsilon_0 m_j}\frac{1}{\omega_j^2 - i\gamma_j\omega - \omega^2} \Rightarrow \epsilon_r = 1 - \frac{\omega_p^2}{\omega^2} \quad \omega_p^2 = \sum_j \frac{N_j q_j^2}{\epsilon_0 m_j}$$

Metals: $N \sim 10^{26}\,\frac{1}{m^3}$ $q = e$ $m =$ effective mass (depends on band structure)
$\Rightarrow$ typically, UV frequency (~ 100 nm)

Below plasma frequency

$\omega < \omega_p$: ϵ_r is negative $\Rightarrow n$ is imaginary

Total reflection: only evanescent wave in material
Reflection coefficient from Fresnel formula

$$r = \frac{E_{refl}}{E_{in}} = \frac{n\cos\theta - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}}{n\cos\theta + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}} \qquad \text{for } \perp \text{ polarisation}$$

$$= \frac{\frac{\mu}{\mu'}n'^2\cos\theta - n\sqrt{n'^2 - n^2\sin^2\theta}}{\frac{\mu}{\mu'}n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}} \quad \text{for } \parallel \text{ polarisation}$$

$n^2 > 0 > n'^2$: $|r| = 1$: total reflection

Plasma penetration depth

$S(x) \propto \left|\vec{E}(t,\vec{x})\right|^2 = S_0 e^{-\frac{2\omega n_i(\omega)}{c}x} \;\Rightarrow\; \delta = \frac{c}{2\omega n_i}$

For $\omega \ll \omega_p$

$$n_i = \sqrt{\frac{\omega_p^2}{\omega^2} - 1} \approx \frac{\omega_p}{\omega} \quad \Rightarrow \quad \delta = \frac{c}{2\omega_p}$$

the penetration of low-frequency magnetic fields is frequency independent!

For typical laboratory plasmas

$N = 10^{18} - 10^{22}\,\frac{1}{m^3} \Rightarrow \delta = 0.2 - 0.002$ cm

Above plasma frequency

$\omega > \omega_p$: ϵ_r is positive $\Rightarrow$ n is real

Material is completely transparent: metals are transparent to UV light!

Exercise: compute phase and group velocity in plasma for $\omega > \omega_p$. Show that

$$v_{\text{phase}} = \frac{\omega}{k} = \frac{c}{n(\omega)} > c \qquad v_{group} = \frac{d\omega}{dk} = cn(\omega) < c$$

Note that at plasma frequency $v_{phase} = \infty$ but $v_{group} = 0$!

Collective excitation: **plasmon** with dispersion

$$\omega^2 = \omega_p^2 + \frac{3k_BT_e}{m_e}\vec{k}^2$$

Week 6: Dispersion and Kramers-Krönig relations

A few important facts from complex analysis and distribution theory

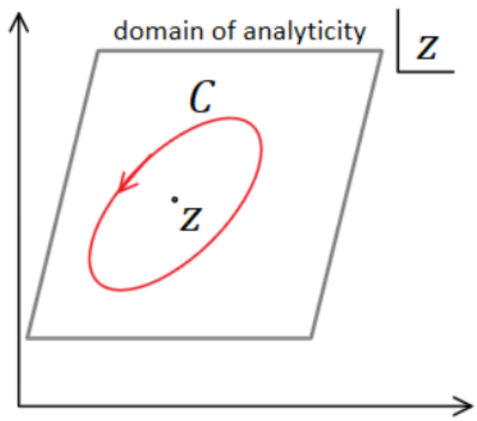
If a function $f(z)$ of a complex variable is differentiable at a point z_0, then it is analytic i.e., infinitely differentiable and its Taylor series at z_0 has a finite convergence radius $a > 0$ and converges to $f(z)$:

$$f(z) = \sum_{n=0}^{\infty} f^{(n)}(z_0)\frac{(z-z_0)^n}{n!} \qquad |z-z_0| < a$$

Any such function satisfies Cauchy's integral formula

$$f(z) = \oint_C \frac{dw}{2\pi i}\frac{f(w)}{w-z}$$

where C is a closed contour around z and the function is analytic in an open domain containing C and its interior.



Cauchy principal value integral

How to define a distribution corresponding to $1/x$?

$\int_{-\infty}^{\infty} dx\,\frac{1}{x}f(x)$ generally does not exist:

$$\int_{\epsilon}^{a} dx\,\frac{f(x)}{x} = \int_{\epsilon}^{a} dx\,\frac{f(0)}{x} + \int_{\epsilon}^{a} dx\,\frac{f(x)-f(0)}{x} = f(0)\log\frac{a}{\epsilon} + \text{terms finite for } \epsilon \to 0$$

Cauchy's idea: cut out the singularity symmetrically, then the divergence cancels between left and right:

$$\mathcal{P}\int_{-\infty}^{\infty} dx\,\frac{1}{x}f(x) = \lim_{\epsilon\to+0}\left\{\int_{\epsilon}^{\infty} dx\,\frac{1}{x}f(x) + \int_{-\infty}^{-\epsilon} dx\,\frac{1}{x}f(x)\right\}$$

A simple way to write it is

$$\mathcal{P}\int_{-\infty}^{\infty} dx\,\frac{1}{x}f(x) = \int_{0}^{\infty} dx\,\frac{f(x)-f(-x)}{x}$$

It can also be defined centred at any point a:

$$\mathcal{P}\int_{-\infty}^{\infty} dx \frac{1}{x-a} f(x) = \int_0^{\infty} dx \frac{f(a+x) - f(a-x)}{x}$$

Magic distribution identity

$$\frac{1}{x-a \pm i\eta} \longrightarrow \mathcal{P}\frac{1}{x-a} \mp i\pi\delta(x-a) \text{ as } \eta \longrightarrow +0$$

Proof: (for + sign; proof similar for - sign)

$$\frac{1}{x-a+i\eta} = \frac{1}{2}\left(\frac{1}{x-a+i\eta} + \frac{1}{x-a-i\eta}\right) + \frac{1}{2}\left(\frac{1}{x-a+i\eta} - \frac{1}{x-a-i\eta}\right)$$
$$= \frac{x-a}{(x-a)^2+\eta^2} - \frac{i\eta}{(x-a)^2+\eta^2}$$

$$\text{Re}\int_{-\infty}^{\infty} dx \frac{1}{x-a+i\eta} f(x) = \int_{-\infty}^{\infty} dx \frac{x-a}{(x-a)^2+\eta^2} f(x) = \int_{-\infty}^{\infty} dx \frac{x}{x^2+\eta^2} f(a+x)$$
$$= \int_{-\infty}^{\infty} dx \frac{x}{x^2+\eta^2} f(a+x) = \int_0^{\infty} dx \frac{x\big(f(a+x) - f(a-x)\big)}{x^2+\eta^2}$$
$$\rightarrow \int_0^{\infty} dx \frac{f(a+x) - f(a-x)}{x}$$
$$= \mathcal{P}\int_{-\infty}^{\infty} dx \frac{1}{x-a} f(x)$$

$$\text{Im}\int_{-\infty}^{\infty} dx \frac{1}{x-a+i\eta} f(x) = -\int_{-\infty}^{\infty} dx \frac{\eta}{(x-a)^2+\eta^2} f(x)$$

$$\frac{\eta}{(x-a)^2+\eta^2} \rightarrow \pi\delta(x-a): \int_{-\infty}^{\infty} dx \frac{\eta}{(x-a)^2+\eta^2} = \pi \text{ and } \frac{\eta}{(x-a)^2+\eta^2} \rightarrow 0 \text{ for } x \neq a$$

$$\text{Im}\int_{-\infty}^{\infty} dx \frac{1}{x-a+i\eta} f(x) = -\pi f(a)$$

Therefore

$$\frac{1}{x-a+i\eta} \longrightarrow \mathcal{P}\frac{1}{x-a} - i\pi\delta(x-a) \text{ as } \eta \longrightarrow +0$$

Similarly

$$\frac{1}{x-a-i\eta} \longrightarrow \mathcal{P}\frac{1}{x-a} + i\pi\delta(x-a) \;\; \text{as} \;\; \eta \longrightarrow +0$$

Kramers-Krönig relations: a consequence of causality

$$P(t) = \int_{-\infty}^{\infty} dt'\, \theta(t-t')G(t-t')E(t') \;\Rightarrow\; \epsilon_r(\omega) = 1 + \int_0^{\infty} dt\, G(t)e^{i\omega t}$$

- it is analytic for $\operatorname{Im}\omega > 0$ if $G(t)$ is bounded for all $t > 0$

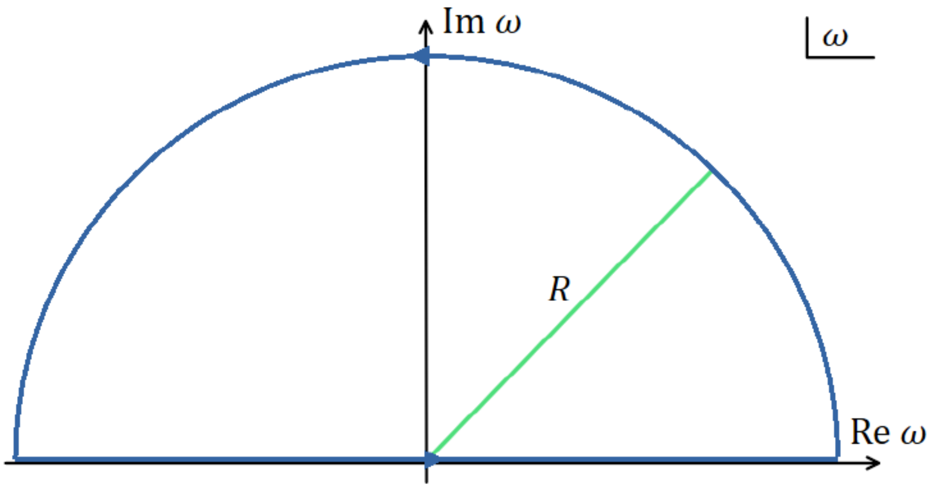
  $\left|e^{i\omega t}\right| = e^{-t\operatorname{Im}\omega}$ makes the integral convergent for all $\operatorname{Im}\omega > 0$

- $\epsilon_r(\omega) - 1 \sim \omega^{-2}$ for large ω

Cauchy's theorem implies that

$$\epsilon_r(\omega + i\eta) - 1 = \frac{1}{2\pi i}\oint_C d\omega' \frac{\epsilon_r(\omega') - 1}{\omega' - \omega - i\eta}$$

with ω real as long as $\eta > 0$, with the contour C given as



Due to $\epsilon_r(\omega) - 1 \sim \omega^{-2}$ for $R \to \infty$ integral over semicircle can be dropped (integral scales as R^{-2}):

$$\epsilon_r(\omega + i\eta) = 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega' \frac{\epsilon_r(\omega') - 1}{\omega' - \omega - i\eta}$$

Applying the magic distribution identity:

$$\frac{1}{\omega'-\omega-i\eta} \longrightarrow \mathcal{P}\frac{1}{\omega'-\omega} + i\pi\delta(\omega'-\omega) \;\; \text{as} \;\; \eta \longrightarrow +0$$

$$\Rightarrow \epsilon_r(\omega) = 1 + \frac{1}{2\pi i}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\epsilon_r(\omega') - 1}{\omega' - \omega} + \frac{1}{2}(\epsilon_r(\omega) - 1)$$

i.e.

$$\epsilon_r(\omega) = 1 + \frac{1}{\pi i}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\epsilon_r(\omega') - 1}{\omega' - \omega}$$

Taking real and imaginary parts:

$$\text{Re}\,\epsilon_r(\omega) = 1 + \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\text{Im}\,\epsilon_r(\omega')}{\omega' - \omega} \qquad , \quad \text{Im}\,\epsilon_r(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\text{Re}\,\epsilon_r(\omega')}{\omega' - \omega}$$

Kramers-Krönig relations

<u>Remark:</u> It is also true that if for a function its Fourier transform is analytic in the upper half plane, then the function vanishes for negative arguments.

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(\omega) e^{-i\omega t}$$

For $t < 0$: the contour can be closed in the upper half plane, since

$$\left|e^{-i\omega t}\right| = e^{+t\text{Im}\omega} \to 0 \ \text{ for } \ \text{Im}\,\omega \to +\infty$$

implying that

$$f(t) = \oint_C \frac{d\omega}{2\pi} f(\omega) e^{-i\omega t} = 0 \ \text{ for } \ t < 0$$

Relation between ϵ and σ

$\rho_P = -\text{div}\,\vec{P}$ but: continuity equation $\partial_t \rho_P + \text{div}\,\vec{J}_P = 0$

$$\Rightarrow \vec{J}_P = \frac{\partial \vec{P}}{\partial \text{t}}$$

Compare now $\vec{J} = -i\omega\vec{P} = \sigma\vec{E}$ and $\vec{P} = \epsilon_0 \chi \vec{E} \ \Rightarrow \ \sigma = -i\epsilon_0\omega\chi(\omega)$ i.e.

$$\epsilon_r(\omega) = 1 + \frac{i\sigma(\omega)}{\epsilon_0 \omega}$$

Drude-Lorenz model

$$\epsilon_r = 1 + \sum_j \frac{N_j q_j^2}{\epsilon_0 m_j} \frac{1}{\omega_j^2 - i\gamma_j \omega - \omega^2}$$

Metals: conduction electrons not bound ⇒ $\omega_j = 0$

$$\gamma_j = \frac{1}{\tau_j} \quad \tau_j\text{: relaxation time}$$

Assuming ω is small (DC conductivity)

$$\sigma = \sum_j \frac{N_j q_j^2 \tau_j}{m_j}$$

This is the Drude model for DC conductivity.

Week 7: Multipole radiation

Review of material from Electrodynamics 1

Maxwell's equations

$$\mathrm{div}\,\vec{B} = 0 \qquad \Rightarrow \qquad \vec{B} = \mathrm{curl}\,\vec{A}$$

$$\mathrm{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \Rightarrow \qquad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \mathrm{grad}\,\Phi$$

Lorentz gauge: $\mathrm{div}\,\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$

$$\mathrm{curl}\,\vec{B} = \mu_0\,\vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \qquad \Rightarrow \qquad \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\,\vec{J}$$

$$\mathrm{div}\,\vec{E} = \frac{\rho}{\epsilon_0} \qquad \Rightarrow \qquad \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -\frac{\rho}{\epsilon_0}$$

Retarded solution

$$\vec{A}(t,\vec{x}) = \frac{\mu_0}{4\pi}\int_V d^3x'\,\frac{\vec{J}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}'\right)}{|\vec{x}-\vec{x}'|}$$

$$\Phi(t,\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x'\,\frac{\rho\left(t - \frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}'\right)}{|\vec{x}-\vec{x}'|}$$

V: source volume - assumed to be localised

Oscillating sources

$$\rho(t,\vec{x}) = \rho(\vec{x})e^{-i\omega t} \quad \vec{J}(t,\vec{x}) = \vec{J}(\vec{x})e^{-i\omega t} \quad \Rightarrow \quad \partial_t \to -i\omega$$

Consequently: $\Phi = -\frac{ic^2}{\omega}\,\mathrm{div}\,\vec{A}$

$$\vec{A}(t,\vec{x}) = \frac{\mu_0}{4\pi}\int_V d^3x'\,\frac{\vec{J}(\vec{x}')e^{-i\omega\left(t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}}{|\vec{x}-\vec{x}'|} = \frac{\mu_0}{4\pi}e^{-i\omega t}\int_V d^3x'\,\frac{\vec{J}(\vec{x}')e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \qquad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

<u>Small, localised source</u>

$|\vec{x}'| < d \ll \lambda \qquad d$: diameter of source volume V

Observed from far away:

$r = |\vec{x}| \gg d$ radiation zone

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r} + O\left(\frac{d}{r^2}\right) \qquad |\vec{x}-\vec{x}'| = r - \underbrace{\hat{x}\cdot\vec{x}'}_{O(d)} + O\left(\frac{d^2}{r}\right) \quad \text{with } \hat{x} = \frac{\vec{x}}{r}$$

Small parameter: $\frac{d}{r} \ll 1$

Radiation terms

$$\vec{A}_{rad}(t,\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{-i\omega t+ikr}}{r}\vec{a}(k,\hat{x}) \qquad \vec{a}(k,\hat{x}) = \int_V d^3x'\,\vec{J}(\vec{x}')\,e^{-ik\hat{x}\cdot\vec{x}'}$$

$$\partial_j\frac{1}{r} = -\frac{1}{r^2}\partial_j r = -\frac{\hat{x}_j}{r^2} \qquad \partial_j\hat{x}_k = \partial_j\frac{x_k}{r} = \frac{\delta_{jk}}{r} - \frac{\hat{x}_j\hat{x}_k}{r} \quad \text{suppressed by } \frac{1}{r}\ !$$

$\partial_j e^{ikr} = ik\,\partial_j r\,e^{ikr} = ik\hat{x}_j e^{ikr}$ no suppression!

$\vec{\nabla} \to ik\hat{x}$ as far as only radiation terms are concerned!

Field strengths and radiated power

$$\vec{H}_{rad} = \frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{A}\,\right)_{rad} = \frac{ik}{4\pi}\frac{e^{-i\omega t+ikr}}{r}\hat{x}\times\vec{a}(k,\hat{x})$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = -\frac{i\omega}{c^2}\vec{E} \;\Rightarrow\; \vec{E}_{rad} = \frac{ic^2}{\omega}ik\hat{x}\times\left(\mu_0\vec{H}_{rad}\right) = -c\mu_0\hat{x}\times\vec{H}_{rad}$$

$\vec{E}_{rad} = Z_0\vec{H}_{rad}\times\hat{x} \qquad Z_0 = \mu_0 c = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 376.7\ \Omega$ (ohms) **vacuum impedance**

Time averaged energy current density

$$\vec{S}_{rad} = \frac{1}{2}\mathrm{Re}\left(\vec{E}_{rad}\times\vec{H}^*_{rad}\right) = \hat{x}\frac{1}{2}Z_0\left|\vec{H}_{rad}\right|^2 = \hat{x}\frac{Z_0k^2}{32\pi^2}\frac{1}{r^2}|\vec{a}(k,\hat{x})|^2$$

Angular distribution of radiated power $dP = \hat{x}\cdot\vec{S}_{rad}r^2d\Omega$

$$\frac{dP}{d\Omega} = \frac{Z_0k^2}{32\pi^2}|\hat{x}\times\vec{a}(k,\hat{x})|^2 \qquad \vec{a}(k,\hat{x}) = \int_V d^3x'\,\vec{J}(\vec{x}')\,e^{-ik\hat{x}\cdot\vec{x}'} \qquad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

Multipole expansion for radiation

$$\vec{a}(k,\hat{x}) = \int_V d^3x' \, \vec{J}(\vec{x}') \, e^{-ik\hat{x}\cdot\vec{x}'} \qquad k\hat{x}\cdot\vec{x}' \sim \frac{d}{\lambda} \ll 1$$

⇒ expand in Taylor series

$$\vec{a}(k,\hat{x}) = \int_V d^3x' \, \vec{J}(\vec{x}') \, (1 - ik\hat{x}\cdot\vec{x}' + \cdots)$$

Leading contribution

$$\vec{a}_0(k,\hat{x}) = \int_V d^3x' \, \vec{J}(\vec{x}')$$

$$\begin{aligned}
a_0(k,\hat{x})_j &= \int_V d^3x' \, J_j(\vec{x}') = \int_V d^3x' \, \delta_{jk} J_k(\vec{x}') = \int_V d^3x' \left(\partial'_k \, x'_j\right) J_k(\vec{x}') \\
&= \int_V d^3x' \left(\partial'_k \, (x'_j J_k(\vec{x}')) - x'_j \partial'_k J_k(\vec{x}')\right) \\
&= \underbrace{\oiint_{\partial V} d^2 f_k \; x'_j \, J_k(\vec{x}')}_{\substack{\text{vanishes if } V \text{ is large} \\ \text{enough to contain all} \\ \text{the source inside}}} - \int_V d^3x' \, x'_j \, \underbrace{\partial'_k J_k(\vec{x}')}_{-\partial_t \rho} \\
&= -i\omega \int_V d^3x' \, x'_j \rho(\vec{x}')
\end{aligned}$$

End result for leading contribution

$$\vec{a}_0(k,\hat{x}) = -i\omega\vec{p} \qquad \text{electric dipole radiation}$$

Sub-leading contribution

$$\vec{a}_1(k,\hat{x}) = -ik \int_V d^3x' \, \hat{x}\cdot\vec{x}' \vec{J}(\vec{x}')$$

$$a_1(k,\hat{x})_l = -ik \, \hat{x}_j M_{jl} \qquad M_{jl} = \int_V d^3x' \, x'_j \, J_l(\vec{x}')$$

Antisymmetric part

$$m_i = \frac{1}{2} \, \epsilon_{ijk} M_{jk}$$

$$\frac{1}{2}\left(M_{jl} - M_{lj}\right) = \epsilon_{jln} m_n \quad \text{with} \quad \vec{m} = \frac{1}{2}\int_V d^3x' \, \vec{x}' \times \vec{J}(\vec{x}')$$

magnetic dipole moment

Symmetric part

$$M_{jl} = \int_V d^3x'\, x'_j\, J_l(\vec{x}') = \int_V d^3x'\, x'_j (\partial_k{}' x'_l)\, J_k(\vec{x}')$$

$$= \underbrace{\int_V d^3x'\, \partial_k{}' \left(x'_j x'_l\, J_k(\vec{x}') \right)}_{\text{surface term} - \text{can be thrown away}} - \int_V d^3x'\, x'_l\, \underbrace{(\partial'_k x'_j)}_{\delta_{kj}} J_k(\vec{x}') - \int_V d^3x'\, x'_l x'_j \partial_k{}'\, J_k(\vec{x}')$$

$$= -M_{lj} - i\omega \int_V d^3x'\, x'_l x'_j \rho(\vec{x}')$$

$$\frac{1}{2}\left(M_{jl} + M_{lj}\right) = -\frac{i\omega}{6} Q_{jl} + \frac{i\omega}{2}\delta_{jl} \int_V d^3x'\, r'^2 \rho(\vec{x}') \qquad Q_{jl} = \int_V d^3x' \left(x'_l x'_j - 3\delta_{jl} r'^2\right)\rho(\vec{x}')$$

electric quadrupole moment

Summarising the result

$$a(k,\hat{x})_l = -i\omega p_l - ik\, \hat{x}_j M_{jl} = -i\omega p_l - ik\hat{x}_j \frac{1}{2}\left(M_{jl} - M_{lj}\right) - ik\hat{x}_j \frac{1}{2}\left(M_{jl} + M_{lj}\right)$$

$$= -i\omega p_l - ik\hat{x}_j \epsilon_{jln} m_n - \frac{k\omega}{6}\hat{x}_j Q_{jl} + \underbrace{\frac{k\omega}{2}\hat{x}_l \int_V d^3x'\, r'^2 \rho(\vec{x}')}_{\hat{x}-\text{term}}$$

This must be inserted into

$$\vec{A}_{rad}(t,\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{-i\omega t + ikr}}{r} \vec{a}(k,\hat{x})$$

Note that

$$\partial_l \frac{e^{ikr}}{r} = ik\hat{x}_l \frac{e^{ikr}}{r} + O\left(\frac{1}{r^2}\right)$$

the radiation part of the $\hat{x}$-term is a pure gauge! Therefore, we can safely omit it and get the following
results for the radiation:

$$\vec{a}(k,\hat{x}) = -i\omega\left(\vec{p} + \frac{1}{c}\vec{m} \times \hat{x}\right) - \frac{\omega^2}{6c} \underline{Q}\hat{x}$$

$$\vec{H}_{rad} = \frac{ik}{4\pi} \frac{e^{-i\omega t + ikr}}{r} \hat{x} \times \vec{a}(k,\hat{x}) \qquad \vec{E}_{rad} = Z_0 \vec{H}_{rad} \times \hat{x}$$

$$\frac{dP}{d\Omega} = \frac{Z_0 \omega^2}{32\pi^2 c^2} |\hat{x} \times \vec{a}(k,\hat{x})|^2$$

Homework: quadrupole radiation - angular dependence
Assuming cylindrical symmetry around axis 3

$$Q_{33} = Q_0\,, \quad Q_{11} = Q_{22} = -\frac{Q_0}{2}\,, \quad \text{while } Q_{jk} = 0 \text{ for } j \neq k$$

