CHAPTER SIX

Quasiperiodicity

6.1 Frequency spectrum and attractors

In Chapter 1 we introduced three kinds of dynamical motions for continuous time systems: steady states (as in Figure 1.10(*a*)), periodic motion (as in Figure 1.10(*b*)), and chaotic motion (as in Figure 1.2). In addition to these three, there is another type of dynamical motion that is common; namely, *quasiperiodic* motion. Quasiperiodic motion is especially important in Hamiltonian systems where it plays a central role (see Chapter 7). Furthermore, in dissipative systems quasiperiodic *attracting* motions frequently occur.

Let us contrast quasiperiodic motion with periodic motion. Say we have a system of differential equations with a limit cycle attractor (Figure 1.10(*b*)). For orbits on the attractor, a dynamical variable, call it $f(t)$, will vary periodically with time. This means that there is some smallest time $T > 0$ (the period) such that $f(t) = f(t + T)$. Correspondingly, the Fourier transform of $f(t)$,

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\exp(\mathrm{i}\omega t)\,\mathrm{d}t, \tag{6.1}$$

consists of delta function spikes of varying strength located at integer multiples of the fundamental frequency $\Omega = 2\pi/T$,

$$\hat{f}(\omega) = 2\pi \sum_n a_n \delta(\omega - n\Omega). \tag{6.2}$$

Basically, quasiperiodic motion can be thought of as a mixture of periodic motions of several different fundamental frequencies. We speak of N-frequency quasiperiodicity when the number of fundamental frequencies that are 'mixed' is N. In the case of N-frequency quasiperiodic motion a dynamical variable $f(t)$ can be represented in terms of a function of N independent variables, $G(t_1, t_2, \ldots, t_N)$, such that G is periodic in each of its N independent variables. That is,

$$G(t_1, t_2, \ldots, t_i + T_i, \ldots, t_N) = G(t_1, t_2, \ldots, t_i, \ldots, t_N), \tag{6.3}$$

where, for each of the N variables, there is a period T_i. Furthermore, the N frequencies $\Omega_i \equiv 2\pi/T_i$ are *incommensurate*. This means that no one of the frequencies Ω_i can be expressed as a linear combination of the others using coefficients that are rational numbers. In particular, a relation of the form

$$m_1\Omega_1 + m_2\Omega_2 + \cdots + m_N\Omega_N = 0 \tag{6.4}$$

does not hold for *any* set of integers, $m_1, m_2, \ldots, m_N$ (negative integers are allowed), except for the trivial solution $m_1 = m_2 = \cdots = m_N = 0$. In terms of the function G, an N-frequency quasiperiodic dynamical variable $f(t)$ can be represented as

$$f(t) = G(t, t, t, \ldots, t). \tag{6.5}$$

That is, f is obtained from G by setting all its N variables equal to t; $t_1 = t_2 = \cdots = t_N = t$. Due to the periodicity of G, it can be represented as an N-tuple Fourier series of the form

$$G = \sum_{n_1, n_2, \ldots, n_N} a_{n_1 \ldots n_N} \exp[i(n_1\Omega_1 t_1 + n_2\Omega_2 t_2 + \cdots + n_N\Omega_N t_N)].$$

Thus setting $t = t_1 = t_2 = \cdots = t_N$ and taking the Fourier transform we obtain,

$$\hat{f}(\omega) = 2\pi \sum_{n_1, n_2, \ldots, n_N} a_{n_1 \ldots n_N} \delta(\omega - (n_1\Omega_1 + n_2\Omega_2 + \cdots + n_N\Omega_N)) \tag{6.6}$$

Hence the Fourier transform of a dynamical variable $\hat{f}(\omega)$ consists of delta functions at all integer linear combinations of the N fundamental frequencies $\Omega_1, \ldots, \Omega_N$.

Figure 6.1 shows the magnitude squared of the Fourier transform (i.e., the frequency power spectrum) of a dynamical variable for three experimental situations: (*a*) a case with a limit cycle attractor, (*b*) a case with a two frequency quasiperiodic attractor, and (*c*) a case with a chaotic attractor. These results (Swinney and Gollub, 1978) were obtained for an experiment on Couette–Taylor flow (see Figure 3.10(*a*)). The three spectra shown correspond to three values of the rotation rate of the inner cylinder in Figure 3.10(*a*), with (*a*) corresponding to the smallest rate and (*c*) corresponding to the largest rate. Note that for the quasiperiodic case the frequencies $n_1\Omega_1 + n_2\Omega_2$ are dense on the ω-axis, but, since their amplitudes decrease with increasing n_1 and n_2, peaks at frequencies corresponding to very large values of n_1 and n_2 are eventually below the overall noise level of the experiment. In the chaotic case, Figure 6.1(*c*), we see that peaks at the two basic frequencies Ω_1 and Ω_2 are present, but that the spectrum has also developed a broad continuous component. (Note that the broad continuous component in Figure 6.1(*c*) is far above the noise level of $\sim 10^{-4}$ evident in Figure 6.1(*a*).) The situation in Figure 6.1(*c*) is in contrast to that in Figure 6.1(*b*), where the only apparent frequency components are discrete (namely $n_1\Omega_1 + n_2\Omega_2$). The presence