Show that

$$\frac{dP}{d\Omega} = \frac{Z_0 \omega^6}{512\pi c^4} Q_0^2 \sin^2\theta \cos^2\theta$$

Total radiated power

$$\frac{dP}{d\Omega} = \frac{Z_0 \omega^4}{32\pi^2 c^2} \left| \hat{x} \times \vec{p} + \frac{1}{c} \hat{x} \times (\vec{m} \times \hat{x}) - \frac{i\omega}{6c} \hat{x} \times \underline{Q} \hat{x} \right|^2$$

We want to obtain $P = \int d\Omega \frac{dP}{d\Omega} = \text{squared terms} + \text{mixed terms}$

Mixed terms vanish

(different multipoles are orthogonal vector-valued spherical harmonics)

$$\int d\Omega (\hat{x} \times \vec{p}) \cdot \big(\hat{x} \times (\vec{m} \times \hat{x})\big) = 0 = \int d\Omega (\hat{x} \times \vec{p}) \cdot \left(\hat{x} \times \underline{Q} \hat{x}\right)$$

since integrands are odd under spatial reflection $\hat{x} \to -\hat{x}$

More complicated case

$$\int d\Omega \big(\hat{x} \times (\vec{m} \times \hat{x})\big) \cdot \left(\hat{x} \times \underline{Q} \hat{x}\right) = \int d\Omega (\vec{m} - (\hat{x} \cdot \vec{m}) \hat{x}) \cdot \left(\hat{x} \times \underline{Q} \hat{x}\right) = \int d\Omega \vec{m} \cdot \left(\hat{x} \times \underline{Q} \hat{x}\right)$$

Choose Cartesian basis so that $\vec{m} = m\vec{e}_3$:

$$\int d\Omega \vec{m} \cdot \left(\hat{x} \times \underline{Q} \hat{x}\right) = m \int d\Omega \left(\hat{x} \times \underline{Q} \hat{x}\right)_3$$

$$= m \int d\Omega\, \hat{x}_1 (Q_{21} \hat{x}_1 + Q_{22} \hat{x}_2 + Q_{23} \hat{x}_3) - m \int d\Omega\, \hat{x}_2 (Q_{11} \hat{x}_1 + Q_{12} \hat{x}_2 + Q_{13} \hat{x}_3)$$

Lemma

$$\int d\Omega \hat{x}_i \hat{x}_j = \frac{4\pi}{3} \delta_{ij}$$

Proof:

$i \neq j$ e.g. $\int d\Omega \hat{x}_1 \hat{x}_2 = 0$ since the integrand is odd under reflection $\hat{x}_1 \to -\hat{x}_1, \hat{x}_{2,3} \to \hat{x}_{2,3}$

By rotation symmetry

$$\int d\Omega \hat{x}_1 \hat{x}_1 = \int d\Omega \hat{x}_2 \hat{x}_2 = \int d\Omega \hat{x}_3 \hat{x}_3 = C$$

and

$$3C = \int d\Omega (\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2) = \int d\Omega = 4\pi \,. \qquad \text{Q.e.d.}$$

So, we get

$$\int d\Omega \big(\hat{x} \times (\vec{m} \times \hat{x})\big) \cdot \left(\hat{x} \times \underline{Q} \hat{x}\right) = \frac{4\pi}{3} m (Q_{21} - Q_{12}) = 0$$

Computing the square terms

$$P = \frac{Z_0\omega^4}{32\pi^2 c^2}\int d\Omega \left\{|\hat{x}\times\vec{p}|^2 + \frac{1}{c^2}|\hat{x}\times(\vec{m}\times\hat{x})|^2 + \frac{\omega^2}{36c^2}\left|\hat{x}\times \underline{Q}\hat{x}\right|^2\right\}$$

Dipole contributions

$$\int d\Omega\, |\hat{x}\times\vec{p}|^2 = 2\pi\int_{-1}^{1} d(\cos\,\theta)p^2\sin^2\theta = 2\pi p^2\int_{-1}^{1} dx(1-x^2) = \frac{8\pi p^2}{3}$$
$$\int d\Omega\, |\hat{x}\times(\vec{m}\times\hat{x})|^2 = \int d\Omega\, |(\vec{m}\times\hat{x})|^2 = \frac{8\pi m^2}{3}$$

For the quadrupole

$$\left|\hat{x}\times\underline{Q}\hat{x}\right|^2 = \left(\hat{x}\times\underline{Q}\hat{x}\right)\cdot\left(\hat{x}\times\underline{Q}\hat{x}\right) = -\underline{Q}\hat{x}\cdot\left(\hat{x}\times\left(\hat{x}\times\underline{Q}\hat{x}\right)\right)$$
$$= -\underline{Q}\hat{x}\cdot\left(\hat{x}\left(\hat{x}\cdot\underline{Q}\hat{x}\right)-\underline{Q}\hat{x}\right) = \left|\underline{Q}\hat{x}\right|^2 - \left(\hat{x}\cdot\underline{Q}\hat{x}\right)^2$$

(1) $\int d\Omega \left|\underline{Q}\hat{x}\right|^2 = \int d\Omega\, Q_{ij}\hat{x}_j Q_{il}\hat{x}_l = \frac{4\pi}{3}Q_{ij}Q_{ij} = \frac{4\pi}{3}Q_{ij}Q_{ji} = \frac{4\pi}{3}\mathrm{Tr}\,\underline{Q}^2$

(2) $\int d\Omega\left(\hat{x}\cdot\underline{Q}\hat{x}\right)^2 = \int d\Omega\, Q_{ij}Q_{kl}\hat{x}_i\hat{x}_j\hat{x}_k\hat{x}_l$

Lemma

$$\int d\Omega\, \hat{x}_i\hat{x}_j\hat{x}_k\hat{x}_l = \frac{4\pi}{15}\left(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)$$

Proof:

Again, parity under spatial reflections imply that the integral vanishes unless the indices are pairwise equal. The integral is also independent of the selection of the pairs. Therefore, we already know that

$$\int d\Omega\, \hat{x}_i\hat{x}_j\hat{x}_k\hat{x}_l = C\left(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)$$

Now we can compute

$$3C = \int d\Omega\, \hat{x}_3^4 = 2\pi\int_{-1}^{1} d(\cos\,\theta)\cos^4\theta = 2\pi\int_{-1}^{1} dx\, x^4 = 2\pi\frac{2}{5} \qquad \text{Q. e. d.}$$

So

$$\int d\Omega\left(\hat{x}\cdot\underline{Q}\hat{x}\right)^2 = \int d\Omega\, Q_{ij}Q_{kl}\hat{x}_i\hat{x}_j\hat{x}_k\hat{x}_l = \frac{4\pi}{15}\left(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)Q_{ij}Q_{kl}$$

$$= \frac{4\pi}{15}\left(\left(\mathrm{Tr}\,\underline{Q}\right)^2 + 2\mathrm{Tr}\,\underline{Q}^2\right) = \frac{8\pi}{15}\mathrm{Tr}\,\underline{Q}^2 \text{ since } \mathrm{Tr}\,\underline{Q} = 0$$

Therefore

$$\int d\Omega\left|\hat{x} \times \underline{Q}\hat{x}\right|^2 = \left(\frac{4\pi}{3} - \frac{8\pi}{15}\right)\mathrm{Tr}\,\underline{Q}^2 = \frac{4\pi}{5}\mathrm{Tr}\,\underline{Q}^2$$

Recalling the previous results

$$\int d\Omega\,|\hat{x} \times \vec{p}|^2 = \frac{8\pi p^2}{3} \qquad \int d\Omega\,|\hat{x} \times (\vec{m} \times \hat{x})|^2 = \frac{8\pi m^2}{3}$$

the end result for the total radiated power is

$$P = \frac{Z_0\omega^4}{32\pi^2c^2}\int d\Omega\left\{|\hat{x} \times \vec{p}|^2 + \frac{1}{c^2}|\hat{x} \times (\vec{m} \times \hat{x})|^2 + \frac{\omega^2}{36c^2}\left|\hat{x} \times \underline{Q}\hat{x}\right|^2\right\}$$

$$= \frac{Z_0\omega^4}{12\pi^2c^2}\left(p^2 + \frac{m^2}{c^2}\right) + \frac{Z_0\omega^6}{1440\pi c^4}\mathrm{Tr}\,\underline{Q}^2$$

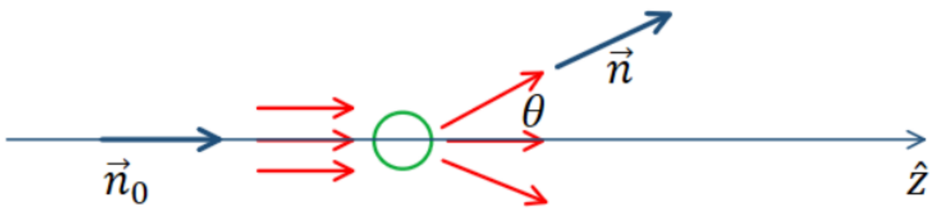
Week 8: Scattering of EM waves

Review of material from Electrodynamics 1

3 scatterer sizes:

1. $d \ll \lambda$: dipole dominated
2. $d \sim \lambda$: higher multipoles needed (Mie scattering)
3. $d \gg \lambda$: optics (diffraction, geometric optics)

Here we only consider 1.



$$\vec{E}_{in} = \vec{e}_0 E_0 e^{ik\vec{n}_0\cdot\vec{x}} \qquad \vec{H}_{in} = \frac{1}{Z_0}\vec{n}_0 \times \vec{E}_{in}$$

$$\vec{E}_{sc} = \frac{1}{4\pi\epsilon_0}k^2\frac{e^{ikr}}{r}\left[(\vec{n}\times\vec{p})\times\vec{n} - \frac{1}{c}\vec{n}\times\vec{m}\right] \qquad \vec{H}_{sc} = \frac{1}{Z_0}\vec{n}\times\vec{E}_{sc}$$

(HW: check these formulae using last weeks material)

$$\frac{d\sigma}{d\Omega} = \frac{\text{power radiated in unit solid angle in given direction } (W)}{\text{incoming energy flux } \left(\frac{W}{m^2}\right)}$$

$$= \frac{r^2|\vec{S}_{sc}|}{|\vec{S}_{in}|} \qquad \vec{S} = \frac{1}{2}\text{Re}\left(\vec{E}\times\vec{H}^*\right) = \frac{1}{2Z_0}\left|\vec{E}\right|^2\vec{n}$$

If we follow the polarisation, we must project the field as $\vec{E} \to \vec{e}^* \cdot \vec{E}$:

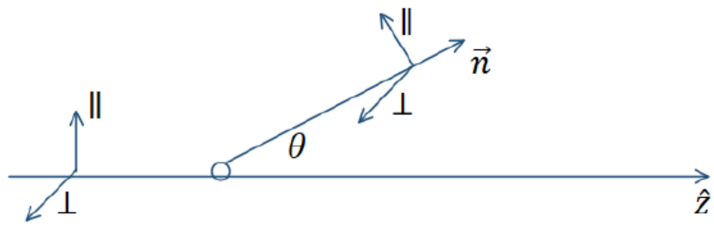
$$\frac{d\sigma}{d\Omega} = \frac{r^2\left|\vec{e}^*\cdot\vec{E}_{sc}\right|^2}{E_0^2}$$

$$= \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\vec{e}^*\cdot\vec{p} + \frac{1}{c}(\vec{n}\times\vec{e}^*)\cdot\vec{m}\right|^2 \quad \text{using } \vec{e}\perp\vec{n} \text{ and } |\vec{e}| = 1 = |\vec{e}_0|$$

Example: scattering on a small dielectric sphere of radius a

$d \ll \lambda$: polarisation can be computed using homogeneous field

$$\vec{p} = 4\pi\epsilon_0\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2 a^3\vec{E}_{in} \;\Rightarrow\; \frac{d\sigma}{d\Omega} = k^4a^6\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2|\vec{e}^*\cdot\vec{e}_0|^2$$



Incoming polarisations: $\vec{e}_{0\parallel} = (1,0,0)$ $\vec{e}_{0\perp} = (0,1,0)$

Outgoing direction: $\vec{n} = (\sin\theta, 0, \cos\theta)$

Outgoing polarisations: $\vec{e}_{\parallel} = (cos\,\theta, 0, -sin\ \theta)$ $\vec{e}_{\perp} = (0,1,0)$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\parallel\to\parallel} = k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2 \cos^2\theta \qquad \left(\frac{d\sigma}{d\Omega}\right)_{\perp\to\perp} = k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2$$
$$\left(\frac{d\sigma}{d\Omega}\right)_{\parallel\to\perp} = \left(\frac{d\sigma}{d\Omega}\right)_{\perp\to\parallel} = 0$$

Assume incoming light is unpolarized: <u>average over incoming polarisation</u>
outgoing polarisation is not observed: <u>sum over outgoing polarisation</u>

$$\frac{d\bar{\sigma}}{d\Omega} = \underbrace{\left[\frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_{\parallel\to\parallel} + \frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_{\perp\to\parallel}\right]}_{\text{outgoing }\parallel,\ \text{averaged over incoming}} + \underbrace{\left[\frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_{\parallel\to\perp} + \frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_{\perp\to\perp}\right]}_{\text{outgoing }\perp,\ \text{averaged over incoming}}$$

$$= \frac{1}{2} k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2 (1 + \cos^2\theta)$$

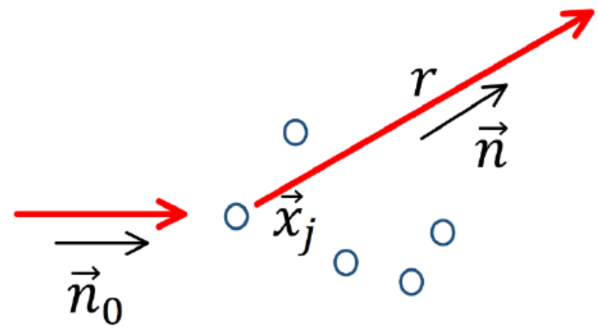
Even if incoming light is unpolarised, the outgoing light is polarised.

Polarisation depends on direction:

$$\Pi(\theta) = \frac{\left[\frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_{\parallel\to\parallel} + \frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_{\perp\to\parallel}\right] - \left[\frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_{\parallel\to\perp} + \frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_{\perp\to\perp}\right]}{\left[\frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_{\parallel\to\parallel} + \frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_{\perp\to\parallel}\right] + \left[\frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_{\parallel\to\perp} + \frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_{\perp\to\perp}\right]} = \frac{\sin^2\theta}{1 + \cos^2\theta}$$

Form factor

Consider many scattering centers. On the j^{th} center



$\vec{E}_{in}^{(j)} \propto e^{ik\vec{n}_0\cdot\vec{x}_j} \qquad |\vec{x}-\vec{x}_j| \sim r - \vec{n}\cdot\vec{x}_j$

$\vec{E}_{sc}^{(j)} \propto \vec{E}_{1,sc} e^{ik\vec{n}_0\cdot\vec{x}_j - ik\vec{n}\cdot\vec{x}_j}$

$\vec{E}_{1,sc}$: field of a single scatterer at origin

Denote $\vec{q} = k\vec{n}_0 - k\vec{n} = \Delta\vec{k}$

$$\vec{E}_{sc} = \sum_j \vec{E}_{sc}^{(j)} = \vec{E}_{1,sc}\sum_j e^{i\vec{q}\cdot\vec{x}_j}$$

$$\frac{d\sigma}{d\Omega} = \frac{r^2\left|\vec{e}^*\cdot\vec{E}_{sc}\right|^2}{E_0^2} = \left(\frac{d\sigma}{d\Omega}\right)_1 F(q) \qquad \left(\frac{d\sigma}{d\Omega}\right)_1 \text{: cross section for single scatterer}$$

Form factor:

$$F(\vec{q}) = \left|\sum_j e^{i\vec{q}\cdot\vec{x}_j}\right|^2$$

Random medium

$$F(\vec{q}) = \sum_{j,k} e^{i\vec{q}\cdot(\vec{x}_j-\vec{x}_k)} = N \quad \text{number of scatterers}$$

Crystal lattice (cubic)

$$\vec{x}_j \to a\sum_{i=1}^{3} n_i\vec{e}_i \qquad n_i = -N_i, -N_i+1, \dots N_i \qquad \text{volume } V = (2N_1+1)(2N_2+1)(2N_3+1)a^3$$

$$f(\vec{q}) = \sum_j e^{i\vec{q}\cdot\vec{x}_j} = \sum_{n_1,n_2,n_3}\prod_{l=1}^{3} e^{iq_l n_l a} = \prod_{l=1}^{3}\sum_{n_l} e^{iq_l n_l a} = \prod_{l=1}^{3}\frac{\sin\frac{2N_l+1}{2}q_l a}{\sin\frac{q_l a}{2}}$$

$$F(\vec{q}) = \prod_{l=1}^{3}\frac{\sin^2\frac{2N_l+1}{2}q_l a}{\sin^2\frac{q_l a}{2}} \quad \text{typically } O(1) \text{ so } \frac{1}{V}F(\vec{q}) \to 0 \text{ for } V\to\infty$$

Exception: Bragg condition

$$q_l = \frac{2\pi}{a} m_l \quad m_l = \text{integer} \;\Rightarrow\; \frac{\sin \frac{2N_l+1}{2}\; q_l a}{\sin \frac{q_l a}{2}} \to 2N_l + 1$$

$F(\vec{q}) \propto N^2$: amplification (Bragg peak) when $\vec{q} \in$ reciprocal lattice

Typically: $a \sim 10^{-10}\, m \;\Rightarrow\;$ X − ray crystallography

Long wavelength: $\lambda \gg a$

but: $\vec{q} = \frac{2\pi}{\lambda}(\vec{n} - \vec{n}_0) \;\Rightarrow\; |q_l| \leq \frac{4\pi}{\lambda} \;\Rightarrow\;$ only Bragg peak at $m_l = 0$

i.e., only $\vec{q} = 0$: forward scattering

Perturbative theory of scattering

Wave equation in inhomogeneous medium

$$\epsilon(\vec{x}) = \bar{\epsilon} + \delta\epsilon(\vec{x}) \qquad \mu(\vec{x}) = \bar{\mu} + \delta\mu(\vec{x})$$

$\bar{\epsilon}, \bar{\mu}$ average values $\qquad \delta\epsilon(\vec{x}), \delta\mu(\vec{x})$ small inhomogeneities
$\langle \delta\epsilon \rangle = \langle \delta\mu \rangle = 0$

$$\vec{D} = \bar{\epsilon}\vec{E} + \delta\epsilon\vec{E} \quad \vec{B} = \bar{\mu}\vec{H} + \delta\mu\vec{H}$$

Maxwell's equations (no free sources, just the medium)

$$\text{curl}\,\vec{H} = \partial_t \vec{D} \quad \text{curl}\,\vec{E} = -\partial_t \vec{B} \quad \text{div}\,\vec{D} = 0 = \text{div}\,\vec{B}$$

$$\text{curl curl}\left(\vec{D} - \bar{\epsilon}\vec{E}\right) = \text{grad div}\,\vec{D} - \Delta\vec{D} + \bar{\epsilon}\,\partial_t\,\text{curl}\,\vec{B}$$

$$\text{curl}\,\vec{B} = \text{curl}\left(\vec{B} - \bar{\mu}\vec{H}\right) + \bar{\mu}\,\text{curl}\,\vec{H} = \text{curl}\left(\vec{B} - \bar{\mu}\vec{H}\right) + \bar{\mu}\partial_t\,\vec{D}$$

$$\text{curl curl}\left(\vec{D} - \bar{\epsilon}\vec{E}\right) = -\Delta\vec{D} + \bar{\epsilon}\,\partial_t \text{curl}\left(\vec{B} - \bar{\mu}\vec{H}\right) + \bar{\epsilon}\bar{\mu}\partial_t^2\,\vec{D}$$

We obtain a wave equation with velocity $\bar{c}$

$$\left(\Delta - \frac{1}{\bar{c}^2}\partial_t\right)\vec{D} = \underbrace{\bar{\epsilon}\,\partial_t \text{curl}\left(\vec{B} - \bar{\mu}\vec{H}\right) - \text{curl curl}\left(\vec{D} - \bar{\epsilon}\vec{E}\right)}_{\text{sources: inhomegeneities}} \qquad \bar{c}^2 = \frac{1}{\bar{\epsilon}\bar{\mu}}$$

HW: derive the wave equation for the magnetic field

$$\left(\Delta - \frac{1}{\bar{c}^2}\partial_t\right)\vec{B} = -\bar{\mu}\,\partial_t \text{curl}\left(\vec{D} - \bar{\epsilon}\vec{E}\right) - \text{curl curl}\left(\vec{B} - \bar{\mu}\vec{H}\right)$$

Consider a single monochromatic component

$t -$ dependence $\propto e^{-i\omega t} \Rightarrow \partial_t = -i\omega$ and $\Delta - \frac{1}{\bar{c}^2}\partial_t = \Delta + k^2$ with $k = \frac{\omega}{\bar{c}}$

Solving the wave equation with retarded Green's function

$$(\Delta + k^2)\Psi(x) = -f(x)$$

$$\Psi(x) = \Psi_{in}(x) + \frac{1}{4\pi}\int d^3x' \frac{e^{+ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} f(\vec{x}') \text{ retarded solution}$$

$\Psi_{in}(x)$: homogeneous solution (incoming field)

We obtain an integral equation:

$$\vec{D}(x) = \vec{D}_{in}(x) + \frac{1}{4\pi}\int d^3x' \frac{e^{+ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\left\{i\omega\bar{\epsilon}\,\text{curl}\left(\vec{B} - \bar{\mu}\vec{H}\right) + \text{curl curl}\left(\vec{D} - \bar{\epsilon}\vec{E}\right)\right\}$$

(similarly for $\vec{B}$)

Can be solved by iteration.