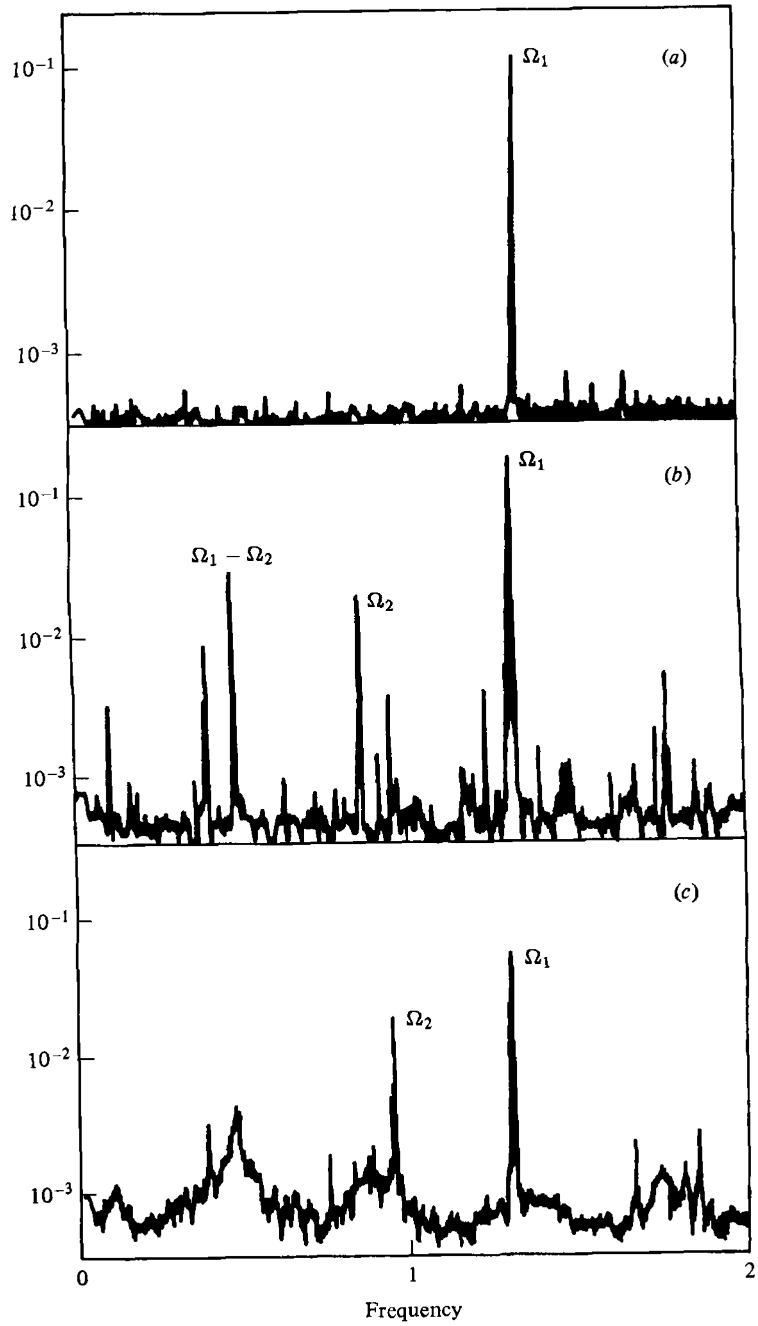


Figure 6.1 Results for frequency power spectra for a Couette–Taylor experiment with increasing rotation rate of the inner cylinders (Gollub and Swinney, 1975).

of a continuous component in a frequency power spectrum is a hallmark of chaotic dynamics.

A simple way to envision the creation of a quasiperiodic signal with a mixture of frequencies is illustrated in Figure 6.2, which shows two sinusoidal voltage oscillators in series with a nonlinear resistive element whose resistance R is a function of the voltage V across it, $R = R(V)$. Since the voltage sources are in series, we have $V = v_1 \sin(\Omega_1 t + \theta_0^{(1)}) + v_2 \sin(\Omega_2 t + \theta_0^{(2)})$. The current through the resistor, $I(t) = V/R(V)$, is a nonlinear function of V and hence will typically have all frequency components $n_1\Omega_1 + n_2\Omega_2$. Assuming that Ω_1 and Ω_2 are intercommensurate, the current $I(t)$ is two frequency quasiperiodic. The situation shown in Figure 6.2 is, in a sense, *too* simple to give very interesting behavior. If, for example, the value of the current I were to effect the dynamics of the voltage source oscillators, then a much richer range of behaviors would be possible, including *frequency locking* and chaos. Frequency locking refers to a situation where the interaction of two nonlinear oscillators causes them to self-synchronize in a coherent way so that their basic frequencies become commensurate (as we shall see, this implies that the motion is periodic) and remain locked in their commensurate relationship over a range of parameters. This will be discussed further shortly.

Let us now specialize to the case of attracting two frequency quasiperiodicity ($N = 2$) and ask, what is the geometrical shape of the attractor in phase space in such a case? To answer this, assume that we have a two frequency quasiperiodic solution of the dynamical system Eq. (1.3). In this case every component $x^{(i)}$ of the vector $\mathbf{x}$ giving the system state can be expressed as

$$x^{(i)}(t) = G^{(i)}(t_1, t_2)|_{t_1=t_2=t}.$$

Since $G^{(i)}$ is periodic in t_1 and t_2, we only need specify the value of t_1 and t_2 modulo T_1 and T_2 respectively. That is, we can regard the $G^{(i)}$ as being functions of two *angle* variables

$$\bar{\theta}_j = \Omega_j t_j \text{ modulo } 2\pi; j = 1, 2. \tag{6.7}$$

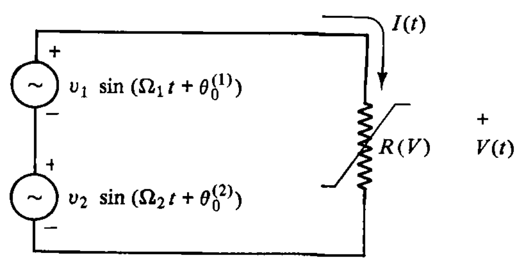


Figure 6.2 Two sinusoidal voltage sources driving a nonlinear resistor.

Thus the system state is specified by specifying two angles,

$$\mathbf{x} = \mathbf{G}(\bar{\theta}_1/\Omega_1, \bar{\theta}_2/\Omega_2), \tag{6.8}$$

where $\mathbf{G}$ is periodic with period 2π in $\bar{\theta}_1$ and $\bar{\theta}_2$. Specification of one angle can be regarded geometrically as specifying a point on a circle. Specification of two angles can be regarded geometrically as specifying a point on a two-dimensional toroidal surface (cf. Figure 6.3). In the full phase space, the attractor is given by Eq. (6.8), which must hence be topologically equivalent to a two-dimensional torus (i.e., it is a distorted version of Figure 6.3). A two frequency quasiperiodic orbit on a toroidal surface in a three-dimensional $\mathbf{x}$ phase space is shown schematically in Figure 6.4. The orbit continually winds around the torus in the short direction (making an average of $\Omega_1/2\pi$ rotations per unit time) and simultaneously continually winds around the torus in the long direction (making an average of $\Omega_2/2\pi$ rotations per unit time). Provided that Ω_1 and Ω_2 are incommensurate, the orbit on the torus never closes on itself, and, as time goes to infinity the orbit will eventually come arbitrarily close to every point on the toroidal surface. If we consider the orbit originating from the initial condition $\mathbf{x}_0$ near (but not on) a *toroidal attractor*, as shown in Figure 6.4, then, as time progresses, the orbit circulates around the torus in the long and short directions and asymptotes to a two frequency quasiperiodic orbit on the torus.

We define the *rotation number* in the short direction as the average number of rotations executed by the orbit in the short direction for each rotation it makes in the long direction,

$$R = \Omega_1/\Omega_2. \tag{6.9}$$

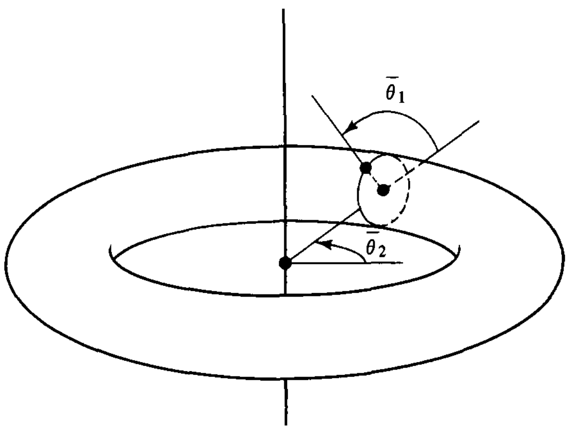


Figure 6.3 A point on a torus specifying the two angles $\bar{\theta}_1$ and $\bar{\theta}_2$.

When R is irrational the orbit fills the torus, never closing on itself. When R is rational, $R = \tilde{p}/\tilde{q}$ with $\tilde{p}$ and $\tilde{q}$ integers that have no common factor, the orbit closes on itself after $\tilde{p}$ rotations the short way and $\tilde{q}$ rotations the long way. Such an orbit is periodic and has period $\tilde{p}T_1 = \tilde{q}T_2$. The case $R = 3$ is illustrated in Figure 6.5, where we see that the orbit closes on itself after three rotations the short way around and one rotation the long way around.

In Figures 6.2–6.4 we have restricted our considerations to two frequency quasiperiodicity. We emphasize, however, that the situation is essentially the same for N-frequency quasiperiodicity. In that case the orbit fills up an N-dimensional torus in the phase space. By an N-dimensional torus we mean an N-dimensional surface on which it is possible to specify uniquely any point by a smooth one to one relationship with the values of N angle variables. We denote the N-dimensional torus by the symbol T^N.