Born approximation

$$\vec{D}(\vec{x}) = \vec{D}_{in}(\vec{x}) + \frac{1}{4\pi}\int d^3x' \frac{e^{+ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\left\{i\omega\bar{\epsilon}\,\text{curl}\,\delta\mu(\vec{x}')\vec{H}_{in}(\vec{x}') + \text{curl curl}\,\delta\epsilon(\vec{x}')\vec{E}_{in}(\vec{x}')\right\}$$

Scattered wave: $r = |\vec{x}| \gg |\vec{x}'|$

$$\vec{D}_{sc}(\vec{x}) = \frac{e^{ikr}}{4\pi r}\int d^3x'\, e^{-ik\hat{x}\cdot\vec{x}'}\left\{i\omega\bar{\epsilon}\,\text{curl}\,\delta\mu(\vec{x}')\vec{H}_{in}(\vec{x}') + \text{curl curl}\,\delta\epsilon(\vec{x}')\vec{E}_{in}(\vec{x}')\right\}$$

Validity of Born approximation: $D_{sc} \ll D_{in}$

Under the Fourier transform: $\vec{\nabla} = ik\hat{x}$ (exact relation, obtained by partial integration)

$$\vec{E}_{sc}(\vec{x}) = \frac{e^{ikr}}{4\pi r}\int d^3x'\, e^{-ik\hat{x}\cdot\vec{x}'}\left\{-\omega k\delta\mu(\vec{x}')\hat{x}\times\vec{H}_{in}(\vec{x}') - k^2\frac{\delta\epsilon(\vec{x}')}{\bar{\epsilon}}\hat{x}\times\left(\hat{x}\times\vec{E}_{in}(\vec{x}')\right)\right\}$$

Incoming wave:

$$\vec{E}_{in}(\vec{x}') = E_0\vec{e}_0 e^{i\vec{k}\cdot\vec{x}'} \quad \vec{H}_{in}(\vec{x}') = \frac{1}{\bar{Z}}\hat{k}\times\vec{E}_{in}(\vec{x}') \qquad \bar{Z} = \sqrt{\frac{\bar{\mu}}{\bar{\epsilon}}} = \bar{c}\bar{\mu}$$

Scattered wave can be written in terms of amplitude function

$$\vec{E}_{sc}(\vec{x}) = \frac{e^{ikr}}{r}\vec{a}(k,\hat{x})$$

$$\vec{a}(k,\hat{x}) = \frac{1}{4\pi}\int d^3x'\, e^{-ik\hat{x}\cdot\vec{x}'}\left\{-\frac{\bar{c}}{\bar{Z}}k^2\delta\mu(\vec{x}')\hat{x}\times(\hat{k}\times E_0\vec{e}_0 e^{i\vec{k}\cdot\vec{x}'}) - k^2\frac{\delta\epsilon(\vec{x}')}{\bar{\epsilon}}\hat{x} \times\left(\hat{x}\times E_0\vec{e}_0 e^{i\vec{k}\cdot\vec{x}'}\right)\right\}$$

$$= \frac{k^2E_0}{4\pi}\int d^3x'\, e^{i\vec{q}\cdot\vec{x}'}\left\{-\frac{\delta\mu(\vec{x}')}{\bar{\mu}}\hat{x}\times\left(\hat{k}\times\vec{e}_0\right) - \frac{\delta\epsilon(\vec{x}')}{\bar{\epsilon}}\hat{x}\times(\hat{x}\times\vec{e}_0)\right\}$$

$$\vec{q} = \vec{k} - k\hat{x}$$

$$= \frac{k^2E_0}{4\pi}\left\{\frac{\widetilde{\delta\mu}(\vec{q})}{\bar{\mu}}\left(\hat{k}\times\vec{e}_0\right)\times\hat{x} + \frac{\widetilde{\delta\epsilon}(\vec{q})}{\bar{\epsilon}}(\hat{x}\times\vec{e}_0)\times\hat{x}\right\}$$

Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{r^2\left|\vec{e}^*\cdot\vec{E}_{sc}\right|^2}{E_0^2} = \frac{|\vec{e}^*\cdot\vec{a}(k,\hat{x})|^2}{E_0^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2}\left|\frac{\widetilde{\delta\mu}(\vec{q})}{\bar{\mu}}\left(\hat{k}\times\vec{e}_0\right)\cdot(\hat{x}\times\vec{e}^*) + \frac{\widetilde{\delta\epsilon}(\vec{q})}{\bar{\epsilon}}\vec{e}^*\cdot\vec{e}_0\right|^2$$

$$\widetilde{\delta\epsilon}(\vec{q}) = \int d^3x'\, e^{i\vec{q}\cdot\vec{x}'}\delta\epsilon(\vec{x}') \quad \widetilde{\delta\mu}(\vec{q}) = \int d^3x'\, e^{i\vec{q}\cdot\vec{x}'}\delta\mu(\vec{x}')$$

Incoming direction: $\hat{k} = (0,0,1)$

Incoming polarisations: $\vec{e}_{0\parallel} = (1,0,0)$ $\quad\vec{e}_{0\perp} = (0,1,0)$

Outgoing direction: $\vec{n} = (\sin\theta, 0, \cos\theta)$

Outgoing polarisations: $\vec{e}_{\parallel} = (\cos\theta, 0, -\sin\theta)$ $\quad\vec{e}_{\perp} = (0,1,0)$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\parallel\to\parallel} = \frac{k^4}{16\pi^2}\left|\frac{\widetilde{\delta\mu}(\vec{q})}{\bar{\mu}} + \frac{\widetilde{\delta\epsilon}(\vec{q})}{\bar{\epsilon}}\cos\theta\right|^2$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\perp\to\perp} = \frac{k^4}{16\pi^2}\left|\frac{\widetilde{\delta\mu}(\vec{q})}{\bar{\mu}}\cos\theta + \frac{\widetilde{\delta\epsilon}(\vec{q})}{\bar{\epsilon}}\right|^2$$

Scattering on a random medium

Scattering on a dilute gas

Molecular polarisation

$$\vec{p} = \epsilon_0\gamma_{mol}\vec{E}$$

$$\bar{\epsilon} \approx \epsilon_0 \qquad \delta\epsilon(\vec{x}) \approx \epsilon_0 \sum_j \gamma_{mol}\, \delta\big(\vec{x} - \vec{x}_j\big) \quad \text{(dilute approx. of Clausius} - \text{Mossotti)}$$

$$\bar{\mu} \approx \mu_0 \qquad \delta\mu \approx 0$$

$$\widetilde{\delta\epsilon}(\vec{q}) = \epsilon_0 \gamma_{mol} \sum_j e^{i\vec{q}\cdot\vec{x}_j}$$

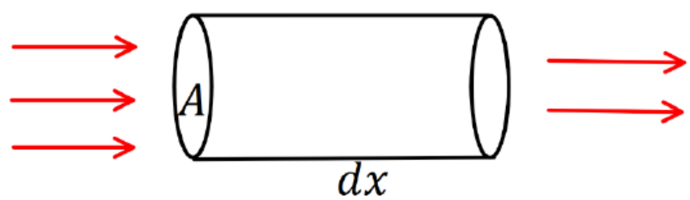
$$\left|\frac{\widetilde{\delta\epsilon}(\vec{q})}{\bar{\epsilon}}\right|^2 = \gamma_{mol}^2 \underbrace{\left(\sum_j e^{i\vec{q}\cdot\vec{x}_j}\right)^2}_{F(\vec{q})} = \gamma_{mol}^2 N \quad \text{(gas = random medium)}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\parallel\to\parallel} = \frac{k^4 \gamma_{mol}^2}{16\pi^2} \rho V \cos^2\theta \qquad \left(\frac{d\sigma}{d\Omega}\right)_{\perp\to\perp} = \frac{k^4 \gamma_{mol}^2}{16\pi^2} \rho V$$

Total cross section for unpolarised incoming light

$$\sigma = \int d\Omega \left\{ \frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_{\parallel\to\parallel} + \frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_{\perp\to\perp} \right\} = \frac{k^4 \gamma_{mol}^2}{6\pi} \rho V$$

Attenuation coefficient:



$$\alpha = \frac{\sigma}{V} = \frac{k^4 \gamma_{mol}^2}{6\pi} \rho$$

$$\begin{aligned} dP = -\sigma S &= -\alpha S A dx \\ &= -\alpha P dx \end{aligned}$$

$$\Rightarrow P(x) = P_0 e^{-\alpha x} \qquad \text{Opacity: } \kappa = \frac{\alpha}{\rho} = \frac{k^4 \gamma_{mol}^2}{6\pi} \qquad \text{Rayleigh scattering}$$

$$\rho(\vec{x}) = \sum_j \delta\big(\vec{x} - \vec{x}_j\big) \qquad \epsilon_r \approx 1 + \rho\gamma_{mol} \qquad n = \sqrt{\epsilon_r} = 1 + \frac{\rho\gamma_{mol}}{2}$$

$$\Rightarrow \gamma_{mol} = 2\,\frac{n-1}{\rho}$$

$$\alpha = \frac{2k^4}{3\pi\rho}(n-1)^2 \quad \text{Rayleigh formula (explanation of blue sky)}$$

Critical opalescence

Yet another way to express σ:

$$\sigma = \frac{k^4}{6\pi} \left|\frac{\widetilde{\delta\epsilon}(\vec{q})}{\bar{\epsilon}}\right|^2$$

$$\epsilon_r \approx 1 + \rho\gamma_{mol} \Rightarrow \ \delta\epsilon_r = \frac{\epsilon_r - 1}{\rho}\delta\rho = 2\frac{n-1}{\rho}\delta\rho$$

$$\frac{\widetilde{\delta\epsilon}(\vec{q})}{\bar{\epsilon}} = 2\frac{n-1}{\rho}\int d^3x\, e^{i\vec{q}\cdot\vec{x}}\delta\rho(\vec{x})$$

$$\sigma = \frac{2k^4(n-1)^2}{3\pi\rho^2}\int d^3x \int d^3y\ e^{-i\vec{q}\cdot(\vec{x}-\vec{y})}\langle\delta\rho(\vec{x})\delta\rho(\vec{y})\rangle$$

$\langle\dots\rangle$: statistical ensemble average

Typically, there is a finite correlation length ξ:
$\langle\delta\rho(\vec{x})\delta\rho(\vec{y})\rangle \sim e^{-|\vec{x}-\vec{y}|/\xi}$

Translation invariance: $\langle\delta\rho(\vec{x})\delta\rho(\vec{y})\rangle$ only depends on $\vec{x}-\vec{y}$

$$\sigma = \frac{2k^4(n-1)^2}{3\pi\rho^2}V\underbrace{\int d^3x\ e^{-i\vec{q}\cdot\vec{x}}\langle\delta\rho(\vec{x})\delta\rho(0)\rangle}_{\langle\delta\rho^2\rangle V_0} \quad \text{with } V_0 \sim \xi^3$$

Scattering on density fluctuations

$$\alpha = \frac{\sigma}{V} = \frac{2k^4(n-1)^2}{3\pi\rho^2}\langle\delta\rho^2\rangle V_0$$

Critical points: fluctuations grow - material becomes opaque.

Supplementary Material: Determining the density fluctuations

Let's take a piece of volume V_0 , with $N = V_0\rho$ the number of particles in V_0. The partition function is

$$Z = \sum_i e^{-\beta(E_i-\mu N_i)} \quad \Rightarrow \quad \frac{\partial Z}{\partial\mu} = \beta\sum_i N_i e^{-\beta(E_i-\mu N_i)} \qquad \frac{\partial^2 Z}{\partial\mu^2} = \beta^2\sum_i N_i^2 e^{-\beta(E_i-\mu N_i)}$$

from which we can compute the average particle number and its fluctuation:

$$\beta\langle N\rangle = \frac{1}{Z}\frac{\partial Z}{\partial\mu} \qquad \beta\frac{\partial}{\partial\mu}\langle N\rangle = \frac{1}{Z}\frac{\partial^2 Z}{\partial^2\mu} - \left(\frac{1}{Z}\frac{\partial Z}{\partial\mu}\right)^2 = \beta^2(\langle N^2\rangle - \langle N\rangle^2) \ \Rightarrow\ \langle\delta N^2\rangle = k_B T\frac{\partial\langle N\rangle}{\partial\mu}$$

Now we use some thermodynamic relations:

$$\left(\frac{\partial N}{\partial\mu}\right)_{T,V} = \left(\frac{\partial N}{\partial p}\right)_{T,V}\left(\frac{\partial p}{\partial\mu}\right)_{T,V}$$

$$\text{Gibbs} - \text{Duhem:}\ N d\mu + S dT - V dp = 0 \Rightarrow \left(\frac{\partial p}{\partial\mu}\right)_{T,V} = \frac{N}{V}$$

These imply

$$\left(\frac{\partial N}{\partial \mu}\right)_{T,V} = \frac{N}{V}\left(\frac{\partial N}{\partial p}\right)_{T,V} = N\left(\frac{\partial (N/V)}{\partial p}\right)_{T,V}$$

Using that $N/V = f(p,T)$ we can write

$$\left(\frac{\partial N}{\partial \mu}\right)_{T,V} = N\left(\frac{\partial (N/V)}{\partial p}\right)_{T,N} = -\frac{N^2}{V^2}\left(\frac{\partial V}{\partial p}\right)_{T,N}$$

Isotherm compressibility is defined as

$$\beta_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{T,N} \quad \text{so} \quad \left(\frac{\partial N}{\partial \mu}\right)_{T,V} = \frac{N^2}{V}\beta_T \;\Rightarrow\; \langle \delta N^2 \rangle = k_B T \frac{N^2}{V_0}\beta_T \;\; (\text{recall } V = V_0)$$

Opacity:

$$\alpha = \frac{\sigma}{V} = \frac{2k^4(n-1)^2}{3\pi\rho^2}\langle \delta\rho^2 \rangle V_0$$

$$\frac{\langle \delta\rho^2 \rangle}{\rho^2} = \frac{\langle \delta N^2 \rangle}{N^2} = \frac{k_B T \beta_T}{V_0}$$

$$\alpha = \frac{2k^4(n-1)^2}{3\pi} k_B T \beta_T$$

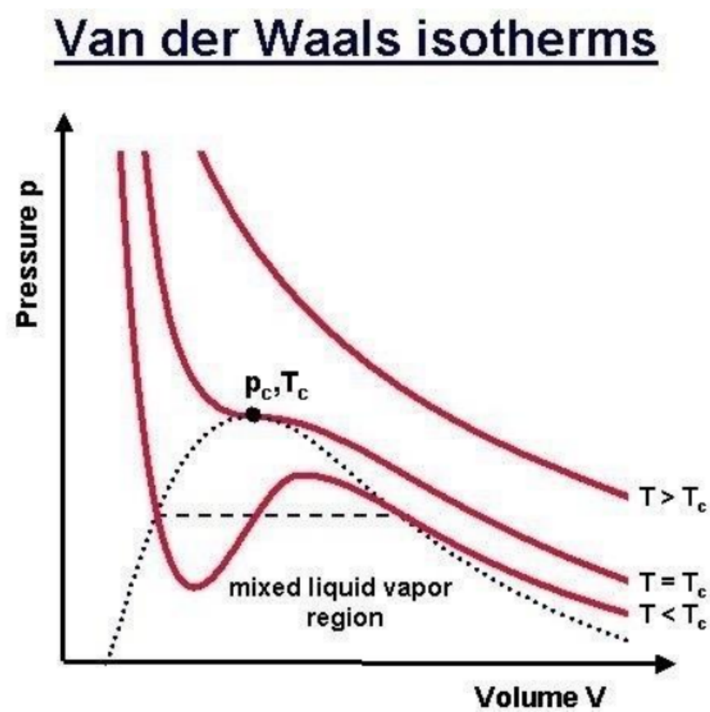
Critical point:

$$\frac{\partial p}{\partial V} = 0 \Rightarrow \; \beta_T = \infty$$

The correlation length is divergent: $\xi = \infty$

But experimentally the opacity is finite!

Note that our calculation is only valid when the scatterer is small, i.e., when $\xi \ll \lambda$: in the vicinity of the critical point, different approach is needed.



Week 9: EM field of a moving charge

Lienard-Wiechert potentials

Trajectory of charge: $\vec{\gamma}(t)$

$$\Rightarrow \rho(\vec{x},t) = q\delta^{(3)}\big(\vec{x}-\vec{\gamma}(t)\big)$$

$$\vec{J}(\vec{x},t) = q\vec{v}(t)\delta^{(3)}\big(\vec{x}-\vec{\gamma}(t)\big) \qquad \vec{v}(t) = \frac{d\vec{\gamma}(t)}{dt}$$

Retarded potential:

$$\Phi(\vec{x},t) = \frac{q}{4\pi\epsilon_0}\int d^3x' \left(\frac{\delta^{(3)}\big(\vec{x}'-\vec{\gamma}(t')\big)}{|\vec{x}-\vec{x}'|}\right)_{t'=t-\frac{|\vec{x}-\vec{x}'|}{c}}$$

Trick: insert

$$1 = \int dt' \;\delta\left(t'-t+\frac{|\vec{x}-\vec{x}'|}{c}\right)$$

so

$$\Phi(\vec{x},t) = \frac{q}{4\pi\epsilon_0}\int dt' \int d^3x' \;\delta\left(t'-t+\frac{|\vec{x}-\vec{x}'|}{c}\right)\frac{\delta^{(3)}\big(\vec{x}'-\vec{\gamma}(t')\big)}{|\vec{x}-\vec{x}'|}$$

$$= \frac{q}{4\pi\epsilon_0}\int dt' \;\delta\left(t'-t+\frac{|\vec{x}-\vec{\gamma}(t')|}{c}\right)\frac{1}{|\vec{x}-\vec{\gamma}(t')|}$$

Retarded time $\bar{t}$: solution of

$$\bar{t}-t+\frac{|\vec{x}-\vec{\gamma}(\bar{t})|}{c} = 0$$

Displacement to retarded position of the source:

$$\vec{R} = \vec{x}-\vec{\gamma}(\bar{t}) \qquad R = |\vec{x}-\vec{\gamma}(\bar{t})| = c(t-\bar{t})$$

$$\Phi(\vec{x},t) = \frac{q}{4\pi\epsilon_0 R}\int dt' \;\delta\left(t'-t+\frac{|\vec{x}-\vec{\gamma}(t')|}{c}\right)$$

Now we use

$$\int dx\, \delta\big(f(x)\big) = \frac{1}{|f'(x_0)|} \quad \text{where } f(x_0) = 0$$

$$\frac{d}{dt'}\left(t'-t+\frac{|\vec{x}-\vec{\gamma}(t')|}{c}\right)_{t'=\bar{t}} = \left(1+\frac{1}{c}\left(-\frac{d\vec{\gamma}(t')}{dt'}\right)\frac{\vec{x}-\vec{\gamma}(t')}{|\vec{x}-\vec{\gamma}(t')|}\right)_{t'=\bar{t}}$$

$$= 1-\vec{\beta}(\bar{t})\cdot\frac{\vec{R}}{R} > 0 \qquad \vec{\beta} = \frac{\vec{v}}{c}$$

Therefore

$$\Phi(\vec{x},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{R - \vec{\beta}\cdot\ \vec{R}}$$

HW: derive

$$\vec{A}(\vec{x},t) = \frac{\mu_0 q}{4\pi}\frac{\vec{v}(\bar{t})}{R - \vec{\beta}\cdot\ \vec{R}}$$

Field strengths

Complicated due to retardation

$$\vec{E} = -\vec{\nabla}\Phi - \partial_t\vec{A} \qquad \vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A}$$

Computing retarded derivatives

We need time and space derivatives of the retarded position and time first!

$$\bar{t} + \frac{1}{c}R = t \quad \vec{R} = \vec{x} - \vec{\gamma}(\bar{t}) \ \text{ so } \ \bar{t} = \bar{t}(t,\vec{x}) \quad \text{(a complicated function)}$$

$$\partial_t\!: \ \frac{\partial\bar{t}}{\partial t} + \frac{1}{c}\frac{\partial R}{\partial t} = 1$$

$$\frac{\partial R}{\partial t} = \frac{\partial\bar{t}}{\partial t}\frac{\partial R}{\partial\bar{t}} = \frac{\partial\bar{t}}{\partial t}\frac{\partial R}{\partial R_i}\frac{\partial R_i}{\partial\bar{t}} = \frac{\partial\bar{t}}{\partial t}\hat{R}_i\left(-\frac{\partial\gamma_i}{\partial\bar{t}}\right) = -\frac{\partial\bar{t}}{\partial t}\hat{R}\cdot\vec{v}$$

$$\Rightarrow \ \frac{\partial\bar{t}}{\partial t} - \frac{\partial\bar{t}}{\partial t}\hat{R}\cdot\vec{\beta} = 1$$

$$\frac{\partial\bar{t}}{\partial t} = \frac{1}{1 - \hat{R}\cdot\vec{\beta}}$$

$$\frac{\partial}{\partial x_i}\!: \ \frac{\partial\bar{t}}{\partial x_i} + \frac{1}{c}\frac{\partial R}{\partial x_i} = 0$$

$$\frac{\partial R}{\partial x_i} = \hat{R}_j\frac{\partial R_j}{\partial x_i} = \hat{R}_j\left(\delta_{ji} - \frac{d\gamma_j}{d\bar{t}}\frac{\partial\bar{t}}{\partial x_i}\right) = \hat{R}_i - \hat{R}\cdot\vec{v}\,\frac{\partial\bar{t}}{\partial x_i}$$

$$\Rightarrow \ \frac{\partial\bar{t}}{\partial x_i} + \frac{1}{c}\hat{R}_i - \hat{R}\cdot\vec{\beta}\,\frac{\partial\bar{t}}{\partial x_i} = 0$$

$$\frac{\partial\bar{t}}{\partial x_i} = -\frac{1}{c}\frac{\hat{R}_i}{1 - \hat{R}\cdot\vec{\beta}}$$

$$\frac{\partial R}{\partial t} = \frac{\partial R}{\partial\bar{t}}\frac{\partial\bar{t}}{\partial t} = \left(-\hat{R}\cdot\vec{v}\right)\frac{1}{1 - \hat{R}\cdot\vec{\beta}}$$

$$\frac{\partial R}{\partial t} = -\frac{\hat{R}\cdot\vec{\beta}c}{1 - \hat{R}\cdot\vec{\beta}}$$

$$\frac{\partial R}{\partial x_i} = \hat{R}_i - \hat{R}\cdot\vec{v}\frac{\partial \bar{t}}{\partial x_i} = \hat{R}_i - \hat{R}\cdot\vec{v}\left(-\frac{1}{c}\frac{\hat{R}_i}{1-\hat{R}\cdot\vec{\beta}}\right) = \hat{R}_i\left(1+\frac{\hat{R}\cdot\vec{\beta}}{1-\hat{R}\cdot\vec{\beta}}\right)$$

$$\frac{\partial R}{\partial x_i} = \frac{\hat{R}_i}{1-\hat{R}\cdot\vec{\beta}}$$

Now recall $\vec{R} = \vec{x} - \vec{\gamma}(\bar{t})$

$$\frac{\partial R_i}{\partial t} = -\frac{\partial \gamma_i}{\partial \bar{t}}\frac{\partial \bar{t}}{\partial t} = -\frac{v_i}{1-\hat{R}\cdot\vec{\beta}}$$

$$\frac{\partial R_i}{\partial t} = -\frac{\beta_i c}{1-\hat{R}\cdot\vec{\beta}}$$

$$\frac{\partial R_j}{\partial x_i} = \delta_{ij} - \frac{d\gamma_j}{d\bar{t}}\frac{\partial \bar{t}}{\partial x_i} = \delta_{ij} - c\beta_j\left(-\frac{1}{c}\frac{\hat{R}_i}{1-\hat{R}\cdot\vec{\beta}}\right)$$

$$\frac{\partial R_j}{\partial x_i} = \delta_{ij} + \frac{\hat{R}_i\beta_j}{1-\hat{R}\cdot\vec{\beta}}$$

Collecting all the derivatives of retarded quantities

$$\frac{\partial \bar{t}}{\partial t} = \frac{1}{1-\hat{R}\cdot\vec{\beta}} \qquad \frac{\partial R}{\partial t} = -\frac{\hat{R}\cdot\vec{\beta}c}{1-\hat{R}\cdot\vec{\beta}} \qquad \frac{\partial R_i}{\partial t} = -\frac{\beta_i c}{1-\hat{R}\cdot\vec{\beta}}$$

$$\frac{\partial \bar{t}}{\partial x_i} = -\frac{1}{c}\frac{\hat{R}_i}{1-\hat{R}\cdot\vec{\beta}} \qquad \frac{\partial R}{\partial x_i} = \frac{\hat{R}_i}{1-\hat{R}\cdot\vec{\beta}} \qquad \frac{\partial R_j}{\partial x_i} = \delta_{ij} + \frac{\hat{R}_i\beta_j}{1-\hat{R}\cdot\vec{\beta}}$$

Electric field strength

$E_i = -\partial_i \Phi - \partial_t A_i$

$$\partial_i \Phi = \partial_i\left(\frac{q}{4\pi\epsilon_0}\frac{1}{R-\vec{\beta}\cdot\vec{R}}\right) = -\frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\vec{\beta}\cdot\vec{R}\right)^2}\,\partial_i\left(R-\vec{\beta}(\bar{t})\cdot\vec{R}\right)$$

$$\begin{aligned}\partial_i\left(R-\vec{\beta}(\bar{t})\cdot\vec{R}\right) &= \partial_i\left(R-\beta_j(\bar{t})\cdot R_j\right) = \partial_i R - \beta_j(\bar{t})\partial_i R_j - \partial_i\beta_j(\bar{t})R_j \\ &= \frac{\hat{R}_i}{1-\hat{R}\cdot\vec{\beta}} - \beta_j\left(\delta_{ij} + \frac{\hat{R}_i\beta_j}{1-\hat{R}\cdot\vec{\beta}}\right) - R_j\frac{\partial\beta_j}{\partial\bar{t}}\frac{\partial\bar{t}}{\partial x_i} \\ &= \frac{\hat{R}_i}{1-\hat{R}\cdot\vec{\beta}} - \beta_i - \frac{\hat{R}_i\beta^2}{1-\hat{R}\cdot\vec{\beta}} - R_j\frac{1}{c}\underbrace{\frac{\partial v_j}{\partial\bar{t}}}_{a_j}\left(-\frac{1}{c}\frac{\hat{R}_i}{1-\hat{R}\cdot\vec{\beta}}\right) \\ &= \frac{\hat{R}_i}{1-\hat{R}\cdot\vec{\beta}} - \beta_i - \frac{\hat{R}_i\beta^2}{1-\hat{R}\cdot\vec{\beta}} + \frac{a_j}{c^2}\frac{R_j\hat{R}_i}{1-\hat{R}\cdot\vec{\beta}}\end{aligned}$$

$$= \frac{1}{1 - \hat{R}\cdot\vec{\beta}}\left(\hat{R}_i - \beta_i\left(1 - \hat{R}\cdot\vec{\beta}\right) - \hat{R}_i\beta^2 + \frac{\left(\vec{a}\cdot\vec{R}\right)\hat{R}_i}{c^2}\right)$$
$$= \frac{1}{R - \vec{R}\cdot\vec{\beta}}\left(R_i - \beta_i\left(R - \vec{R}\cdot\vec{\beta}\right) - R_i\beta^2 + \frac{\left(\vec{a}\cdot\vec{R}\right)R_i}{c^2}\right)$$

$$\partial_i\Phi = -\frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \vec{\beta}\cdot\vec{R}\right)^3}\left(R_i(1-\beta^2) - \beta_i\left(R - \vec{R}\cdot\vec{\beta}\right) + \frac{\left(\vec{a}\cdot\vec{R}\right)R_i}{c^2}\right)$$

$$\partial_t A_i = \partial_t\left(\frac{\mu_0 q}{4\pi}\frac{v_i(\bar{t})}{R - \vec{\beta}\cdot\vec{R}}\right) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\partial\bar{t}}{\partial t}\frac{\partial}{\partial\bar{t}}\left(\frac{v_i(\bar{t})}{R - \vec{\beta}\cdot\vec{R}}\right)$$
$$= \frac{q}{4\pi\epsilon_0 c^2}\frac{R}{R - \vec{R}\cdot\vec{\beta}}\left(\frac{a_i(\bar{t})}{R - \vec{\beta}\cdot\vec{R}} + v_i(\bar{t})\frac{\partial}{\partial\bar{t}}\left(\frac{1}{R - \vec{\beta}\cdot\vec{R}}\right)\right)$$

$$\frac{\partial}{\partial\bar{t}}\left(\frac{1}{R - \vec{\beta}\cdot\vec{R}}\right) = -\frac{1}{\left(R - \vec{\beta}\cdot\vec{R}\right)^2}\left(\frac{\partial R}{\partial\bar{t}} - \frac{1}{c}\vec{a}\cdot\vec{R} - \vec{\beta}\cdot\frac{\partial\vec{R}}{\partial\bar{t}}\right)$$
$$= -\frac{1}{\left(R - \vec{\beta}\cdot\vec{R}\right)^2}\left(-\hat{R}\cdot\vec{\beta}c - \frac{1}{c}\vec{a}\cdot\vec{R} + \vec{\beta}\cdot\vec{\beta}c\right)$$

$$\partial_t A_i = \frac{q}{4\pi\epsilon_0 c^2}\frac{R}{R - \vec{R}\cdot\vec{\beta}}\left(\frac{a_i}{R - \vec{\beta}\cdot\vec{R}} - \frac{\beta_i c}{\left(R - \vec{\beta}\cdot\vec{R}\right)^2}\left(-\hat{R}\cdot\vec{\beta}c - \frac{1}{c}\vec{a}\cdot\vec{R} + \vec{\beta}\cdot\vec{\beta}c\right)\right)$$
$$= \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \vec{R}\cdot\vec{\beta}\right)^3}\left(\frac{a_i}{c^2}R\left(R - \vec{\beta}\cdot\vec{R}\right) - \beta_i R\left(-\hat{R}\cdot\vec{\beta} - \frac{1}{c^2}\vec{a}\cdot\vec{R} + \vec{\beta}^2\right)\right)$$
$$= \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \vec{R}\cdot\vec{\beta}\right)^3}\left(\frac{Ra_i\left(R - \vec{\beta}\cdot\vec{R}\right)}{c^2} + \beta_i\left(\vec{\beta}\cdot\vec{R}\right) + \frac{\beta_i R\left(\vec{a}\cdot\vec{R}\right)}{c^2} - \beta_i R\vec{\beta}^2\right)$$

$$E_i = -\partial_i\Phi - \partial_t A_i$$
$$= \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \vec{\beta}\cdot\vec{R}\right)^3}\left(R_i(1-\beta^2) - \beta_i\left(R - \vec{R}\cdot\vec{\beta}\right) + \frac{\left(\vec{a}\cdot\vec{R}\right)R_i}{c^2}\right)$$
$$- \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \vec{R}\cdot\vec{\beta}\right)^3}\left(\frac{Ra_i\left(R - \vec{\beta}\cdot\vec{R}\right)}{c^2} + \beta_i\left(\vec{\beta}\cdot\vec{R}\right) + \frac{\beta_i R\left(\vec{a}\cdot\vec{R}\right)}{c^2} + \beta_i R\vec{\beta}^2\right)$$
$$= \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \vec{\beta}\cdot\vec{R}\right)^3}\left(R_i(1-\beta^2) - \beta_i\left(R - \vec{R}\cdot\vec{\beta}\right) + \frac{\left(\vec{a}\cdot\vec{R}\right)R_i}{c^2}\right.$$
$$\left. - \frac{Ra_i\left(R - \vec{\beta}\cdot\vec{R}\right)}{c^2} - \beta_i\left(\vec{\beta}\cdot\vec{R}\right) - \frac{\beta_i R\left(\vec{a}\cdot\vec{R}\right)}{c^2} - \beta_i R\vec{\beta}^2\right)$$

Collecting the terms

$$
\begin{aligned}
E_i &= \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\vec{\beta}\cdot\vec{R}\right)^3}\left(R_i(1-\beta^2)-\beta_i R+\beta_i R\vec{\beta}^2\right) \\
&\quad+\frac{q}{4\pi\epsilon_0 c^2}\frac{1}{\left(R-\vec{\beta}\cdot\vec{R}\right)^3}\left((\vec{a}\cdot\vec{R})R_i-Ra_i\left(R-\vec{\beta}\cdot\vec{R}\right)-\beta_i R(\vec{a}\cdot\vec{R})\right) \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R-\vec{\beta}\cdot\vec{R}\right)^3}\left(R_i(1-\beta^2)-\beta_i R\left(1-\vec{\beta}^2\right)\right) \\
&\quad+\frac{q}{4\pi\epsilon_0 c^2}\frac{1}{\left(R-\vec{\beta}\cdot\vec{R}\right)^3}\left((\vec{a}\cdot\vec{R})(R_i-\beta_i R)-Ra_i\left(R-\vec{\beta}\cdot\vec{R}\right)\right) \\
&= \frac{q}{4\pi\epsilon_0}\frac{R_i-\beta_i R}{\left(R-\vec{\beta}\cdot\vec{R}\right)^3}(1-\beta^2)+\frac{q}{4\pi\epsilon_0 c^2}\frac{1}{\left(R-\vec{\beta}\cdot\vec{R}\right)^3}\left(\vec{R}\times\left(\left(\vec{R}-R\vec{\beta}\right)\times\vec{a}\right)\right)_i
\end{aligned}
$$

Here we used

$$
\vec{R}\times\left(\left(\vec{R}-R\vec{\beta}\right)\times\vec{a}\right)=\left(\vec{R}-R\vec{\beta}\right)(\vec{a}\cdot\vec{R})-\vec{a}\underbrace{\left(\vec{R}\cdot\left(\vec{R}-R\vec{\beta}\right)\right)}_{R\left(R-\vec{\beta}\cdot\vec{R}\right)}
$$

Final result for the electric field

$$
\vec{E}=\underbrace{\frac{q}{4\pi\epsilon_0}\frac{\vec{R}-R\vec{\beta}}{\left(R-\vec{\beta}\cdot\vec{R}\right)^3}(1-\beta^2)}_{\text{Coulomb field of moving charge}}+\underbrace{\frac{\mu_0 q}{4\pi}\frac{\vec{R}\times\left(\left(\vec{R}-R\vec{\beta}\right)\times\vec{a}\right)}{\left(R-\vec{\beta}\cdot\vec{R}\right)^3}}_{\text{radiation term}}
$$

Magnetic field strength

$$
H_i=\frac{1}{\mu_0}\epsilon_{ijk}\partial_j A_k=\epsilon_{ijk}\partial_j\left(\frac{q}{4\pi}\frac{\vec{v}(\bar{t})}{R-\vec{\beta}\cdot\vec{R}}\right)
$$

HW: follow the method above to show that

$$
\vec{H}=\frac{1}{Z_0}\hat{R}\times\vec{E}
$$

The field of an arbitrarily moving point charge

$$\Phi(\vec{x},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{R - \vec{\beta}(\bar{t})\cdot\vec{R}} \qquad \vec{A}(\vec{x},t) = \frac{\mu_0 q}{4\pi}\frac{\vec{v}(\bar{t})}{R - \vec{\beta}(\bar{t})\cdot\vec{R}}$$

$$\vec{E}(\vec{x},t) = \underbrace{\frac{q}{4\pi\epsilon_0}\frac{\vec{R} - R\vec{\beta}(\bar{t})}{\left(R - \vec{\beta}(\bar{t})\cdot\vec{R}\right)^3}(1-\beta^2(\bar{t}))}_{\text{Coulomb field of moving charge}} + \underbrace{\frac{\mu_0 q}{4\pi}\frac{\vec{R}\times\left(\left(\vec{R} - R\vec{\beta}(\bar{t})\right)\times\vec{a}(\bar{t})\right)}{\left(R - \vec{\beta}(\bar{t})\cdot\vec{R}\right)^3}}_{\text{radiation term: due to acceleration}}$$

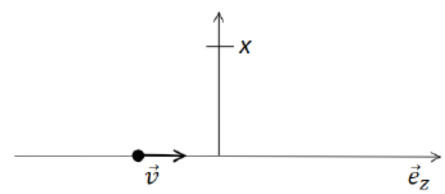
$$\vec{H}(\vec{x},t) = \frac{1}{Z_0}\hat{R}\times\vec{E}(\vec{x},t)$$

Retardation:

$$\bar{t}:\ \bar{t} - t + \frac{|\vec{x} - \vec{\gamma}(\bar{t})|}{c} = 0 \qquad \vec{R} = \vec{x} - \vec{\gamma}(\bar{t}) \qquad R = |\vec{x} - \vec{\gamma}(\bar{t})| = c(t - \bar{t})$$

Case of uniform motion

$$\vec{\gamma}(t) = \vec{v}t + \vec{x}_0'$$



Choose coordinate system so that

- $\vec{v} = v\vec{e}_z$ (by rotation of axis)
- $\vec{x}_0' = x_0'\vec{e}_z$ (by shift of origin in xy plane)

So $\vec{\gamma}(t) = \vec{v}(t - t_0)$

Observation point:

- $\vec{x} = (x, 0, 0)$

(shift in z direction to set $x_3 = 0$, rotate around z to set $x_2 = 0$)

Origin of time measurement:

- $t_0 = 0$

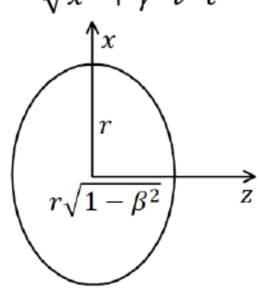
$$\vec{\gamma}(t) = (0,0,vt) \qquad \vec{x} = (x,0,0) \qquad \vec{R} = \vec{x} - \vec{\gamma}(\bar{t}) = (x, 0, -v\bar{t})$$

Retardation: $R = \sqrt{x^2 + v^2\bar{t}^2} = c(t - \bar{t}) \Rightarrow \bar{t} = \gamma^2\left(t - \frac{1}{\gamma c}\sqrt{x^2 + \gamma^2 v^2 t^2}\right)$

$$R - \vec{\beta}\cdot\vec{R} = c(t - \bar{t}) + \frac{v^2}{c}\bar{t} = ct - \frac{\gamma^2}{c}\bar{t} = \frac{1}{\gamma}\sqrt{x^2 + \gamma^2 v^2 t^2}$$

$$\Phi(t,\vec{x}) = \frac{q}{4\pi\epsilon_0}\frac{\gamma}{\sqrt{x^2 + \gamma^2 v^2 t^2}} \qquad \vec{A}(t,\vec{x}) = \frac{\mu_0 q}{4\pi}\frac{\gamma v}{\sqrt{x^2 + \gamma^2 v^2 t^2}}\vec{e}_z$$

Equipotential surfaces: $x^2 + \gamma^2 \underbrace{v^2 t^2}_{z^2} = r^2$



HW: derive the potentials using Lorentz transformation!

In rest frame: $\Phi(t,\vec{x}) = \frac{q}{4\pi\epsilon_0|\vec{x}|}$ $\quad \vec{A}(t,\vec{x}) = 0$

Transformation to coordinate system moving with u in direction z:

$$t' = \gamma\left(t - \frac{uz}{c^2}\right) \quad x' = x \quad y' = y \quad z' = \gamma(z - ut)$$

$$\Phi' = \gamma(\Phi - uA_z) \quad A'_x = A_x \quad A'_y = A_y \quad A'_z = \gamma\left(A_z - \frac{u\Phi}{c^2}\right)$$

Hint: we need $\Phi'(t',x')$ and $\vec{A}'(t',\vec{x}')$!

Field strengths

$$\vec{\gamma}(t) = (0,0,vt) \qquad \vec{x} = (x,0,0) \qquad \vec{R} = \vec{x} - \vec{\gamma}(\bar{t}) = (x,0,-v\bar{t})$$

$$\vec{R} - R\vec{\beta} = (x,0,-v\bar{t}) - c(t-\bar{t})\frac{1}{c}(0,0,v) = (x,0,-vt)$$

$$R - \vec{\beta}\cdot\vec{R} = \frac{1}{\gamma}\sqrt{x^2 + \gamma^2 v^2 t^2}$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{R} - R\vec{\beta}}{\left(R - \vec{\beta}\cdot\vec{R}\right)^3}(1-\beta^2) = \frac{q}{4\pi\epsilon_0}\frac{\gamma}{(x^2+\gamma^2v^2t^2)^{3/2}}\ (x,0,-vt)$$

$$\vec{H} = \frac{1}{Z_0}\hat{R}\times\vec{E}$$

$$\hat{R}\times(x,0,-vt) = \frac{1}{R}(x,0,-v\bar{t})\times(x,0,-vt) = \frac{1}{c(t-\bar{t})}(0,-vx(t-\bar{t}),0) = \frac{1}{c}(0,-vx,0)$$

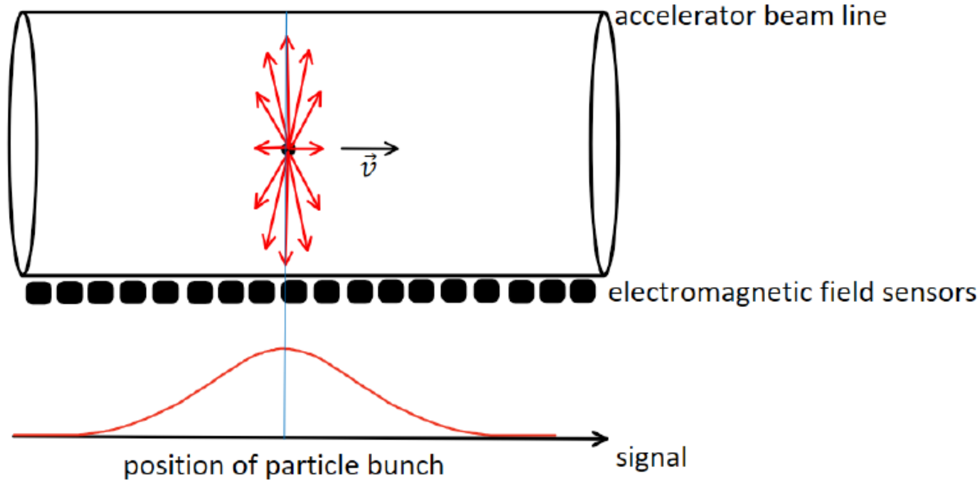
$$\vec{H} = \frac{q}{4\pi}\frac{\gamma}{(x^2+\gamma^2v^2t^2)^{3/2}}(0,-vx,0) \qquad \text{using} \quad c\epsilon_0 = \frac{1}{Z_0}$$

How does the electric field look like?

Along direction of motion: $(x = 0, z = -vt = r)$: $E = \frac{1}{\gamma^2}\frac{q}{4\pi\epsilon_0 r^2}$

Perpendicular to motion: $(x = r, z = 0)$: $E = \gamma\frac{q}{4\pi\epsilon_0 r^2}$

Ultra-relativistic case $\gamma \gg 1$: "squashed" in direction of motion - almost purely transversal!



It can monitor not just position along beamline, but also displacement in transversal plane by comparing fields around circumference - important for beam direction and focusing.

HW: derive the fields using Lorentz transformation!

In rest frame: $\vec{E}(t,\vec{x}) = \frac{q}{4\pi\epsilon_0}\frac{\vec{x}}{|\vec{x}|^3} \qquad \vec{B}(t,\vec{x}) = 0$

Transformation to coordinate system moving with u in direction z:

$$t' = \gamma\left(t - \frac{uz}{c^2}\right) \qquad x' = x \qquad y' = y \qquad z' = \gamma(z - ut)$$

$$E_x' = \gamma\left(E_x - uB_y\right) \qquad E_y' = \gamma\left(E_y + uB_x\right) \qquad E_z' = E_z$$

$$B_x' = \gamma\left(B_x + \frac{uE_y}{c^2}\right) \qquad B_y' = \gamma\left(B_y - \frac{uE_x}{c^2}\right) \qquad B_z' = B_z$$

Hint: we need $\vec{E}'(t',x')$ and $\vec{B}'(t',\vec{x}')$!

Week 10: Radiation field of accelerated charge

Field of a point charge in general motion

1. Retardation

$$\bar{t}:\ \bar{t} - t + \frac{|\vec{x} - \vec{\gamma}(\bar{t})|}{c} = 0 \qquad \vec{R} = \vec{x} - \vec{\gamma}(\bar{t}) \qquad R = |\vec{x} - \vec{\gamma}(\bar{t})| = c(t - \bar{t})$$

$$\frac{\partial \bar{t}}{\partial t} = \frac{1}{1 - \hat{R}\cdot\vec{\beta}} \qquad \frac{\partial R}{\partial t} = -\frac{\hat{R}\cdot\vec{\beta}c}{1 - \hat{R}\cdot\vec{\beta}} \qquad \frac{\partial R_i}{\partial t} = -\frac{\beta_i c}{1 - \hat{R}\cdot\vec{\beta}}$$

$$\frac{\partial \bar{t}}{\partial x_i} = -\frac{1}{c}\frac{\hat{R}_i}{1 - \hat{R}\cdot\vec{\beta}} \qquad \frac{\partial R}{\partial x_i} = \frac{\hat{R}_i}{1 - \hat{R}\cdot\vec{\beta}} \qquad \frac{\partial R_j}{\partial x_i} = \delta_{ij} + \frac{\hat{R}_i \beta_j}{1 - \hat{R}\cdot\vec{\beta}}$$

2. Fields

$$\vec{E}(\vec{x},t) = \underbrace{\frac{q}{4\pi\epsilon_0}\frac{\vec{R} - R\vec{\beta}(\bar{t})}{\left(R - \vec{\beta}(\bar{t})\cdot\vec{R}\right)^3}\left(1 - \beta^2(\bar{t})\right)}_{\text{Coulomb field of moving charge}} + \underbrace{\frac{\mu_0 q}{4\pi}\frac{\vec{R}\times\left(\left(\vec{R} - R\vec{\beta}(\bar{t})\right)\times\vec{a}(\bar{t})\right)}{\left(R - \vec{\beta}(\bar{t})\cdot\vec{R}\right)^3}}_{\text{radiation term: due to acceleration}}$$

$$\vec{H}(\vec{x},t) = \frac{1}{Z_0}\hat{R}\times\vec{E}(\vec{x},t)$$

Angular distribution of radiated power

$$\vec{S}_{rad} = \vec{E}_{rad}\times\vec{H}_{rad} = \frac{1}{Z_0}\hat{R}E_{rad}^2 \quad \vec{E}_{rad} = \frac{Z_0 q}{4\pi c R}\frac{\hat{R}\times\left(\left(\hat{R} - \vec{\beta}\right)\times\vec{a}\right)}{\left(1 - \vec{\beta}\cdot\hat{R}\right)^3}$$

$$= \frac{Z_0 q}{4\pi R}\frac{\hat{R}\times\left(\left(\hat{R} - \vec{\beta}\right)\times\dot{\vec{\beta}}\right)}{\left(1 - \vec{\beta}\cdot\hat{R}\right)^3}$$

Choice of coordinate origin: retarded position of charge $\vec{\gamma}(\bar{t}) = 0 \Rightarrow \vec{R} = \vec{x}$

Radiated power computed per unit retarded time (W=energy):

$$dP = \frac{dW}{d\bar{t}} = \frac{dW}{dt}\frac{dt}{d\bar{t}} = R^2 d\Omega\,\hat{R}\cdot\vec{S}(1 - \hat{R}\cdot\vec{\beta})$$

$$\frac{dP}{d\Omega} = \frac{Z_0 q^2}{16\pi^2}\frac{\left|\hat{x}\times\left(\left(\hat{x} - \vec{\beta}\right)\times\dot{\vec{\beta}}\right)\right|^2}{\left(1 - \vec{\beta}\cdot\hat{x}\right)^5}$$

Example 1: charge accelerating along a straight line

$$\vec{\beta}\parallel\dot{\vec{\beta}} \ \Rightarrow\ \hat{x}\times\left(\left(\hat{x} - \vec{\beta}\right)\times\dot{\vec{\beta}}\right) = \hat{x}\times\left(\hat{x}\times\dot{\vec{\beta}}\right)$$

θ: angle between line motion $\vec{\beta} \parallel \dot{\vec{\beta}}$ and direction of observation $\hat{x}$

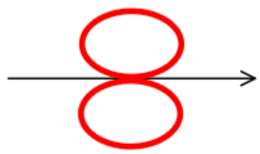
$$\left|\hat{x} \times \left(\hat{x} \times \dot{\vec{\beta}}\right)\right| = \left|\hat{x} \times \dot{\vec{\beta}}\right| = \dot{\beta} \sin\theta = \frac{a}{c} \sin\ \theta$$

$$1 - \vec{\beta} \cdot \hat{x} = 1 - \beta \cos\ \theta$$

$$\frac{dP}{d\Omega} = \frac{Z_0 q^2 a^2}{16\pi^2 c^2} \frac{\sin^2\theta}{(1 - \beta\cos\theta)^5}$$

<u>Non-relativistic limit</u>: dipole pattern

$\beta = 0$:



direction of acceleration

<u>Ultra-relativistic limit</u>:

$$\gamma = (1 - \beta^2)^{-1/2} \gg 1 \ \Rightarrow\ \beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2}$$

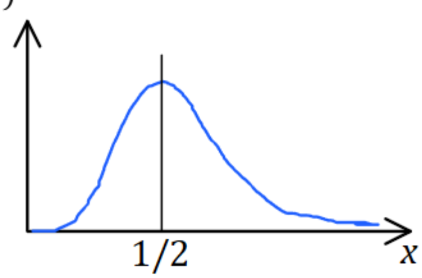
Radiation is dominated by direction $\theta \sim 0$:

$$\sin\theta \sim \theta \quad \cos\theta \sim 1 - \frac{\theta^2}{2}$$

$$1 - \beta\cos\theta \sim \frac{1}{2\gamma^2} + \frac{\theta^2}{2} = \frac{1}{2\gamma^2}(1 + (\gamma\theta)^2)$$

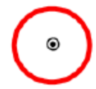
$$\frac{dP}{d\Omega} = \frac{2Z_0 q^2 a^2 \gamma^8}{\pi^2 c^2} \frac{(\gamma\theta)^2}{(1 - (\gamma\theta)^2)^5}$$

Function $f(x) = \frac{x^2}{(1 + x^2)^5}$:



Note also that $I_{max} \propto \gamma^8$!

$$\theta_{max} = \frac{1}{2\gamma}$$



As seen from the front

Example 2: charge in circular motion

$$\frac{dP}{d\Omega} = \frac{Z_0 q^2}{16\pi^2} \frac{\left| \hat{x} \times \left((\hat{x} - \vec{\beta}) \times \dot{\vec{\beta}} \right) \right|^2}{(1 - \vec{\beta} \cdot \hat{x})^5}$$