In some situations it is possible to rule out the possibility of quasiperiodicity. As an example, consider the system of equations studied by Lorenz, Eqs. (2.30). It was shown in the paper by Lorenz (1963) that all orbits eventually enter a spherical region, $X^2 + Y^2 + Z^2 <$ (constant), from which they never leave. Thus, X, Y and Z are bounded, and we may regard the phase space as Cartesian with axes X, Y and Z. A two frequency quasiperiodic orbit fills up a two-dimensional toroidal surface

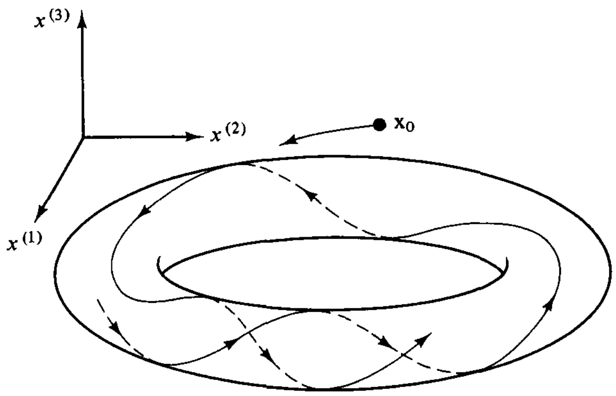


Figure 6.4 Two frequency quasiperiodic orbit on a torus lying in a three-dimensional phase space $x = (x^{(1)}, x^{(2)}, x^{(3)})$.

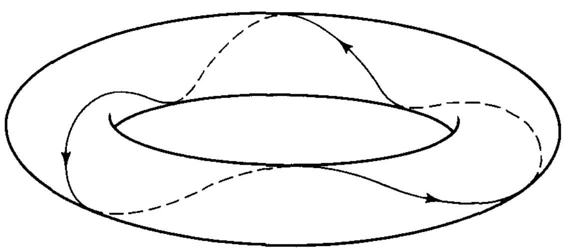


Figure 6.5 An orbit with a rotation number of $R = 3$.

in this space. Thus the toroidal surface is invariant under the flow. That is, evolving every point on the surface forward in time by any fixed amount maps the surface to itself. Furthermore, the volume inside the torus must also be invariant by the continuity of the flow. However, we have seen in Section 2.4.1 that following the points on a closed surface forward in time, the Lorenz equations contract the enclosed phase space volumes exponentially in time. Thus two frequency quasiperiodic motion is impossible for this system of equations.

6.2 The circle map

The system illustrated in Figure 6.2 is particularly simple. Since $\Omega_1 t$ and $\Omega_2 t$ appear only as the argument in sinusoids, we regard them as angles $\theta^{(1)}(t) = \Omega_1 t + \theta_0^{(1)}$ and $\theta^{(2)}(t) = \Omega_2 t + \theta_0^{(2)}$. In these terms the dynamical system reduces to

$$d\theta^{(1)}/dt = \Omega_1 \text{ and } d\theta^{(2)}/dt = \Omega_2.$$

Now taking a surface of section at ($\theta^{(2)}$ modulo 2π) = (const.), we obtain a one-dimensional map for $\theta_n = \theta^{(1)}(t_n)$ modulo 2π (where t_n denotes the time at the nth piercing of the surface of section),

$$\theta_{n+1} = (\theta_n + w) \text{ modulo } 2\pi, \tag{6.10}$$

where $w = 2\pi\Omega_1/\Omega_2$. Geometrically, the map Eq. (6.10) can be thought of as a rigid rotation of the circle by the angle w. For incommensurate frequencies, Ω_1/Ω_2 is irrational, and for any initial condition, the orbit obtained from the map (6.10) densely fills the circle, creating a uniform invariant density of orbit points in the limit as time goes to infinity. On the other hand, if $\Omega_1/\Omega_2 = \tilde{p}/\tilde{q}$ is rational, then the orbit is periodic with period $\tilde{q}$ ($\theta_{n+\tilde{q}} = (\theta_n + \tilde{q}w)$ modulo $2\pi = \theta_n$). Thus there is only a zero Lebesgue measure set of w (namely, the rationals) for which periodic motion (as opposed to two frequency quasiperiodic motion) applies. Let us now ask, what would we expect to happen if the two voltage oscillators in Figure 6.2 were allowed to couple nonlinearly? Would the quasiperiodicity be destroyed and immediately be replaced by periodic orbits? Since the rationals are dense, and coupling is known to induce frequency locking, this question deserves some serious consideration. To answer this Arnold (1965) considered a model that addresses the main points. In particular, the effect of such coupling of the oscillator dynamics is to add nonlinearity to Eq. (6.10). Thus Arnold introduced the map,

$$\theta_{n+1} = (\theta_n + w + k\sin\theta_n) \text{ modulo } 2\pi, \tag{6.11}$$

where the term $k\sin\theta$ models the effect of the nonlinear oscillator coupling. This map is called the *circle map*. In what follows we take w to lie in the range $[0, 2\pi]$.