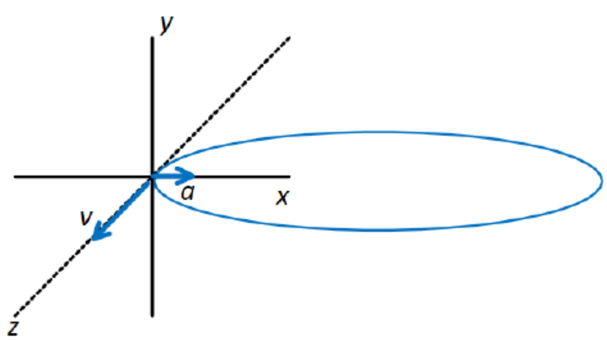
$$\vec{\beta} \perp \dot{\vec{\beta}} \Rightarrow \ \hat{x} \times \left((\hat{x} - \vec{\beta}) \times \dot{\vec{\beta}} \right) = (\hat{x} - \vec{\beta})\hat{x} \cdot \dot{\vec{\beta}} - \dot{\vec{\beta}}(1 - \hat{x} \cdot \vec{\beta})$$

$$|\ldots|^2 = (1 - 2\hat{x} \cdot \vec{\beta} + \beta^2)\left(\hat{x} \cdot \dot{\vec{\beta}}\right)^2 + \dot{\vec{\beta}}^2 (1 - \hat{x} \cdot \vec{\beta})^2 - 2\hat{x} \cdot \dot{\vec{\beta}}(1 - \hat{x} \cdot \vec{\beta})\left(\dot{\vec{\beta}} \cdot \hat{x} - \underbrace{\dot{\vec{\beta}} \cdot \vec{\beta}}_{0}\right)$$

$$= (-1 + \beta^2)\left(\hat{x} \cdot \dot{\vec{\beta}}\right)^2 + \dot{\vec{\beta}}^2 (1 - \hat{x} \cdot \vec{\beta})^2$$

$$= -\frac{1}{\gamma^2}\left(\hat{x} \cdot \dot{\vec{\beta}}\right)^2 + \dot{\vec{\beta}}^2 (1 - \hat{x} \cdot \vec{\beta})^2$$

Choice of coordinates:



$$\vec{\beta} = \beta \vec{e}_z \quad \dot{\vec{\beta}} = \frac{a}{c}\vec{e}_x \quad \hat{x} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\vec{\beta} \cdot \hat{x} = \beta\cos\theta \quad \dot{\vec{\beta}} \cdot \hat{x} = \frac{a}{c}\sin\theta\cos\phi$$

θ: angle of line of sight to velocity (direction of motion z)

ϕ: angle of line of sight to plane xz of circular motion

$$\frac{dP}{d\Omega} = \frac{Z_0 q^2 a^2}{16\pi^2 c^2} \frac{1}{(1 - \beta\cos\theta)^3}\left\{1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1 - \beta\cos\theta)^2}\right\}$$

$$\propto \ \frac{1}{(1 + \gamma^2\theta^2)^3}\left\{1 - \frac{4\ \gamma^2\theta^2\cos^2\phi}{(1 + \gamma^2\theta^2)^2}\right\} \text{ for ultrarelativistic motion}$$

Concentrated in a cone around direction of velocity ($\theta = 0$)

With half opening angle $\sim 1/\gamma$

But: depends also on direction relative to plane of motion (ϕ)

Radiated power: relativistic generalisation of Larmor formula

In system where the charge is at (instantaneous) rest ($\vec{\beta} = 0$):

$$\frac{dP}{d\Omega} = \frac{Z_0 q^2}{16\pi^2} \frac{\left| \hat{x} \times \left(\left(\hat{x} - \vec{\beta} \right) \times \dot{\vec{\beta}} \right) \right|^2}{\left(1 - \vec{\beta} \cdot \hat{x} \right)^5} = \frac{Z_0 q^2}{16\pi^2} \left| \hat{x} \times \dot{\vec{\beta}} \right|^2 = \frac{Z_0 q^2 a^2}{16\pi^2 c^2} \sin^2 \theta$$

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{Z_0 q^2 a^2}{6\pi c^2} \quad \text{Larmor formula}$$

Relativistic generalisation

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{Z_0 q^2}{6\pi c^2} \gamma^6 \left(\vec{a}^2 - \left(\vec{\beta} \times \vec{a} \right)^2 \right)$$

Complicated derivation: perform integral

Simple derivation: use Lorentz-invariance

In reference system $v = 0$: time derivative of radiated energy W given by

$$P = \frac{dW}{dt} = \frac{Z_0 q^2 a^2}{6\pi c^2}$$

In system where $v \neq 0$

$t' = \gamma t \quad W' = \gamma W$ so $\dfrac{dW'}{dt'} = \dfrac{dW}{dt}$

We only need a Lorentz invariant expression for a^2 where $\vec{a}$ is acceleration measured in particle's own reference system.

Proper time: $\mathrm{d}\tau = \mathrm{dt}/\gamma$ (relativistic invariant)

World-line: $x^\mu = x^\mu(t) = \left(ct, \vec{x}(t) \right)$

Four velocity: contravariant vector

$u^\mu = \dfrac{dx^\mu}{d\tau} = \gamma \dfrac{dx^\mu}{dt} = (\gamma c, \gamma \vec{v})$ Note: $u^\mu u_\mu = \gamma^2 c^2 - \gamma^2 \vec{v}^2 = c^2$

Four acceleration: again, contravariant vector

$$a^\mu = \frac{du^\mu}{d\tau} = \gamma \frac{da^\mu}{dt} = \gamma \frac{d\gamma}{dt} (c, \vec{v}) + \gamma (0, \gamma \vec{a}) \qquad \vec{a}\text{: usual } 3 - \text{acceleration}$$

$$\frac{d\gamma}{dt} = \frac{d}{dt} \left(1 - \frac{v^2}{c^2} \right)^{-\frac{1}{2}} = -\frac{1}{2} \left(1 - \frac{v^2}{c^2} \right)^{-\frac{3}{2}} \left(-2 \frac{\vec{v} \cdot \vec{a}}{c^2} \right) = \gamma^3 \frac{\vec{v} \cdot \vec{a}}{c^2}$$

$$a^\mu = \gamma^4 \frac{\vec{v} \cdot \vec{a}}{c^2} (c, \vec{v}) + \gamma (0, \vec{a}) = \left(\gamma^4 \vec{\beta} \cdot \vec{a},\ \gamma^4 \vec{\beta} \left(\vec{\beta} \cdot \vec{a} \right) + \gamma^2 \vec{a} \right)$$

In $\vec{\beta} = 0$ system

$a^\mu = (0, \vec{a}) \quad \Rightarrow \quad a^\mu a_\mu = -\vec{a}^2$

So, the covariant formula for the total radiate power is

$$P = -\frac{Z_0 q^2}{6\pi c^2} a^\mu a_\mu$$

Now we need to express the invariant $a^\mu a_\mu$ in the moving system:

$$\begin{aligned}
a^\mu a_\mu &= \left(\gamma^4 \vec{\beta} \cdot \vec{a}\right)^2 - \left(\gamma^4 \vec{\beta}\left(\vec{\beta} \cdot \vec{a}\right) + \gamma^2 \vec{a}\right)^2 \\
&= \gamma^8 \left(\vec{\beta} \cdot \vec{a}\right)^2 - \gamma^8 \vec{\beta}^2 \left(\vec{\beta} \cdot \vec{a}\right)^2 - \gamma^4 \vec{a}^2 - 2\gamma^6 \left(\vec{\beta} \cdot \vec{a}\right)^2 \\
&= \underbrace{\gamma^8 (1 - \beta^2)}_{\gamma^6} \left(\vec{\beta} \cdot \vec{a}\right)^2 - \gamma^4 \vec{a}^2 - 2\gamma^6 \left(\vec{\beta} \cdot \vec{a}\right)^2 \\
&= -\gamma^4 \vec{a}^2 - \gamma^6 \left(\vec{\beta} \cdot \vec{a}\right)^2 = -\gamma^6 \left((1 - \beta^2)\vec{a}^2 + \left(\vec{\beta} \cdot \vec{a}\right)^2\right) \\
&= -\gamma^6 \left(\vec{a}^2 - \left(\beta^2 \vec{a}^2 - \left(\vec{\beta} \cdot \vec{a}\right)^2\right)\right) \\
&= -\gamma^6 \left(\vec{a}^2 - \left(\vec{\beta} \times \vec{a}\right)^2\right)
\end{aligned}$$

So, we finally get the relativistic Larmor formula as promised

$$P = \frac{Z_0 q^2}{6\pi c^2} \gamma^6 \left(\vec{a}^2 - \left(\vec{\beta} \times \vec{a}\right)^2\right) \qquad \text{Q.e.d.}$$

Note that

$$\gamma = (1 - \beta^2)^{-\frac{1}{2}} = 1 + \frac{\beta^2}{2} + O(\beta^4)$$

Therefore

$$P = \frac{Z_0 q^2}{6\pi c^2} \vec{a}^2 \left(1 + O(\beta^2)\right)$$

So, the relativistic corrections to Larmor formula are $O(\beta^2)$.

Radiation in accelerators

(details in exercise class and homework)
Assume ultra-relativistic motion $\gamma \gg 1$

1. Circular motion (synchrotron)

$W = \gamma mc^2 \quad p = m\beta c\gamma \approx \frac{W}{c}$

$P \approx \frac{Z_0 q^2 c^2}{6\pi r^2}\left(\frac{W}{mc^2}\right)^4 \quad T \approx \frac{2\pi r}{c}$

Under one revolution

$$\delta W = \frac{Z_0 q^2 c}{3r}\left(\frac{W}{mc^2}\right)^4$$

Advantage of accelerating protons:

$\left(\frac{m_e}{m_p}\right)^4 \approx 6 \cdot 10^{-14}$

LEP2: 100 GeV LHC: 7 TeV beam energy in same tunnel!
Collision energy: 200 GeV vs. 14 TeV

Synchrotrons are also used as radiation source (e.g., ESRF Grenoble)

2. Linear motion (linear accelerator, linac)

$P = \frac{Z_0 q^2}{6\pi m^2 c^2}\left(\frac{dp}{dt}\right)^2 = \frac{Z_0 q^2}{6\pi m^2 c^2}\left(\frac{dW}{dx}\right)^2 \quad \frac{dp}{dt} = \frac{dW}{dx} = \frac{1}{c}\frac{dW}{dt}$

$\frac{P}{\partial_t W} = \frac{Z_0 q^2}{6\pi m^2 c^3}\frac{dW}{dx}$

Radiation loss prevents further acceleration when

$\frac{P}{\partial_t W} = 1$

For electrons the necessary field strength is

$E \approx 2.7 \cdot 10^{20}\,\frac{V}{m}$ Maximum available today: $10^{12}\,\frac{V}{m}$

Theoretical maximum possible electric field strength

$10^{18}\,\frac{V}{m}$ (Schwinger limit; spontanous pair creation)

Radiation loss in linear accelerators is always negligible!

Week 11: Radiation of a moving charge: distribution in frequency and angle of observation

Reminder from two weeks ago

1. Retardation

$$\bar{t}:\ \bar{t} - t + \frac{|\vec{x} - \vec{\gamma}(\bar{t})|}{c} = 0 \qquad \vec{R} = \vec{x} - \vec{\gamma}(\bar{t}) \qquad R = |\vec{x} - \vec{\gamma}(\bar{t})| = c(t - \bar{t})$$

$$\frac{\partial \bar{t}}{\partial t} = \frac{1}{1 - \hat{R}\cdot\vec{\beta}} \qquad \frac{\partial R}{\partial t} = -\frac{\hat{R}\cdot\vec{\beta}c}{1 - \hat{R}\cdot\vec{\beta}} \qquad \frac{\partial R_i}{\partial t} = -\frac{\beta_i c}{1 - \hat{R}\cdot\vec{\beta}}$$

$$\frac{\partial \bar{t}}{\partial x_i} = -\frac{1}{c}\frac{\hat{R}_i}{1 - \hat{R}\cdot\vec{\beta}} \qquad \frac{\partial R}{\partial x_i} = \frac{\hat{R}_i}{1 - \hat{R}\cdot\vec{\beta}} \qquad \frac{\partial R_j}{\partial x_i} = \delta_{ij} + \frac{\hat{R}_i\beta_j}{1 - \hat{R}\cdot\vec{\beta}}$$

2. Fields

$$\vec{E}(\vec{x},t) = \underbrace{\frac{q}{4\pi\epsilon_0}\frac{\vec{R} - R\vec{\beta}(\bar{t})}{\left(R - \vec{\beta}(\bar{t})\cdot\vec{R}\right)^3}\left(1 - \beta^2(\bar{t})\right)}_{\text{Coulomb field of moving charge}} + \underbrace{\frac{\mu_0 q}{4\pi}\frac{\vec{R}\times\left(\left(\vec{R} - R\vec{\beta}(\bar{t})\right)\times\vec{a}(\bar{t})\right)}{\left(R - \vec{\beta}(\bar{t})\cdot\vec{R}\right)^3}}_{\text{radiation term: due to acceleration}}$$

$$\vec{H}(\vec{x},t) = \frac{1}{Z_0}\hat{R}\times\vec{E}(\vec{x},t)$$

$$\vec{S}_{rad} = \vec{E}_{rad}\times\vec{H}_{rad} = \frac{1}{Z_0}\hat{R}E_{rad}^2$$

$$\vec{E}_{rad}(\vec{x},t) = \frac{Z_0 q}{4\pi c R}\frac{\hat{R}\times\left(\left(\hat{R} - \vec{\beta}(\bar{t})\right)\times\vec{a}(\bar{t})\right)}{\left(1 - \vec{\beta}(\bar{t})\cdot\hat{R}\right)^3} = \frac{Z_0 q}{4\pi R}\frac{\hat{R}\times\left(\left(\hat{R} - \vec{\beta}(\bar{t})\right)\times\dot{\vec{\beta}}(\bar{t})\right)}{\left(1 - \vec{\beta}(\bar{t})\cdot\hat{R}\right)^3}$$

Now we look at what the distant observer sees (in observation time *t*)

$$dP = \frac{dW}{dt} = R^2 d\Omega\,\hat{R}\cdot\vec{S}_{rad} = \left|\vec{A}(t)\right|^2 d\Omega$$

$$\frac{dP}{d\Omega} = \left|\vec{A}(t)\right|^2 \qquad \vec{A}(t) = \frac{1}{\sqrt{Z_0}}\left[R\vec{E}_{rad}\right]_{ret}$$

Emitted energy (angular distribution):

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt\,\left|\vec{A}(t)\right|^2 = \int_{-\infty}^{\infty} d\omega\,\left|\vec{A}(\omega)\right|^2 \qquad \vec{A}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\,\vec{A}(t)e^{i\omega t}$$

$$= 2\int_0^\infty d\omega\, \left|\vec{A}(\omega)\right|^2 \quad \text{since} \quad \vec{A}(-\omega) = \vec{A}(\omega)^*$$

Angular and frequency distribution of emitted energy:

$$\frac{d^2W}{d\omega d\Omega} = 2\left|\vec{A}(\omega)\right|^2$$

$$\vec{A}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, \vec{A}(t) e^{i\omega t} \qquad \vec{A}(t) = \frac{1}{\sqrt{Z_0}}\left[R\vec{E}_{rad}\right]_{ret}$$

$$\vec{E}_{rad} = \frac{Z_0 q}{4\pi R}\frac{\hat{R}\times\left(\left(\hat{R}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right)}{\left(1-\vec{\beta}\cdot\hat{R}\right)^3}$$

$$\vec{A}(\omega) = \frac{q\sqrt{Z_0}}{\sqrt{32\pi^3}}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\left[\frac{\hat{R}\times\left(\left(\hat{R}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right)}{\left(1-\vec{\beta}\cdot\hat{R}\right)^3}\right]_{ret}$$

Change integration variable to retarded time:

$$t = \bar{t} + \frac{R(\bar{t})}{c} \qquad dt = \left(1-\vec{\beta}\cdot\hat{R}\right)d\bar{t}$$

$$\vec{A}(\omega) = \frac{q\sqrt{Z_0}}{\sqrt{32\pi^3}}\int_{-\infty}^{\infty} d\bar{t}\, e^{i\omega\left(\bar{t}+\frac{R(\bar{t})}{c}\right)}\frac{\hat{R}\times\left(\left(\hat{R}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right)}{\left(1-\vec{\beta}\cdot\hat{R}\right)^2}$$

Large distance: $r = |\vec{x}| \gg \left|\vec{\xi}(\bar{t})\right|$

$R(\bar{t}) = \left|\vec{x}-\vec{\xi}(\bar{t})\right| = r - \hat{x}\cdot\vec{\xi}(\bar{t}) + O(r^{-1}) \qquad \hat{R} = \hat{x} + O(r^{-1})$

$$\vec{A}(\omega) = \frac{q\sqrt{Z_0}}{\sqrt{32\pi^3}} e^{i\omega r/c}\int_{-\infty}^{\infty} d\bar{t}\, e^{i\omega\left(\bar{t}-\frac{\hat{x}\cdot\vec{\xi}(\bar{t})}{c}\right)}\frac{\hat{x}\times\left(\left(\hat{x}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right)}{\left(1-\hat{x}\cdot\vec{\beta}\right)^2}$$

We can relabel the integration variable $\bar{t}\to t$ and write

$$\frac{d^2W}{d\omega d\Omega} = 2\left|\vec{C}(\omega)\right|^2 \qquad \vec{C}(\omega) = \frac{q\sqrt{Z_0}}{\sqrt{32\pi^3}}\int_{-\infty}^{\infty} dt\, e^{i\omega\left(t-\frac{\hat{x}\cdot\vec{\xi}(t)}{c}\right)}\frac{\hat{x}\times\left(\left(\hat{x}-\vec{\beta}(t)\right)\times\dot{\vec{\beta}}(t)\right)}{\left(1-\hat{x}\cdot\vec{\beta}(t)\right)^2}$$

Funny relation

$$\frac{\hat{x}\times\left(\left(\hat{x}-\vec{\beta}(t)\right)\times\dot{\vec{\beta}}(t)\right)}{\left(1-\hat{x}\cdot\vec{\beta}(t)\right)^2}=\frac{d}{dt}\left(\frac{\hat{x}\times\left(\hat{x}\times\vec{\beta}(t)\right)}{1-\hat{x}\cdot\vec{\beta}(t)}\right)$$

<u>Proof</u>

$$\frac{d}{dt}\left(\frac{\hat{x}\times\left(\hat{x}\times\vec{\beta}\right)}{1-\hat{x}\cdot\vec{\beta}}\right)=\frac{\hat{x}\times\left(\hat{x}\times\dot{\vec{\beta}}\right)\left(1-\hat{x}\cdot\vec{\beta}\right)-\hat{x}\times\left(\hat{x}\times\vec{\beta}\right)\left(-\hat{x}\cdot\dot{\vec{\beta}}\right)}{\left(1-\hat{x}\cdot\vec{\beta}\right)^2}$$

Now we can compute

$$\hat{x}\times\left(\hat{x}\times\dot{\vec{\beta}}\right)\left(1-\hat{x}\cdot\vec{\beta}\right)-\hat{x}\times\left(\hat{x}\times\vec{\beta}\right)\left(-\hat{x}\cdot\dot{\vec{\beta}}\right)$$

$$=\hat{x}\times\left(\hat{x}\times\dot{\vec{\beta}}\right)-\left(\underline{\hat{x}\left(\hat{x}\cdot\dot{\vec{\beta}}\right)}-\dot{\vec{\beta}}\right)\left(\hat{x}\cdot\vec{\beta}\right)+\left(\underline{\hat{x}\left(\hat{x}\cdot\vec{\beta}\right)}-\vec{\beta}\right)\left(\hat{x}\cdot\dot{\vec{\beta}}\right)$$

The underlined terms cancel

$$=\hat{x}\times\left(\hat{x}\times\dot{\vec{\beta}}\right)+\dot{\vec{\beta}}\left(\hat{x}\cdot\vec{\beta}\right)-\vec{\beta}\left(\hat{x}\cdot\dot{\vec{\beta}}\right)=\hat{x}\times\left(\hat{x}\times\dot{\vec{\beta}}\right)-\hat{x}\times\left(\vec{\beta}\times\dot{\vec{\beta}}\right)$$

$$=\hat{x}\times\left(\left(\hat{x}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right)\qquad\text{Q. e. d.}$$

So, we arrive at

$$\vec{C}(\omega)=\frac{q\sqrt{Z_0}}{\sqrt{32\pi^3}}\int_{-\infty}^{\infty}dt\,e^{i\omega\left(t-\frac{\hat{x}\cdot\vec{\xi}(t)}{c}\right)}\frac{d}{dt}\left(\frac{\hat{x}\times\left(\hat{x}\times\vec{\beta}(t)\right)}{1-\hat{x}\cdot\vec{\beta}(t)}\right)$$

Partial integration gives

$$\vec{C}(\omega)=-\frac{q\sqrt{Z_0}}{\sqrt{32\pi^3}}\int_{-\infty}^{\infty}dt\left(\frac{\hat{x}\times\left(\hat{x}\times\vec{\beta}(t)\right)}{1-\hat{x}\cdot\vec{\beta}(t)}\right)\frac{d}{dt}e^{i\omega\left(t-\frac{\hat{x}\cdot\vec{\xi}(t)}{c}\right)}$$

$$=-\frac{q\sqrt{Z_0}}{\sqrt{32\pi^3}}\int_{-\infty}^{\infty}dt\left(\frac{\hat{x}\times\left(\hat{x}\times\vec{\beta}(t)\right)}{1-\hat{x}\cdot\vec{\beta}(t)}\right)i\omega\left(1-\frac{\hat{x}\cdot\vec{v}(t)}{c}\right)e^{i\omega\left(t-\frac{\hat{x}\cdot\vec{\xi}(t)}{c}\right)}$$

$$=-\frac{i\omega q\sqrt{Z_0}}{\sqrt{32\pi^3}}\int_{-\infty}^{\infty}dt\ \hat{x}\times\left(\hat{x}\times\vec{\beta}(t)\right)e^{i\omega\left(t-\frac{\hat{x}\cdot\vec{\xi}(t)}{c}\right)}$$

Recall

$$\frac{d^2W}{d\omega d\Omega}=2\left|\vec{C}(\omega)\right|^2$$

If we also observe outgoing polarisation $\vec{e}$:

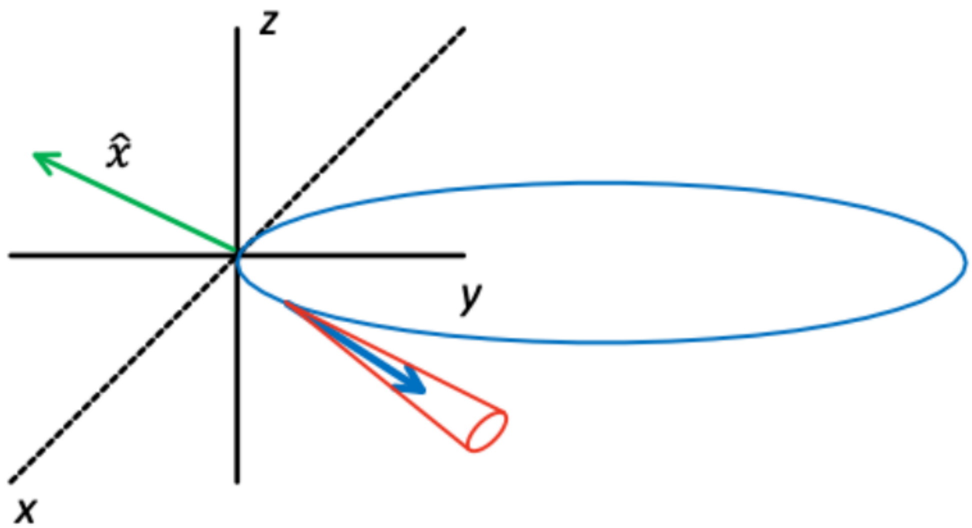
$$\frac{d^2W}{d\omega d\Omega}=2\left|\vec{e}^*\cdot\vec{C}(\omega)\right|^2$$

So, the end result is

Angular and frequency distribution of radiation by a moving charge

$$\frac{d^2W}{d\omega d\Omega} = \frac{Z_0 q^2 \omega^2}{16\pi^2} \left| \vec{e}^* \cdot \int_{-\infty}^{\infty} dt \ \hat{x} \times \left(\hat{x} \times \vec{\beta}(t) \right) e^{i\omega\left(t - \frac{\hat{x}\cdot\vec{\xi}(t)}{c}\right)} \right|^2$$

Example: synchrotron radiation



$$\xi(t) = (\rho \sin \omega_0 t\,, \rho(1 - \cos \omega_0 t), 0) \ \beta(t) = \frac{1}{c}(\rho\omega_0 \cos \omega_0 t\,, \rho\omega_0 \sin \omega_0 t\,, 0)$$
$$\hat{x} = (\cos\theta\,, 0, \sin\theta) \quad \vec{e}_{\parallel} = (0,1,0) \quad \vec{e}_{\perp} = (-\sin\theta\,, 0, \cos\theta)$$

UR limit: only very small θ cone contributes

$$\omega_0 = \frac{2\pi}{T} \quad T = \frac{2\pi\rho}{c} \quad \Rightarrow \quad \omega_0 = \frac{c}{\rho}$$

$\vec{e}_{\parallel} = (0,1,0)$ polarization in plane of motion
$\vec{e}_{\perp} \approx (0,0,1)$ polarization perpendicular to plane of motion

We get a complicated integral (cf. Jackson) which evaluates to

$$\frac{d^2W_{\parallel}}{d\omega d\Omega} = \frac{Z_0 q^2}{12\pi^2}\left(\frac{\omega\rho}{c}\right)^2 \left(\frac{1}{\gamma^2} + \theta^2\right)^2 K_{2/3}(\xi)^2 \quad \xi = \frac{\omega\rho}{3c}\left(\frac{1}{\gamma^2} + \theta^2\right)^{3/2}$$
$$\frac{d^2W_{\perp}}{d\omega d\Omega} = \frac{Z_0 q^2}{12\pi^2}\left(\frac{\omega\rho}{c}\right)^2 \left(\frac{1}{\gamma^2} + \theta^2\right) \theta^2 K_{1/3}(\xi)^2$$

A few properties of synchrotron radiation

Frequency dependence:

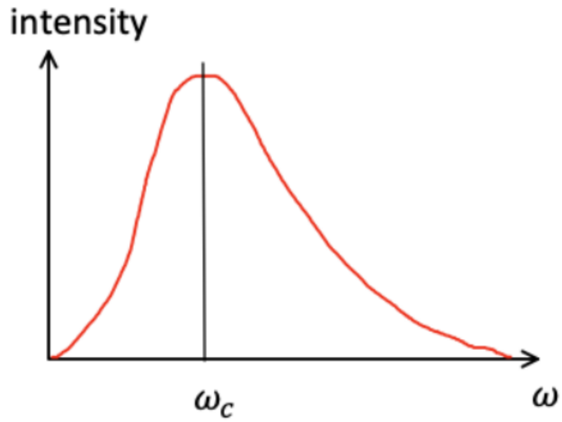
$$\frac{dW}{d\omega} = \int d\Omega \left(\frac{d^2 W_{\parallel}}{d\omega d\Omega} + \frac{d^2 W_{\perp}}{d\omega d\Omega} \right)$$

$$\propto \omega^{1/3} \qquad \omega \ll \omega_c$$

$$\propto \sqrt{\frac{\omega}{\omega_c}} e^{-\omega/\omega_c} \qquad \omega \gg \omega_c$$

where the critical frequency is

$$\omega_c = \frac{3}{2} \gamma^3 \frac{c}{\rho}$$



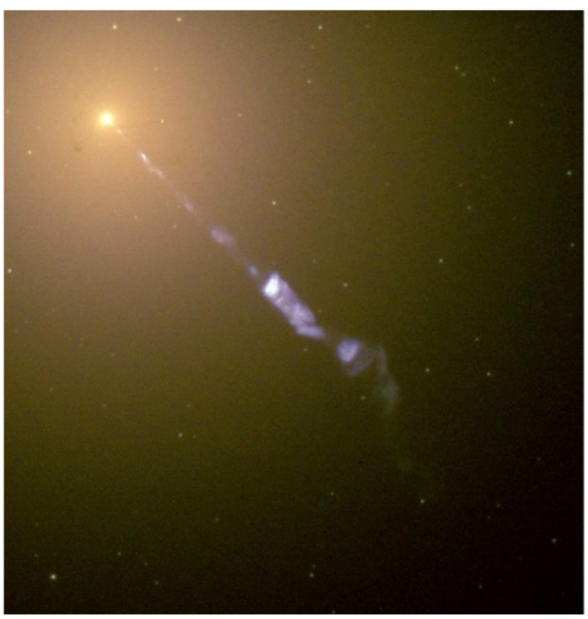
Ratio of total intensities: $\parallel : \perp = 7:1$ allows to determine plane of motion!

Typically, circular motion happens in magnetic field: polarisation allows to determine direction of magnetic field, frequency dependence allows to estimate magnitude!

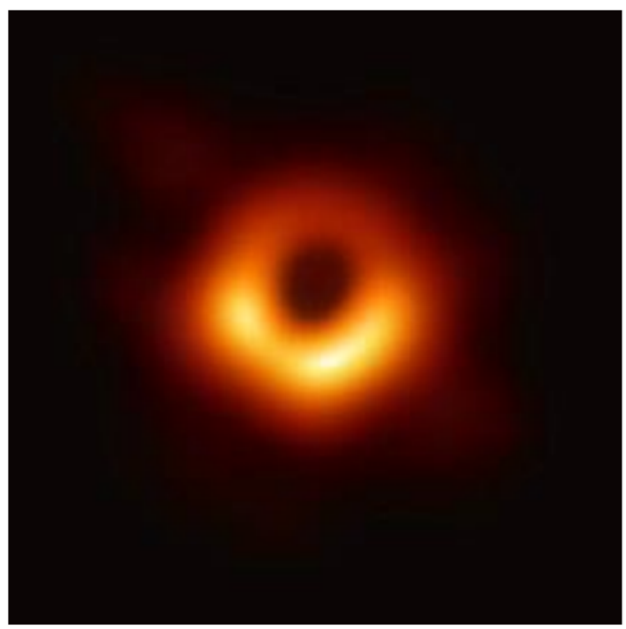
Examples:

1. Messier 87 active galactic nucleus (supermassive black hole) - relativistic jet

$$M_* \approx 6 \cdot 10^9 \; \times \text{ (mass of the Sun)}$$

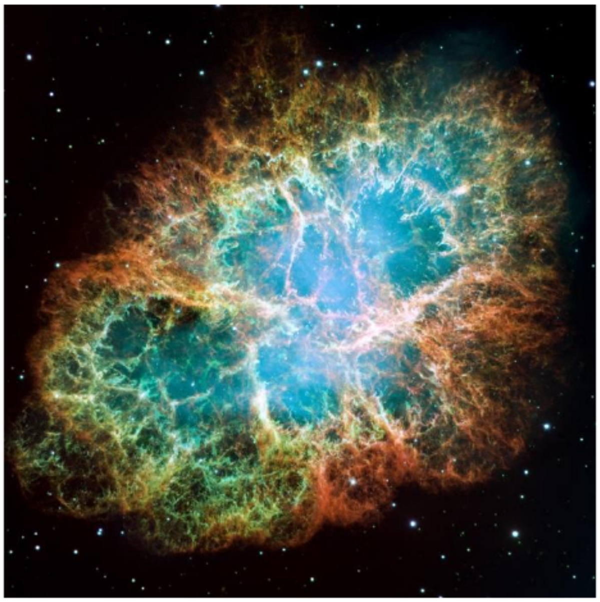


The relativistic jet from M87*
(Hubble Telescope)



Black hole shadow and photon ring
(Event Horizon Telescope)

2. Crab Nebula: pulsar wind from supernova remnant (1054 AD)

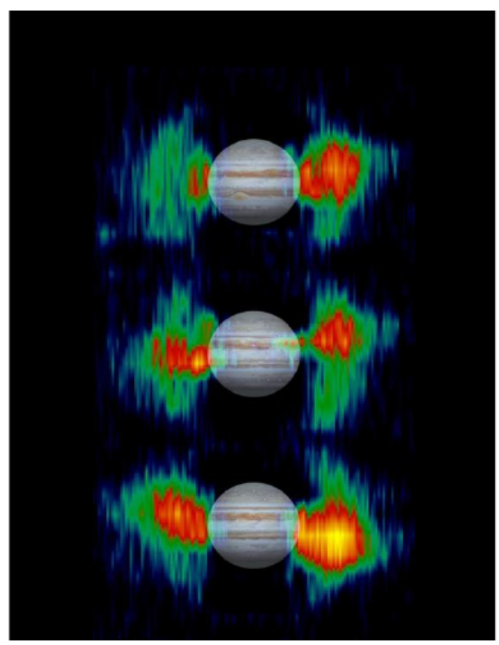


Hubble Telescope image

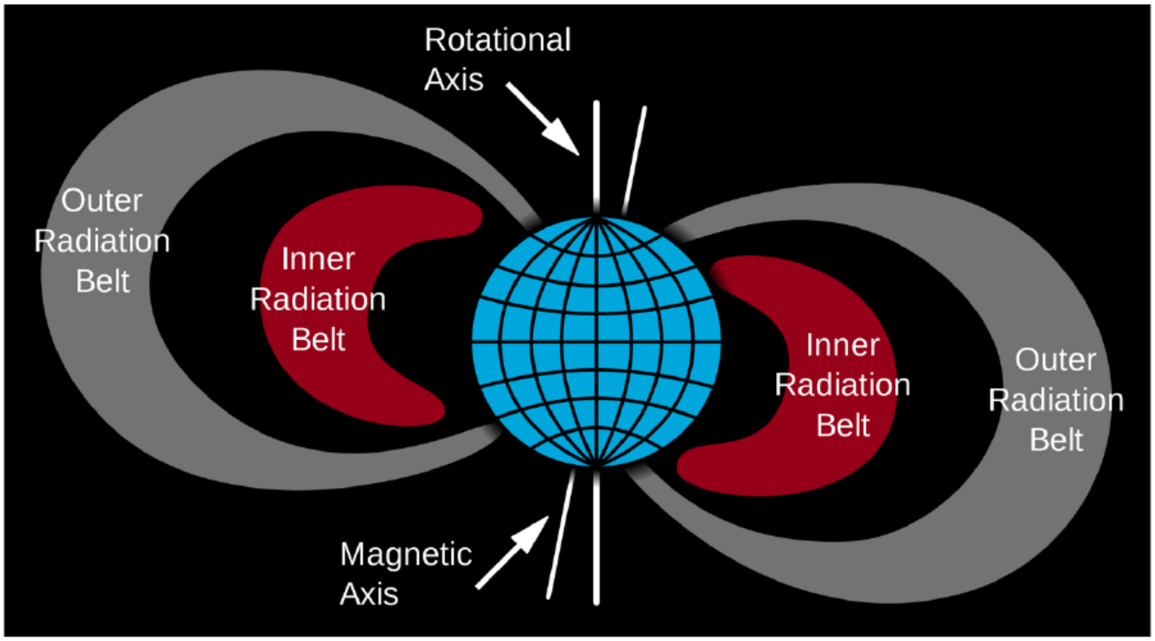
Hubble (optical, red) + Chandra (X-ray, blue) composite image

3. Jupyter's magnetosphere and Earth's van Allen belts

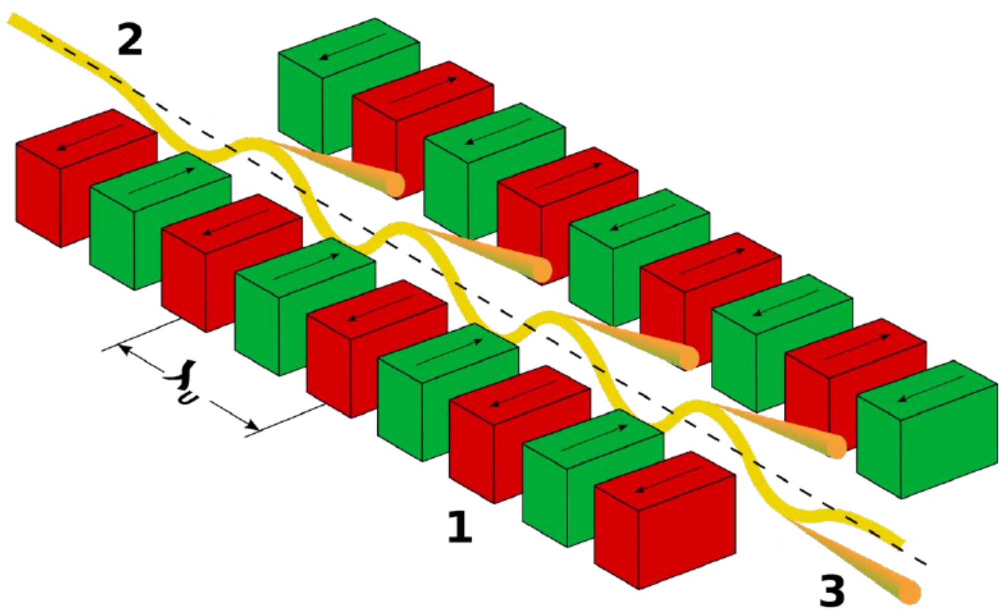
Synchrotron radiation (radio waves) from electrons trapped in Jupyter's magnetic field. It appears to wobble due to the planet's rotation and misalignment of magnetic field with the rotation axis.



A cross section of Earth's Van Allen radiation belts



4. Undulator as synchrotron radiation source



Further information: https://en.wikipedia.org/wiki/Synchrotron_radiation

E.g., ESRF (European Synchrotron Radiation Facility) Grenoble, France.

The most intense source of synchrotron light: X-rays 10^{11} times brighter than in hospitals. Produced in an electron storage ring 844 metres in circumference.

Physical explanation of the critical frequency

How long does the cone point towards us?
In retarded time it is given by the interval it takes for the charge to traverse the cone opening angle $\Delta\theta$:

$$\Delta\bar{t} = \frac{\rho\Delta\theta}{c} \propto \frac{\rho}{\gamma c}$$

But the observed length of this pulse is

$$\Delta t = \left(1 - \hat{x}\cdot\vec{\beta}\right)\Delta\bar{t} \approx (1-\beta)\Delta\bar{t} \approx \frac{1}{2\gamma^2}\Delta\bar{t} = \frac{\rho}{2\gamma^3 c}$$

So, we observe flashes of light every $2\pi\rho/c$ seconds, each of them an impulse of length Δt.
Then the typical frequency dominating the signal is

$$\omega \propto \frac{1}{\Delta t} = \frac{2\gamma^3 c}{\rho} \quad \text{which is consistent with} \quad \omega_c = \frac{3}{2}\gamma^3\frac{c}{\rho}$$

A simple explanation of the $1-\beta$ factor

Charge travels towards us with velocity *v* and we are at distance *r* from it.

Leading edge of pulse leaves charge at time 0, arrives at us at $t_0 = r/c$.

Trailing edge of pulse leaves charge at time $\Delta\bar{t}$, but by then distance is $r - v\Delta\bar{t}$, so it arrives at us at $t_0 + \Delta t = \Delta\bar{t} + (r - v\Delta\bar{t})/c$.

Therefore

$$\Delta t = \Delta\bar{t} + \frac{r - v\Delta\bar{t}}{c} - \frac{r}{c} = (1-\beta)\Delta\bar{t}$$

Field of a uniformly moving charge in a homogeneous medium

$$\rho = q\delta^{(3)}(\vec{x} - \vec{v}t) \qquad \vec{J} = q\vec{v}\delta^{(3)}(\vec{x} - \vec{v}t)$$

$$(\Delta - \mu\epsilon\partial_t^2)\Phi = -\frac{\rho}{\epsilon} \quad (\Delta - \mu\epsilon\partial_t^2)\vec{A} = -\mu\vec{J} = \mu\epsilon\vec{v}\left(-\frac{\rho}{\epsilon}\right)$$

Take Fourier transform of all fields:

$$f(t,\vec{x}) = \frac{1}{(2\pi)^4}\int d\omega d^3k\, f(\omega,\vec{k})e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

$$f(\omega,\vec{k}) = \int dt d^3x\, f(t,\vec{x})e^{-i\vec{k}\cdot\vec{x} + i\omega t}$$

For the charge density

$$\rho(\omega,\vec{k}) = q\int dt d^3x\, \delta^{(3)}(\vec{x} - \vec{v}t)e^{-i\vec{k}\cdot\vec{x} + i\omega t}$$

$$= q\int dt\, e^{i(\omega - \vec{k}\cdot\vec{v})t} = 2\pi q\delta(\omega - \vec{k}\cdot\vec{v})$$

$$(-k^2 + \mu\epsilon\omega^2)\Phi(\omega,\vec{k}) = -\frac{2\pi q}{\epsilon}\delta(\omega - \vec{k}\cdot\vec{v})$$

Potentials in Fourier space

$$\Phi(\omega,\vec{k}) = \frac{2\pi q}{\epsilon}\frac{\delta(\omega - \vec{k}\cdot\vec{v})}{k^2 - \mu\epsilon\omega^2} \qquad \vec{A}(\omega,\vec{k}) = \mu\epsilon\vec{v}\Phi(\omega,\vec{k})$$

$$\vec{E} = -\vec{\nabla}\Phi - \partial_t\vec{A} \;\Rightarrow \vec{E}(\omega,\vec{k}) = -i\vec{k}\Phi(\omega,\vec{k}) + i\omega\vec{A}(\omega,\vec{k})$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \qquad \Rightarrow \vec{B}(\omega,\vec{k}) = i\vec{k}\times\vec{A}(\omega,\vec{k})$$

Fields in Fourier space

$$\vec{E}(\omega,\vec{k}) = \frac{2\pi i q}{\epsilon}(\omega\mu\epsilon\vec{v} - \vec{k})\frac{\delta(\omega - \vec{k}\cdot\vec{v})}{k^2 - \mu\epsilon\omega^2} \qquad \vec{B}(\omega,\vec{k}) = \mu\epsilon\vec{v}\times\vec{E}(\omega,\vec{k})$$

Next time we transform back to real space and compute Cherenkov radiation:
(This is also the starting point for transition radiation):

Frequency decomposition of field of charge in uniform motion in a medium

$$E_x(\omega,\vec{x}) = \frac{q\lambda}{4\pi\epsilon v}2K_1(\lambda r) \quad E_y(\omega,\vec{x}) = 0 \qquad E_z(\omega,\vec{x}) = -\frac{iq\lambda^2}{4\pi\epsilon\omega v}2K_0(\lambda r)$$

$$B_x(\omega,\vec{x}) = 0 \qquad B_y(\omega,\vec{x}) = \mu\epsilon v E_x(\omega,\vec{x}) \qquad B_z(\omega,\vec{x}) = 0$$

$$\lambda = \frac{\omega}{v}\sqrt{1 - \mu\epsilon v^2} = \frac{\omega}{v}\sqrt{1 - \frac{v^2}{c_n^2}} \qquad \vec{v} = (0,0,v) \quad r\text{: distance from line of motion}$$

Week 12: Cherenkov radiation, Brehmsstrahlung, transition radiation

Field of a uniformly moving charge in a homogeneous medium

$$\rho = q\delta^{(3)}(\vec{x} - \vec{v}t) \qquad \vec{J} = q\vec{v}\delta^{(3)}(\vec{x} - \vec{v}t)$$

$$(\Delta - \mu\epsilon\partial_t^2)\Phi = -\frac{\rho}{\epsilon} \quad (\Delta - \mu\epsilon\partial_t^2)\vec{A} = -\mu\vec{J} = \mu\epsilon\vec{v}\left(-\frac{\rho}{\epsilon}\right)$$

Take Fourier transform of all fields:

$$f(t,\vec{x}) = \frac{1}{(2\pi)^4}\int d\omega d^3k\, f(\omega,\vec{k})e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

$$f(\omega,\vec{k}) = \int dt d^3x\, f(t,\vec{x})e^{-i\vec{k}\cdot\vec{x} + i\omega t}$$

For the charge density

$$\rho(\omega,\vec{k}) = q\int dt d^3x\, \delta^{(3)}(\vec{x} - \vec{v}t)e^{-i\vec{k}\cdot\vec{x} + i\omega t}$$

$$= q\int dt\, e^{i(\omega - \vec{k}\cdot\vec{v})t} = 2\pi q\delta(\omega - \vec{k}\cdot\vec{v})$$

$$(-k^2 + \mu\epsilon\omega^2)\Phi(\omega,\vec{k}) = -\frac{2\pi q}{\epsilon}\delta(\omega - \vec{k}\cdot\vec{v})$$

<u>Potentials in Fourier space</u>

$$\Phi(\omega,\vec{k}) = \frac{2\pi q}{\epsilon}\frac{\delta(\omega - \vec{k}\cdot\vec{v})}{k^2 - \mu\epsilon\omega^2} \qquad \vec{A}(\omega,\vec{k}) = \mu\epsilon\vec{v}\Phi(\omega,\vec{k})$$

$$\vec{E} = -\vec{\nabla}\Phi - \partial_t\vec{A} \;\Rightarrow \vec{E}(\omega,\vec{k}) = -i\vec{k}\Phi(\omega,\vec{k}) + i\omega\vec{A}(\omega,\vec{k})$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \qquad\qquad \Rightarrow \vec{B}(\omega,\vec{k}) = i\vec{k}\times\vec{A}(\omega,\vec{k})$$

<u>Fields in Fourier space</u>

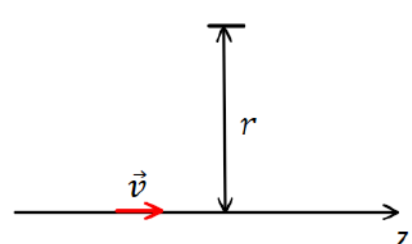
$$\vec{E}(\omega,\vec{k}) = \frac{2\pi iq}{\epsilon}(\omega\mu\epsilon\vec{v} - \vec{k})\frac{\delta(\omega - \vec{k}\cdot\vec{v})}{k^2 - \mu\epsilon\omega^2} \qquad \vec{B}(\omega,\vec{k}) = \mu\epsilon\vec{v}\times\vec{E}(\omega,\vec{k})$$

Back to real space, keeping frequency decomposition

Choose coordinate system so that

$\vec{v} = (0,0,v) \quad \vec{x} = (0,0,r)$

i.e., we compute the frequency decomposition of the field observed at a distance r from straight line trajectory of charge.



$$
\begin{aligned}
E_z(\omega,\vec{x}) &= \frac{2\pi i q}{\epsilon}\int \frac{d^3k}{(2\pi)^3} e^{ik_x r}(\omega\mu\epsilon v - k_z)\frac{\delta(\omega - k_z v)}{k_x^2+k_y^2+k_z^2-\mu\epsilon\omega^2} \\
&= \frac{iq}{4\pi^2\epsilon}\int dk_x dk_y \frac{1}{v} e^{ik_x r}\left(\underbrace{\omega\mu\epsilon v - \frac{\omega}{v}}_{-\lambda^2 v/\omega}\right)\frac{1}{k_x^2+k_y^2+\underbrace{\left(\frac{\omega}{v}\right)^2-\mu\epsilon\omega^2}_{\lambda^2}} \\
&= -\frac{iq\lambda^2}{4\pi^2\epsilon\omega}\int dk_x dk_y\, e^{ik_x r}\frac{1}{k_x^2+k_y^2+\lambda^2} \qquad \lambda = \frac{\omega}{v}\sqrt{1-\mu\epsilon v^2} = \frac{\omega}{v}\sqrt{1-\frac{v^2}{c_n^2}} \\
&= -\frac{iq\lambda^2}{4\pi\epsilon\omega}\int_{-\infty}^{\infty} dk_x\, e^{ik_x r}\frac{1}{\sqrt{k_x^2+\lambda^2}} \qquad \text{using} \quad \int_{-\infty}^{\infty} dx\,\frac{1}{x^2+a^2} = \frac{\pi}{a} \\
&= -\frac{iq\lambda^2}{4\pi\epsilon\omega v}\int_{-\infty}^{\infty} ds\,\frac{e^{is\lambda r}}{\sqrt{s^2+1}} \qquad \text{where} \quad k_x = \lambda s \\
&= -\frac{iq\lambda^2}{4\pi\epsilon\omega} 2K_0(\lambda r)
\end{aligned}
$$