Although deceptively simple in appearance, the circle map (like the logistic map) reveals a wealth of intricate behavior. It is of interest to understand the behavior of this map as a function of both w and the nonlinearity parameter k. A key role is played by the rotation number, which for this case is given by

$$R = \frac{1}{2\pi} \lim_{m \to \infty} \frac{1}{m} \sum_{n=0}^{m-1} \Delta\theta_n, \tag{6.12}$$

where $\Delta\theta_n = w + k\sin\theta_n$. For $k = 0$, we have $R = w/2\pi$ and the periodic orbits (rational values of R) only occur for a set of w of Lebesgue measure zero (i.e., rational values of $w/2\pi$). What are the characters of the sets of w-values yielding rational and irrational R if $k > 0$? Arnold (1965) considered this problem for small k. Specifically, we ask whether the Lebesgue measure of w yielding irrational R (i.e., quasiperiodic motion) immediately becomes zero when k is made nonzero. Arnold proved the fundamental result that quasiperiodicity survives in the following sense. For small k the Lebesgue measure of $w/2\pi$ yielding quasiperiodicity is close to 1 and approaches 1 as $k \to 0$. The set of w-values yielding quasiperiodicity, however, is nontrivial because arbitrarily close to a w-value yielding quasiperiodicity (irrational R) there are *intervals* of w yielding attracting periodic motion (rational R). (The existence of intervals where R is rational is what we mean by the term frequency locking.) Thus, the periodic motions are dense in w. (This corresponds to the fact that rational numbers are dense.) The set of w-values yielding quasiperiodicity is a Cantor set of positive Lebesgue measure (in the terminology of Section 3.9, it is a 'fat fractal'). Arnold's result was an important advance and is closely related to the celebrated KAM theory (for Kolmogorov, Arnold and Moser) for Hamiltonian systems (see Chapter 7). Specifically, in dealing with the circle map, as well as the problem which KAM theory addresses, one has to confront the difficulty of the 'problem of small denominators.' To indicate briefly the nature of this problem, first note that Arnold was examining the case of small k. The natural approach is to do a perturbation expansion around the case $k = 0$ (i.e., the pure rotation, Eq. (6.10)). One problem is that at every stage of the expansion this results in infinite series terms of the form

$$\sum_m \frac{A_m}{1 - \exp(2\pi i m R)} \exp(im\theta).$$

For R any irrational, the number $[(mR)$ modulo $1]$ can be made as small as we wish by a proper choice of the integer m (possibly very large). Hence, the denominator, $1 - \exp(2\pi i m R)$, can become small, and thus there is the concern that the series might not converge. To estimate this effect say

$\exp(2\pi i m R)$ is close to 1 so that the denominator is small. This occurs when mR is close to an integer; call it n. In this case

$$1 - \exp(2\pi i m R) \simeq -2\pi i(mR - n).$$

Thus, the magnitude of a term in the sum is approximately

$$\frac{1}{2\pi m}\frac{|A_m|}{|R - n/m|}.$$

(Clearly, if R is rational, then $R = n/m$ for some n and m, and this expansion fails. But we are here interested in the case of quasiperiodic motion for which R is irrational.) The convergence of the sum will depend on the number R. In particular, R-values satisfying the inequality

$$\left|R - \frac{n}{m}\right| > \frac{K}{m^{(2+\varepsilon)}}$$

for some positive numbers K and ε and all values of the integers m and n ($m \neq 0$) are said to be 'badly approximated by rationals.' It is a basic fact of number theory that the set of numbers (R in our case) that are not badly approximated by rationals has Lebesgue measure zero. The coefficients A_m are obtained from Fourier expansion of an analytic function, and hence the A_m decay exponentially with m, i.e., for some positive numbers σ and c, we have $|A_m| < c\exp(-\sigma|m|)$. Thus,

$$\frac{1}{2\pi m}\frac{|A_m|}{|R - m/n|} < O(m^{(1+\varepsilon)}\exp(-\sigma|m|)).$$

The exponential decay $\exp(-\sigma|m|)$ is much stronger than the power law increase $m^{(1+\varepsilon)}$, and convergence of the sum is therefore obtained for all R-values that are badly approximated by rationals. This, however, is only the beginning of the story since, at each stage of the perturbation expansion, sums of this type appear. While these sums converge, it still remains to show convergence of the perturbation expansion itself. Arnold was able to prove convergence of his perturbation expansion. Thereby he showed that the Lebesgue measure of w in $[0, 2\pi]$ for which there is quasiperiodicity (i.e., irrational R) is not zero for small k and that this measure approaches 2π in the limit $k \to 0$. Thus, for small k, quasiperiodicity survives and occupies most of the Lebesgue measure.

Let us now address the issue of frequency locking. As an example, consider the rotation number $R = 0$. This corresponds to a fixed point of the map. Hence we look for solutions of

$$\theta = (\theta + w + k\sin\theta). \tag{6.13}$$

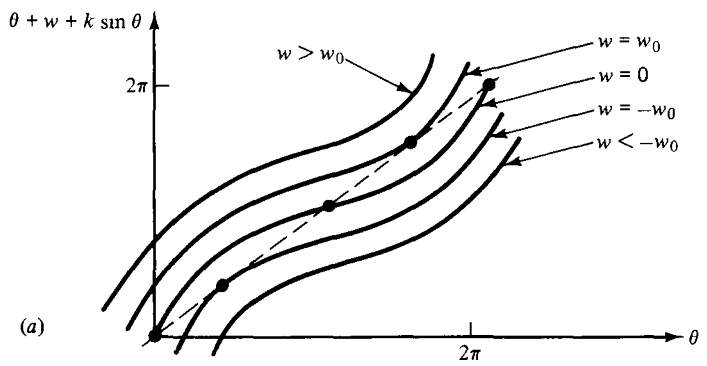
The solution of this equation is demonstrated graphically in Figure 6.6(a) for several values of w. Note that there are no solutions of the fixed point equation, Eq. (6.13), for the value of w labeled $w < -w_0$ in the figure. As w

is increased from $w < -w_0$, the graph of $(\theta + w + k \sin\theta)$ becomes tangent to the dashed 45° line at $w = -w_0$. Thus two fixed point orbits, one stable and one unstable, are born by a tangent bifurcation as w increases through $w = -w_0$. As w is increased further, the two solutions continue to exist, until, as w increases through w_0, they are destroyed in a backward tangent bifurcation. Figure 6.6(*b*) shows the corresponding bifurcation diagram. From Eq. (6.13) we have $w_0 = k$. Thus we see that, for $k > 0$, the stable fixed point ($R = 0$) exists in an *interval* of w values, $k > w > -k$, whereas at $k = 0$ we only have $R = 0$ at the single value $w = 0$. This is what we mean by frequency locking. Similarly, one can show that, for small k, an attracting period two orbit (corresponding to a rotation number $R = \frac{1}{2}$) exists in a range $w_{1/2}^- < w < w_{1/2}^+$, where

$$w_{1/2}^{\pm} = \pi \pm k^2/4 + O(k^3). \tag{6.14}$$

In general, for any rational rotation number $R = \tilde{p}/\tilde{q}$ there is a frequency locking range of w in which the corresponding attracting period $\tilde{q}$ orbit exists, and this range $(w_{\tilde{p}/\tilde{q}}^-, w_{\tilde{p}/\tilde{q}}^+)$ has a width $\Delta w_{\tilde{p}/\tilde{q}} = w_{\tilde{p}/\tilde{q}}^+ - w_{\tilde{p}/\tilde{q}}^-$ which scales as