Homework: derive similarly that

$$
\begin{aligned}
E_x(\omega,\vec{x}) &= \frac{2\pi i q}{\epsilon}\int \frac{d^3k}{(2\pi)^3} e^{ik_x r}(-k_x)\frac{\delta(\omega - k_z v)}{k_x^2+k_y^2+k_z^2-\mu\epsilon\omega^2} \\
&= -\frac{iq\lambda}{4\pi\epsilon v}\int_{-\infty}^{\infty} ds\,\frac{s e^{is\lambda r}}{\sqrt{s^2+1}}
\end{aligned}
$$

$$
E_x(\omega,\vec{x}) = \frac{q\lambda}{4\pi\epsilon v} 2K_1(\lambda r)
$$

Also note that
$E_y(\omega,\vec{x}) = 0 \quad \vec{B}(\omega,\vec{x}) = (0, \mu\epsilon v E_x(\omega,\vec{x}), 0)$

The end result is

Frequency decomposition of field of charge in uniform motion in a medium

$$
E_x(\omega,\vec{x}) = \frac{q\lambda}{4\pi\epsilon v} 2K_1(\lambda r) \quad E_y(\omega,\vec{x}) = 0 \qquad E_z(\omega,\vec{x}) = -\frac{iq\lambda^2}{4\pi\epsilon\omega v} 2K_0(\lambda r)
$$

$$
B_x(\omega,\vec{x}) = 0 \qquad B_y(\omega,\vec{x}) = \mu\epsilon v E_x(\omega,\vec{x}) \qquad B_z(\omega,\vec{x}) = 0
$$

$$
\lambda = \frac{\omega}{v}\sqrt{1-\mu\epsilon v^2} = \frac{\omega}{v}\sqrt{1-\frac{v^2}{c_n^2}}
$$

Large distance asymptotics

$$K_0(\xi) = \sqrt{\frac{\pi}{2\xi}}e^{-\xi}(1+O(\xi^{-1})) \quad K_1(\xi) = \sqrt{\frac{\pi}{2\xi}}e^{-\xi}(1+O(\xi^{-1}))$$

$$E_x(\omega,\vec{x}) \approx \frac{q}{4\pi\epsilon v}\sqrt{\frac{2\pi\lambda}{r}}e^{-\lambda r} \quad E_y(\omega,\vec{x}) = 0 \quad E_z(\omega,\vec{x}) \approx -\frac{i\lambda v}{\omega}E_x(\omega,\vec{x})$$

$$B_x(\omega,\vec{x}) = 0 \quad B_y(\omega,\vec{x}) = \mu\epsilon v E_x(\omega,\vec{x}) \quad B_z(\omega,\vec{x}) = 0$$

Cherenkov radiation

For a given frequency component there are two cases:

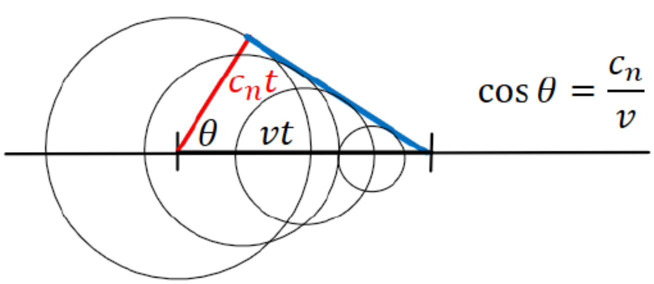
- $v < c_n(\omega)$: fields decay exponentially away from trajectory
- $v > c_n(\omega)$: λ imaginary - fields go as $r^{-1/2}$ - *cylindrical radiation wave!*

$$\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B} \Rightarrow S_x = -\epsilon v E_x E_z \quad S_y = 0 \quad S_z = \epsilon v E_x^2$$

Angle of radiation

$$\tan\theta = \frac{S_x}{S_z} = -\frac{E_z}{E_x} = \frac{i\lambda v}{\omega} \quad \lambda = \frac{\omega}{v}\sqrt{1-\frac{v^2}{c_n^2}} = i\frac{\omega}{v}\sqrt{\frac{v^2}{c_n^2}-1}$$

$$\tan\theta = -\sqrt{\frac{v^2}{c_n^2}-1} \quad \cos^2\theta = \frac{1}{1+\tan^2\theta} = \frac{c_n^2}{v^2}$$

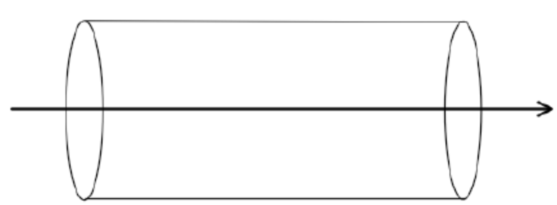


Geometry of wave front emitted by the charge after time *t*

Energy radiated per unit distance travelled

Radiated power:

$$P = \frac{dW}{dt} = \oiint d\vec{f}\cdot\vec{S} = 2\pi r\int_{-\infty}^{\infty} dz\ S_x$$



$$\frac{dW}{dz} = \frac{1}{v}\frac{dW}{dt} = 2\pi r \int_{-\infty}^{\infty} \frac{dz}{v}\, S_x = -2\pi r \int_{-\infty}^{\infty} dt \frac{1}{\mu} E_z(t) B_y(t)$$

Computing the time integral in Fourier space

$$\begin{aligned}\int_{-\infty}^{\infty} dt\, f(t) g(t) &= \int_{-\infty}^{\infty} dt \int \frac{d\omega}{2\pi} e^{i\omega t} f(\omega) \int \frac{d\omega'}{2\pi} e^{i\omega' t} g(\omega') \\ &= \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} 2\pi\delta(\omega+\omega') f(\omega)\, g(\omega') = \int \frac{d\omega}{2\pi} f(\omega)\, g(-\omega) \\ &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(\omega) g(\omega)^*\end{aligned}$$

$$\frac{dW}{dz} = -2r \int_0^{\infty} d\omega \frac{1}{\mu} E_z(\omega) B_y(\omega)^* = -2r \int_0^{\infty} d\omega \frac{1}{\mu} \left(-\frac{i\lambda v}{\omega} E_x(\omega, \vec{x})\right) \mu\epsilon v E_x(\omega, \vec{x})^*$$

$$E_x(\omega, \vec{x}) \approx \frac{q}{4\pi\epsilon v} \sqrt{\frac{2\pi\lambda}{r}} e^{-\lambda r} \qquad \lambda = \frac{\omega}{v}\sqrt{1 - \frac{v^2}{c_n^2}} = i\frac{\omega}{v}\sqrt{\frac{v^2}{c_n^2} - 1}$$

We look at radiation i.e., $r \to \infty$: only frequencies with $c_n(\omega) < v$ contribute!

$$\begin{aligned}\frac{dW}{dz} &= -2r \int_{c_n(\omega)<v} d\omega \left(-\frac{i\lambda\epsilon v^2}{\omega}\right) |E_x(\omega, \vec{x})|^2 \\ &= -2 \int_{c_n(\omega)<v} d\omega \frac{\epsilon v^2}{\omega} \frac{\omega}{v} \sqrt{\frac{v^2}{c_n^2} - 1}\, \frac{q^2}{16\pi^2\epsilon^2 v^2} 2\pi|\lambda| \\ &= -\frac{q^2}{4\pi} \int_{c_n(\omega)<v} d\omega\omega\, \mu(\omega) \left|1 - \frac{c_n^2(\omega)}{v^2}\right|\end{aligned}$$

Frank-Tamm formula

$$\frac{dW}{dz} = -\frac{q^2}{4\pi} \int_{v > \frac{c}{n(\omega)}} d\omega\, \omega\mu(\omega) \left|1 - \frac{c^2}{v^2 n(\omega)^2}\right| \qquad n(\omega) = \sqrt{\epsilon_r(\omega)\mu_r(\omega)}$$

Nobel prize (1958)
Cherenkov: experimental discovery   Frank & Tamm: theoretical explanation

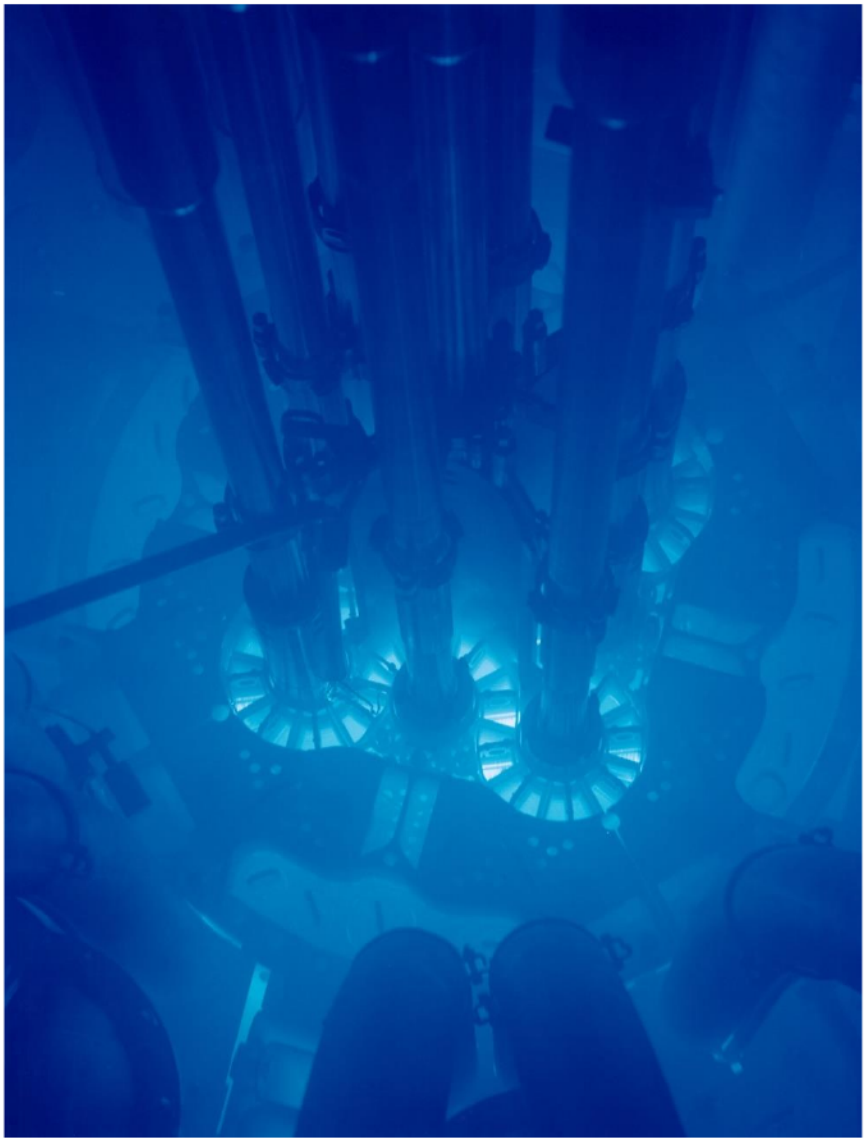
Note: for large frequencies

$$n(\omega) \approx \sqrt{1 - \frac{\omega_p^2}{\omega^2}} < 1$$

So, frequencies above plasma frequency do not contribute.

But: $\int d\omega\, \omega$ is dominated by highest allowed frequencies, typically blue and soft UV

⇒ Cherenkov radiation is normally a faint bluish glow!



Optional material: Brehmsstrahlung

Imagine that the charged particle with charge $q = ze$ has a collision which changes the velocity of the particle from an initial value $c\vec{\beta}$ to a final value $c\vec{\beta}'$ in a time τ.

Last time we derived

$$\frac{d^2W}{d\omega d\Omega} = 2\left|\vec{C}(\omega)\right|^2 \quad \vec{C}(\omega) = \frac{q\sqrt{Z_0}}{\sqrt{32\pi^3}}\int_{-\infty}^{\infty} dt\, e^{i\omega\left(t-\frac{\hat{x}\cdot\vec{\xi}(t)}{c}\right)}\frac{d}{dt}\left(\frac{\hat{x}\times\left(\hat{x}\times\vec{\beta}(t)\right)}{1-\hat{x}\cdot\vec{\beta}(t)}\right)$$

so

$$\frac{d^2W}{d\omega d\Omega} = \frac{Z_0 q^2}{16\pi^3}\left|\vec{e}^*\cdot\int_{-\infty}^{\infty} dt\,\frac{d}{dt}\left(\frac{\hat{x}\times\left(\hat{x}\times\vec{\beta}(t)\right)}{1-\hat{x}\cdot\vec{\beta}(t)}\right)e^{i\omega\left(t-\frac{\hat{x}\cdot\vec{\xi}(t)}{c}\right)}\right|^2$$

Let us compute the low-frequency limit $\omega \ll \tau^{-1}$:

$$\lim_{\omega\to 0}\frac{d^2W}{d\omega d\Omega} = \frac{Z_0 q^2}{16\pi^3}\left|\vec{e}^*\cdot\int_{-\infty}^{\infty} dt\,\frac{d}{dt}\left(\frac{\hat{x}\times\left(\hat{x}\times\vec{\beta}(t)\right)}{1-\hat{x}\cdot\vec{\beta}(t)}\right)\right|^2$$

$$= \frac{Z_0 q^2}{16\pi^3}\left|\vec{e}^*\cdot\left(\frac{\hat{x}\times\left(\hat{x}\times\vec{\beta}'\right)}{1-\hat{x}\cdot\vec{\beta}'} - \frac{\hat{x}\times\left(\hat{x}\times\vec{\beta}\right)}{1-\hat{x}\cdot\vec{\beta}}\right)\right|^2$$

Using $\vec{e}^*\cdot\hat{x} = 0$

$$\lim_{\omega\to 0}\frac{d^2W}{d\omega d\Omega} = \frac{Z_0 q^2}{16\pi^3}\left|\vec{e}^*\cdot\left(\frac{\vec{\beta}'}{1-\hat{x}\cdot\vec{\beta}'} - \frac{\vec{\beta}}{1-\hat{x}\cdot\vec{\beta}}\right)\right|^2$$

Quantum case: we can compute the number of photons by dividing by $\hbar\omega$:

$$\lim_{\omega\to 0}\frac{d^2N}{d\omega d\Omega} = \frac{z^2e^2}{16\pi^3\epsilon_0 c\hbar\omega}\left|\vec{e}^*\cdot\left(\frac{\vec{\beta}'}{1-\hat{x}\cdot\vec{\beta}'} - \frac{\vec{\beta}}{1-\hat{x}\cdot\vec{\beta}}\right)\right|^2$$

i.e.

$$\lim_{\hbar\omega\to 0}\frac{d^2N}{d(\hbar\omega)d\Omega} = \underbrace{\frac{e^2}{4\pi\epsilon_0\hbar c}}_{\alpha\approx\frac{1}{137}}\frac{z^2}{4\pi^2\hbar\omega}\left|\vec{e}^*\cdot\left(\frac{\vec{\beta}'}{1-\hat{x}\cdot\vec{\beta}'} - \frac{\vec{\beta}}{1-\hat{x}\cdot\vec{\beta}}\right)\right|^2$$

We therefore get

$$\lim_{\hbar\omega\to 0}\frac{d^2N}{d(\hbar\omega)d\Omega} = \frac{z^2\alpha}{4\pi^2\hbar\omega}\left|\vec{e}^*\cdot\left(\frac{\vec{\beta}'}{1-\hat{x}\cdot\vec{\beta}'} - \frac{\vec{\beta}}{1-\hat{x}\cdot\vec{\beta}}\right)\right|^2$$

This can be written in a relativistic way:

$$p^\mu = Mc\left(\gamma,\gamma\vec{\beta}\right) \quad p'^\mu = Mc\left(\gamma',\gamma'\vec{\beta}'\right) \quad k^\mu = \frac{\omega}{c}(1,\hat{x}) \quad \epsilon^\mu = (0,\vec{e})$$

$$\epsilon^*\cdot p = Mc\gamma\vec{e}^*\cdot\vec{\beta} \qquad \epsilon^*\cdot p' = Mc\gamma'\vec{e}^*\cdot\vec{\beta}'$$
$$k\cdot p = M\omega\gamma\left(1-\hat{x}\cdot\vec{\beta}\right) \quad k\cdot p' = M\omega\gamma'\left(1-\hat{x}\cdot\vec{\beta}'\right)$$

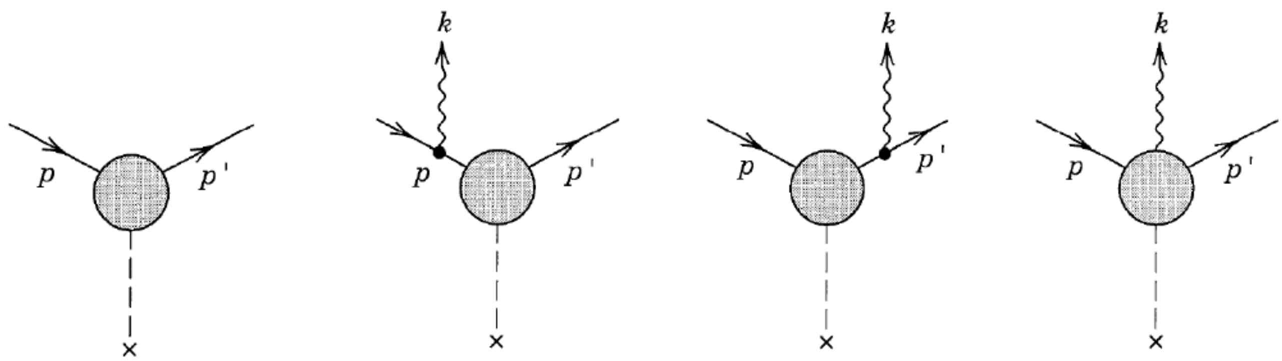
$$\lim_{\hbar\omega\to 0}\frac{d^2N}{d(\hbar\omega)d\Omega} = \frac{z^2\alpha}{4\pi^2\hbar\omega}\frac{\omega^2}{c^2}\left|\frac{\epsilon^*\cdot p'}{k\cdot p'} - \frac{\epsilon^*\cdot p}{k\cdot p}\right|^2$$
$$\lim_{\omega\to 0}\frac{d^2N}{(\omega/c)d(\omega/c)d\Omega} = \frac{z^2\alpha}{4\pi^2}\left|\frac{\epsilon^*\cdot p'}{k\cdot p'} - \frac{\epsilon^*\cdot p}{k\cdot p}\right|^2$$

Now use that $k_0 = \omega/c = |\vec{k}|$ so $d^3k = k_0^2 dk_0 d\Omega$ to write

$$\lim_{\omega\to 0} \frac{d^3N}{d^3k/k_0} = \frac{z^2\alpha}{4\pi^2}\left|\frac{\epsilon^*\cdot p'}{k\cdot p'} - \frac{\epsilon^*\cdot p}{k\cdot p}\right|^2$$

Remark: d^3k/k_0 is a Lorentz invariant combination.

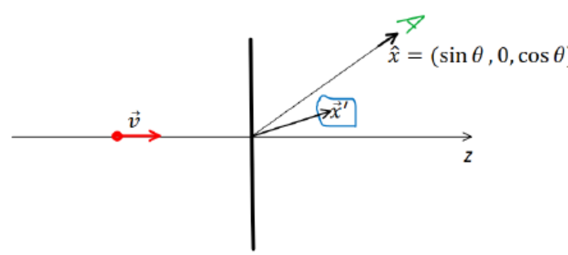
The same can be derived in Quantum Electrodynamics from the Feynman diagrams



Original scattering Emission of soft photons

Optional material: transition radiation

Assume a particle is passing from vacuum to a medium, perpendicular to the boundary surface



The medium on the other side becomes polarised, and a small piece around $\vec{x}'$ produces a radiation of the form

(radiated field observed at point $\vec{x}$) $\propto \vec{P}(\vec{x}')\dfrac{e^{ikr}}{r}e^{-ik\hat{x}\cdot\vec{x}'}$

$\vec{P}(\vec{x}')$: polarisation of the medium at point $\vec{x}' = \underbrace{(\rho'\cos\phi', \rho'\sin\phi', z')}_{\text{cylindrical coordinates}}$

Now we use the frequency decomposition of the field of the charge

$$E_\rho(\omega,\vec{x}) \approx \frac{q}{4\pi\epsilon v}\sqrt{\frac{2\pi\lambda}{\rho}}e^{-\lambda\rho} \quad E_\phi(\omega,\vec{x}) = 0 \quad E_z(\omega,\vec{x}) \approx -\frac{i\lambda v}{\omega}E_\rho(\omega,\vec{x})$$

$$B_\rho(\omega,\vec{x}) = 0 \quad B_\phi(\omega,\vec{x}) = \mu\epsilon v E_\rho(\omega,\vec{x}) \quad B_z(\omega,\vec{x}) = 0$$

$\lambda = \dfrac{\omega}{v}\sqrt{1-\mu_0\epsilon_0 v^2} = \dfrac{\omega}{v}\sqrt{1-\dfrac{v^2}{c^2}} = \dfrac{\omega}{v\gamma}$ note that now this is always real!

Because of the exponential decay away from the z axis, only the cylindrical region

$$\rho' \lesssim \lambda^{-1} = \frac{\gamma v}{\omega}$$

Let us consider the phases!

Incoming field (moving charge):

$$\rho(\omega, \vec{k}) = 2\pi q \delta(\omega - k_z v) \Rightarrow k_z = \frac{\omega}{v}$$

Outgoing field (radiation):

$$\frac{e^{ikr}}{r} e^{-ik\hat{x}\cdot\vec{x}'} = \frac{e^{ikr}}{r} e^{-ik(z'\cos\theta + \rho'\sin\theta\cos\phi)}$$

Total dependence of the phase on the radiator position

$$e^{i\frac{\omega}{v}z'} e^{-i\frac{\omega}{cn(\omega)}(z'\cos\theta + \rho'\sin\theta\cos\phi)} = e^{i\frac{\omega}{c}\left(\frac{1}{\beta} - n(\omega)\cos\theta\right)z'} e^{-i\frac{\omega n(\omega)}{c}\rho'\sin\theta\cos\phi}$$

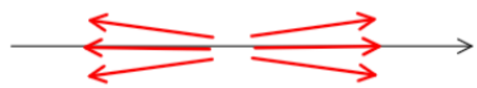
We must integrate over $\vec{x}'$ but: regions with rapidly varying phase are suppressed! This is the stationary phase approximation (useful in optics, and quantum mechanics).

So

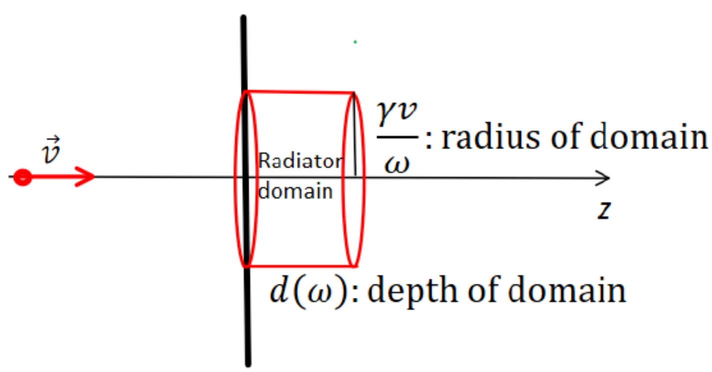
$$\int d\phi' : \ \frac{\omega n(\omega)}{c}\rho' \sin\theta \lesssim 1 \Rightarrow \ \frac{\omega n(\omega)}{c}\frac{\gamma v}{\omega} \sin\theta \lesssim 1$$

UR motion: $v \approx c$ also $n \sim O(1)$

$$\sin\theta \lesssim \frac{1}{\gamma} \ \Rightarrow \theta \lesssim \frac{1}{\gamma} \ \text{ or } \ \pi - \theta \lesssim \frac{1}{\gamma}$$



$$\int dz' : \ \frac{\omega}{c}\left(\frac{1}{\beta} - n(\omega)\cos\theta\right) d(\omega) \simeq 1$$



Typically: high frequencies (X-ray)

$$n(\omega) \approx 1 - \frac{\omega_p^2}{2\omega^2}$$

UR motion:

$$\frac{1}{\beta} \approx 1 + \frac{1}{2\gamma^2} \ \Rightarrow \ d(\omega) = \frac{2\gamma c}{\omega_p}\frac{1}{\nu + \nu^{-1}} \qquad \nu = \frac{\omega}{\gamma\omega_p}$$

Maximum depth where radiation comes from

$$d(\omega) = \frac{\gamma c}{\omega_p}$$

For typical materials: $c/\omega_p \simeq 10^{-8}\, m = 0.01\, \mu m$

Typical frequency is determined from size of radiating region

$$V(\omega) = \pi \rho_{max}(\omega)^2 d(\omega) \simeq 2\pi \left(\frac{c}{\omega_p}\right)^3 \frac{1}{\nu(1+\nu^2)}$$

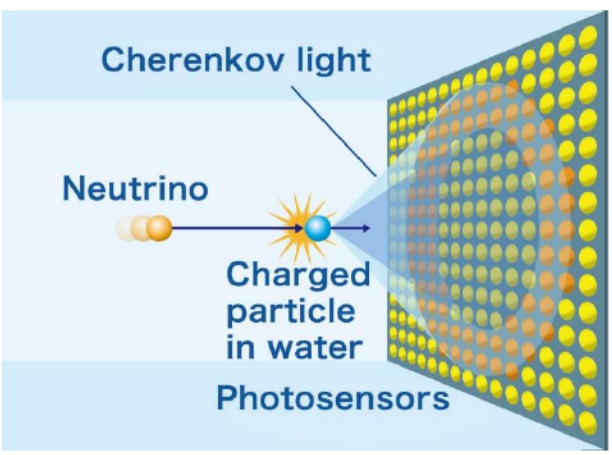
Cut-off frequency: $\nu \sim 1$

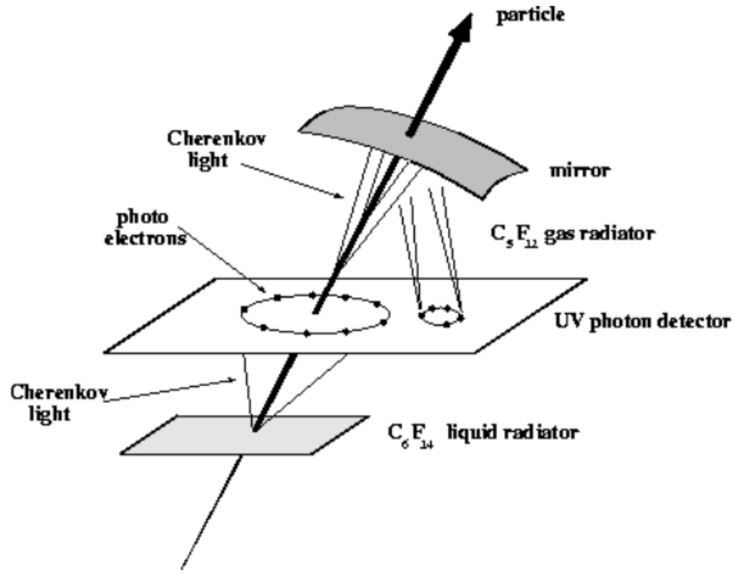
$\omega_{cutoff} \simeq \gamma \omega_p \quad \omega_p\text{: UV and } \gamma \gg 1 \Rightarrow \text{X} - \text{ray!}$

Particle detectors based on Cherenkov and transition radiation

Cherenkov detector

Can measure velocity and direction of motion as well!





Ring imaging Cherenkov detector (RICH), used at the LHC in CERN

Transition radiation detector

Radiation happens both on entry and exit, same intensity: use a stack of thin foils ($10 - 100\, \mu m$) to enhance signal, X-ray photons can be detected by scintillators.

Week 13: Radiation backreaction

Backreaction from radiation

We consider non-relativistic motion for simplicity

Energy loss of an accelerated charge: Larmor formula

$$P_{rad} = \frac{Z_0 q^2 a^2}{6\pi c^2} = \frac{q^2 a^2}{6\pi\epsilon_0 c^3}$$

$$m\frac{d\vec{v}}{dt} = \vec{F}_{ext} + \vec{F}_{rad}$$

$\vec{F}_{rad}$: force of backreaction from radiation energy loss

- Vanishes when $\dot{\vec{v}} = 0$
- Work made by $\vec{F}_{rad}$ accounts for P_{rad}
- Proportional to q^2

$$\int_{t_1}^{t_2} dt\, \vec{F}_{rad} \cdot \vec{v} = -\int_{t_1}^{t_2} dt\, P_{rad} = -\frac{q^2}{6\pi\epsilon_0 c^3}\int_{t_1}^{t_2} dt\, \dot{\vec{v}}^2$$

$$\int_{t_1}^{t_2} dt\, \dot{\vec{v}}^2 = \left[\vec{v}\cdot\dot{\vec{v}}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \ddot{\vec{v}}\cdot\vec{v}$$

Assuming first term is zero (e.g., no acceleration outside interval or periodic motion)

<u>Abraham-Lorentz force</u>

$$\vec{F}_{rad} = \frac{q^2}{6\pi\epsilon_0 c^3}\ddot{\vec{v}} = m\tau\ddot{\vec{v}} \qquad \underbrace{\tau = \frac{q^2}{6\pi\epsilon_0 c^3 m}}_{\text{characteristic time scale}}$$

Particle with longest time scale: electron

$\tau \approx 6.2 \cdot 10^{-24}\ s$

Where does it really come from? Is it possible to give a proper derivation?

Abraham-Lorentz derivation of radiation backreaction

Idea: radiation loss is carried away by EM field of charge

⇒ accelerating charge must interact with its own EM field!

$$\left(\frac{d\vec{p}}{dt}\right)_{mech} = \int d^3x \left(\rho\vec{E} + \vec{J}\times\vec{B}\right)$$

$$\vec{E} = \vec{E}_{ext} + \vec{E}_{self} \qquad \vec{B} = \vec{B}_{ext} + \vec{B}_{self}$$

$$\left(\frac{d\vec{p}}{dt}\right)_{mech} + \left(\frac{d\vec{p}}{dt}\right)_{EM} = \vec{F}_{ext}$$

$$\left(\frac{d\vec{p}}{dt}\right)_{EM} = -\int d^3x \left(\rho\vec{E}_s + \vec{J}\times\vec{B}_s\right)$$

Assumptions:

- Particle is instantaneously at rest (choice of reference system)
- Charge distribution of particle is rigid and spherical occupying a small volume V

$$\left(\frac{d\vec{p}}{dt}\right)_{EM} = -\int_V d^3x\, \rho(t,\vec{x})\vec{E}_s(t,\vec{x}) = \int_V d^3x\, \rho(t,\vec{x})\left(\vec{\nabla}\Phi(t,\vec{x}) + \frac{\partial\vec{A}(t,\vec{x})}{\partial t}\right)$$

Retarded potentials:

$$\Phi(t,\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x'\, \frac{[\rho(t',\vec{x}')]_{ret}}{|\vec{x}-\vec{x}'|} \qquad [\ldots]_{ret}: t' = t - \frac{|\vec{x}-\vec{x}'|}{c}$$

$$\vec{A}(t,\vec{x}) = \frac{\mu_0}{4\pi}\int_V d^3x'\, \frac{\left[\vec{J}(t',\vec{x}')\right]_{ret}}{|\vec{x}-\vec{x}'|}$$

Assume that particle is highly localised: retardation time is short.
During retardation time: motion of charge changes only slightly
⇒ use Taylor expansion to evaluate effect of retardation

$$[\ldots]_{ret} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{R}{c}\right)^n \frac{\partial^n}{\partial t^n}[\ldots]_{t'=t} \qquad \text{notation: } R = |\vec{x}-\vec{x}'|$$

$$\Phi(t,\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x'\, \frac{1}{R}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{R}{c}\right)^n \frac{\partial^n\rho(t,\vec{x}')}{\partial t^n}$$

$$\vec{A}(t,\vec{x}) = \frac{\mu_0}{4\pi}\int_V d^3x'\, \frac{1}{R}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{R}{c}\right)^n \frac{\partial^n\vec{J}(t,\vec{x}')}{\partial t^n}$$

$$\begin{aligned}\left(\frac{d\vec{p}}{dt}\right)_{EM} &= \int_V d^3x\, \rho(t,\vec{x})\left(\vec{\nabla}\Phi(t,\vec{x}) + \frac{\partial\vec{A}(t,\vec{x})}{\partial t}\right)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,c^n 4\pi\epsilon_0}\int_V d^3x\, \int_V d^3x'\, \rho(t,\vec{x})\frac{\partial^n}{\partial t^n}\left[\rho(t,\vec{x}')\vec{\nabla}(R^{n-1}) + \frac{R^{n-1}}{c^2}\frac{\partial\vec{J}(t,\vec{x}')}{\partial t}\right]\end{aligned}$$

First two terms in scalar potential part:

$$n = 0: \frac{1}{4\pi\epsilon_0}\int_V d^3x\, \int_V d^3x'\, \rho(t,\vec{x})\rho(t,\vec{x}')\vec{\nabla}\frac{1}{R}$$

This is the static self-force which vanishes for a spherically symmetric distribution.

$n = 1: 0 \text{ since } \vec{\nabla}(R^0) = 0$

We can then relabel the sum in the scalar part

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n!\, c^n 4\pi\epsilon_0} \int_V d^3x \int_V d^3x'\, \rho(t,\vec{x}) \frac{\partial^n}{\partial t^n} \left[\rho(t,\vec{x}')\vec{\nabla}(R^{n-1})\right]$$

$$\to \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, c^{n+2} 4\pi\epsilon_0} \int_V d^3x \int_V d^3x'\, \rho(t,\vec{x}) \frac{\partial^{n+2}}{\partial t^{n+2}} \left[\frac{\rho(t,\vec{x}')\vec{\nabla}(R^{n+1})}{(n+1)(n+2)}\right]$$

So

$$\left(\frac{d\vec{p}}{dt}\right)_{EM} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, c^{n+2} 4\pi\epsilon_0} \int_V d^3x \int_V d^3x'\, \rho(t,\vec{x}) R^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} [\dots]$$

where

$$[\dots] = \frac{\partial \rho(t,\vec{x}')}{\partial t} \frac{\vec{\nabla}(R^{n+1})}{(n+1)(n+2)R^{n-1}} + \vec{J}(t,\vec{x}')$$

$$= \frac{\partial \rho(t,\vec{x}')}{\partial t} \frac{\vec{R}}{n+2} + \vec{J}(t,\vec{x}') \qquad \text{HW: } \vec{\nabla}(R^{n+1}) = (n+1)R^{n-1}\vec{R}$$

Using the continuity equation:

$$[\dots] = \vec{J}(t,\vec{x}') - \frac{\vec{R}}{n+2} \vec{\nabla}' \cdot \vec{J}(t,\vec{x}')$$

$$\left(\frac{d\vec{p}}{dt}\right)_{EM} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, c^{n+2} 4\pi\epsilon_0} \int_V d^3x \int_V d^3x'\, \rho(t,\vec{x}) R^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} \left[\vec{J}(t,\vec{x}') - \frac{\vec{R}}{n+2} \vec{\nabla}' \cdot \vec{J}(t,\vec{x}')\right]$$

We can compute the last term as

$$-\int_V d^3x'\, R^{n-1} \frac{\vec{R}}{n+2} \partial_i'^{J_i(t,\vec{x}')} = \underbrace{\text{surface term}}_{0:\text{ localised source}} + \int_V d^3x' \partial_i'(R^{n-1}\vec{R}) \frac{1}{n+2} J_i(t,\vec{x}')$$

$$= +\frac{1}{n+2} \int_V d^3x'\, \partial_i'(R^{n-1}\vec{R}) J_i(t,\vec{x}')$$

$$= -\frac{1}{n+2} \int_V d^3x' R^{n-1} \left(\vec{J}(t,\vec{x}') + (n-1)\frac{\vec{J}(t,\vec{x}')\cdot\vec{R}}{R^2}\vec{R}\right)$$

$$HW: \partial_i'\left(R^{n-1}R_j\right) = -R^{n-1}\left(\frac{(n-1)R_i}{R^2}R_j + \delta_{ij}\right)$$

So, we can substitute

$$[\dots] \to \vec{J}(t,\vec{x}') + \left\{-\frac{1}{n+2}\left(\vec{J}(t,\vec{x}') + (n-1)\frac{\vec{J}(t,\vec{x}')\cdot\vec{R}}{R^2}\vec{R}\right)\right\}$$

$$= \frac{n+1}{n+2}\vec{J}(t,\vec{x}') - \frac{n-1}{n+2}\frac{\vec{J}(t,\vec{x}')\cdot\vec{R}}{R^2}\vec{R}$$

into

$$\left(\frac{d\vec{p}}{dt}\right)_{EM} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,c^{n+2}4\pi\epsilon_0} \int_V d^3x \int_V d^3x'\,\rho(t,\vec{x})R^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}[\ldots]$$

Furthermore, for a rigid charge density moving with velocity $\vec{v}(t)$

$$\vec{J}(t,\vec{x}') = \rho(t,\vec{x}')\vec{v}(t)$$

so

$$\left(\frac{d\vec{p}}{dt}\right)_{EM} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,c^{n+2}4\pi\epsilon_0} \int_V d^3x\;\rho(t,\vec{x})\frac{\partial^{n+1}}{\partial t^{n+1}} \int_V d^3x'\,R^{n-1}\left[\frac{n+1}{n+2}\rho(t,\vec{x}')\vec{v}(t) - \frac{n-1}{n+2}\frac{\rho(t,\vec{x}')\vec{v}(t)\cdot\vec{R}}{R^2}\vec{R}\right]$$

Consider the second integral term

$$\int_V d^3x'\,R^{n-1}\frac{\rho(t,\vec{x}')\vec{v}(t)\cdot\vec{R}}{R^2}\vec{R} = \int_V d^3x'\,R^{n-1}\frac{\rho(t,\vec{x}')\vec{v}\cdot\vec{R}}{R^2}\left(\frac{\vec{v}\cdot\vec{R}}{v^2}\vec{v} + \vec{R}_\perp\right) \qquad \vec{R}_\perp = \vec{R} - \frac{\vec{v}\cdot\vec{R}}{v^2}\vec{v}$$

Now since ρ is spherically symmetric, the integral still has symmetry of rotation around direction of $\vec{v} \Rightarrow$ the integral over the term containing the component $\vec{R}_\perp$ perpendicular to $\vec{v}$ vanishes.

$$\left(\frac{d\vec{p}}{dt}\right)_{EM} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,c^{n+2}4\pi\epsilon_0} \int_V d^3x\;\rho(t,\vec{x})\frac{\partial^{n+1}}{\partial t^{n+1}} \int_V d^3x'\,R^{n-1}\rho(t,\vec{x}')\vec{v}(t)\left[\frac{n+1}{n+2} - \frac{n-1}{n+2}\left(\frac{\vec{v}\cdot\vec{R}}{vR}\right)^2\right]$$

Due to spherical symmetry, the second term can be replaced by its average over all directions of $\vec{R}$:

$$\left(\frac{\vec{v}\cdot\vec{R}}{vR}\right)^2 \to \frac{1}{3}$$

HW: check that indeed $\frac{1}{4\pi}\int d\Omega\left(\frac{\vec{v}\cdot\vec{R}}{vR}\right)^2 = \frac{1}{3}$

Hint: choose z axis in direction of $\vec{v}$ and use $\vec{R} = R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$\left(\frac{d\vec{p}}{dt}\right)_{EM} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,c^{n+2}4\pi\epsilon_0}\int_V d^3x\ \rho(t,\vec{x})\frac{\partial^{n+1}}{\partial t^{n+1}}\int_V d^3x'\,R^{n-1}\rho(t,\vec{x}')\vec{v}(t)\underbrace{\left[\frac{n+1}{n+2}-\frac{1}{3}\frac{n-1}{n+2}\right]}_{2/3}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,c^{n+2}6\pi\epsilon_0}\int_V d^3x\ \rho(t,\vec{x})\frac{\partial^{n+1}}{\partial t^{n+1}}\int_V d^3x'\,R^{n-1}\rho(t,\vec{x}')\vec{v}(t)$$

Let us expand it to $O(R^0)$, i.e., keep only terms $n=0,1$:

$$\left(\frac{d\vec{p}}{dt}\right)_{EM} = \frac{1}{6\pi c^2\epsilon_0}\int_V d^3x\ \int_V d^3x'\,\frac{\rho(t,\vec{x})\rho(t,\vec{x}')}{R}\dot{\vec{v}} - \frac{1}{6\pi c^3\epsilon_0}\int_V d^3x\ \rho(t,\vec{x})\int_V d^3x'\,\rho(t,\vec{x}')\ddot{\vec{v}}$$

So, the complete equation of motion

$$\left(\frac{d\vec{p}}{dt}\right)_{mech} + \left(\frac{d\vec{p}}{dt}\right)_{EM} = \vec{F}_{ext}$$

becomes

$$(m_0 + m_{EM})\dot{\vec{v}} = \vec{F}_{ext} + \frac{q^2}{6\pi\epsilon_0 c^3}\ddot{\vec{v}}$$

$$m_{EM} = \frac{4}{3}\frac{W_{self}}{c^2}$$

$$W_{self} = \frac{1}{8\pi\epsilon_0}\int_V d^3x\ \int_V d^3x'\,\frac{\rho(t,\vec{x})\rho(t,\vec{x}')}{|\vec{x}-\vec{x}'|} \qquad \text{self} - \text{interaction energy}$$

Problems:

- 4/3 does not conform with special relativity. Although we did a non-relativistic derivation, it must give exact result in instantaneous rest system.
- Taking the limit of a point particle, we get the Abraham-Lorentz force exactly, but the self-energy is divergent.

The Abraham-Lorentz model of the electron

Self-energy for an electron modelled as spherical shell with radius *r* and charge *q*=*e*

$$W_{self} = \int d^3x\,\frac{1}{2}\epsilon_0 E^2$$

HW: compute

$$W_{self} = \frac{e^2}{8\pi\epsilon_0 r}$$

Electron mass is the sum of "bare mass" and electromagnetic mass
$m_e = m_0 + \frac{W_{self}}{c^2}$

This means that
$\frac{W_{self}}{c^2} \le m_e \ \Rightarrow r \ge r_0 = \frac{e^2}{8\pi\epsilon_0 m_e c^2} \approx 1.4 \cdot 10^{-15} m$

Classical electron radius:
$r_{cl} = \frac{e^2}{4\pi\epsilon_0 m_e c^2} \approx 2.8 \cdot 10^{-15} m$
The electron cannot be smaller than $O(r_{cl})$.

Precise numerical coefficient depends on the detailed model of charge distribution.

HW: assuming that the electron is a uniformly charged sphere of radius *r*, compute

$$W_{self} = \frac{3}{5}\frac{e^2}{4\pi\epsilon_0 r}$$

Further problems with Abraham-Lorentz force

1. Self-accelerating solution in absence of external field
   $m\dot{\vec{v}} = \frac{q^2}{6\pi\epsilon_0 c^3}\ddot{\vec{v}} = m\tau\ddot{\vec{v}} \qquad \tau = \frac{q^2}{6\pi\epsilon_0 m^2 c^3}$

   Simple solution:

   $\vec{v} = \vec{v}_0 e^{t/\tau}$ WHAT???

2. Acausal motion a.k.a. signal from the future

   $m(\dot{v}(t) - \tau\ddot{v}(t)) = F_{ext}(t)$

   Note that
   $\dot{v}(t) - \tau\ddot{v}(t) = -\tau e^{t/\tau}\frac{d}{dt}\left(e^{-t/\tau}\dot{v}(t)\right)$

   $-m\tau e^{t/\tau}\frac{d}{dt}\left(e^{-t/\tau}\dot{v}(t)\right) = F_{ext}(t)$

   So, we can rewrite

   $m\frac{d}{dt}\left(e^{-t/\tau}\dot{v}(t)\right) = -\frac{1}{\tau}e^{-t/\tau}F_{ext}(t) \ \Rightarrow \ me^{-t/\tau}\dot{v}(t) = \frac{1}{\tau}\int_t^\infty dt'\, e^{-t'/\tau}F_{ext}(t')$

and so

$$m\dot{v}(t) = \frac{1}{\tau}\int_t^\infty dt'\, e^{-(t'-t)/\tau} F_{ext}(t')$$

Acceleration at time *t* depends on future values of the applied force!

Problem: presence of third time derivative ("jerk")

$$\ddot{v}(t) = \frac{d^3x}{dt^3}$$

incompatible with general principles of mechanics!

Note:

$$\tau = \frac{2}{3}\frac{r_{cl}}{c}$$

Problematic time scale corresponds to classical charge radius!

Improving the derivation

1. Poincaré stress tensor contribution

   Stabilising the charge density needs force of non-EM origin
   ⇒ Poincaré stress - also contributes to rest mass

   Can be done in a covariant way, guaranteeing relativistic consistency (Jackson 16.6)

2. Repairing acausal behaviour

   $$m\dot{\vec{v}} = \vec{F}_{ext} + m\tau\ddot{\vec{v}}$$

   Use zeroth order equation of motion to replace

   $$m\dot{\vec{v}} = \vec{F}_{ext} + \tau\frac{d\vec{F}_{ext}}{dt} = \vec{F}_{ext} + \tau\left(\frac{\partial\vec{F}_{ext}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{F}_{ext}\right)$$

   This is now a second order, fully causal equation of motion for the trajectory $\vec{x}(t)$.
   Relativistically covariant version also exists.

For the spherical shell a fully relativistic version of the Abraham-Lorentz derivation can be found in

A.D. Yaghjian: Relativistic Dynamics of a Charged Sphere
Lecture Notes in Physics m11, Springer-Verlag, 1992.

Radiatively damped oscillator

$$m\frac{d^2x}{dt^2} - m\tau\frac{d^3x}{dt^3} + m\omega_0^2 x = 0$$

Ansatz: $x \propto e^{-i\omega t}$

HW: solve for ω assuming that $\omega_0\tau \ll 1$ and show that

$$\omega = \omega_0 - \frac{5}{8}\omega_0^3\tau^2 - \frac{1}{2}i\omega_0^2\tau + O(\tau^3)$$

Hint: derive the equation for ω. Substitute $\omega = \omega_0 + A\tau + B\tau^2 + O(\tau^3)$ and solve for the coefficient of the $O(\tau)$ and $O(\tau^2)$ terms.

So, we get a damped oscillator

$$x \propto e^{-i\omega_0 t - \frac{1}{2}\omega_0^2\tau t}$$

HW: repeat the argument we used for resonant cavities to find the intensity spectrum

$$I(\omega) \propto \frac{1}{(\omega - \omega_0 - \Delta\omega)^2 + \left(\frac{\Gamma}{2}\right)^2}$$

with the frequency shift and half-width given by

$$\Delta\omega = -\frac{5}{8}\omega_0^3\tau^2 \quad \Gamma = \omega_0^2\tau$$

Note that $\Delta\omega \ll \Gamma$ since $\omega_0\tau \ll 1$ - observations of spectral lines show that this is not the case, and they agree instead with quantum mechanics.

HW: the half-width in angular frequency is given by Γ. Show that the half-width in the wavelength is

$$\Delta\lambda = 2\pi\frac{c}{\omega_0^2}\Gamma = 2\pi c\tau$$

Hint: neglect $\Delta\omega$ and use $\lambda = 2\pi c/\omega$.

In matter, the charges are electrons, so we get

$$\Delta\lambda = 2\pi c\tau = \frac{e^2}{3\epsilon_0 m_e c^2} = 1.2 \cdot 10^{-14}\ \text{m} = 1.2 \cdot 10^{-4}\ \text{Angstrom}$$

This predicts that the line width (measured in wavelength) is universal: this is in contrast to observations, which are explained by quantum mechanics.

Final note

Nature eventually avoids the problems with the Abraham-Lorentz force for the electron (and other charged elementary particles)!!!

At the Compton scale

$$\lambda_C = \frac{\hbar}{m_e c} \approx 3.9 \cdot 10^{-13} \gg r_{cl} = 2.8 \cdot 10^{-15}\text{m}$$

Classical ElectroDynamics is not valid anymore and is replaced by Quantum ElectroDynamics.

CED⟶QED

Note that

$$\frac{r_{cl}}{\lambda_C} = \frac{\frac{e^2}{4\pi\epsilon_0 m_e c^2}}{\frac{\hbar}{m_e c}} = \frac{e^2}{4\pi\epsilon_0 \hbar c} = \alpha \approx \frac{1}{137}$$

is nothing else but the fine structure constant!

Fundamental theory of matter is not in terms of point particles and classical fields, but

Quantum Field Theory