$$\Delta w_{\tilde{p}/\tilde{q}} = O(k^{\tilde{q}}). \tag{6.15}$$



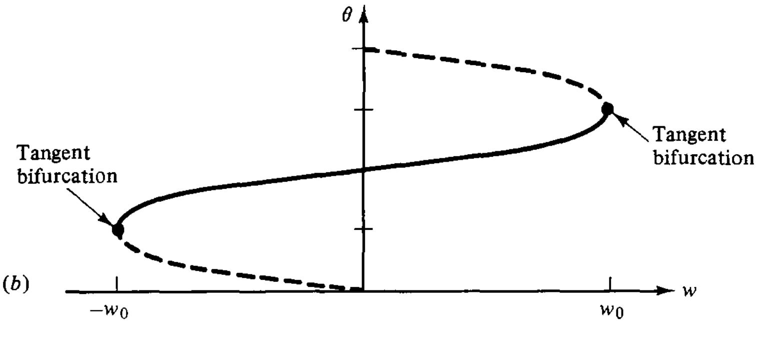


Figure 6.6(*a*) Fixed point solutions of the circle map denoted by dots. (*b*) Bifurcation diagram for the $R = 0$ stable (solid curve) and unstable (dashed curve) orbits.

Furthermore, as w increases through the value $w^-_{\tilde{p}/\tilde{q}}$, the attracting period $\tilde{q}$ orbit with rotation number $\tilde{p}/\tilde{q}$ is born by a forward tangent bifurcation, and as w increases through the value $w^+_{\tilde{p}/\tilde{q}}$ the attracting period $\tilde{q}$ dies by a backward tangent bifurcation.

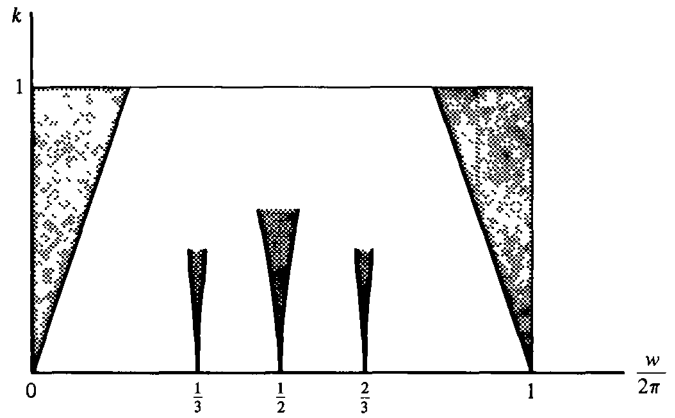
Note that, since the map function is monotonically increasing for $0 \leq k \leq 1$, its derivative and that of its n times composition are positive, $\mathrm{d}M^n(\theta)/\mathrm{d}\theta > 0$. Hence there can be no period doubling bifurcations of a period n orbit for any n in $0 \leq k \leq 1$, since the stability coefficient (slope of M^n) must be -1 at a period doubling bifurcation point.

Consider the total length in w (Lebesgue measure) of all frequency locked intervals in $[0, 2\pi]$,

$$\sum_{r=\tilde{p}/\tilde{q}} \Delta w_r.$$

Arnold's results show that this number is small for small k and decreases to zero in the limit $k \to 0$. Thus the set of w-values yielding quasiperiodic motion has most of the Lebesgue measure of w for small k. This set is a Cantor set of positive Lebesgue measure. (We have previously encountered such a set in Section 2.2 when we considered the set of r-values for which the logistic map yields attracting chaotic motion.) The situation can be illustrated schematically as in Figure 6.7 which shows regions of the w–k plane (called Arnold tongues) in which the rotation numbers $R = 0, \frac{1}{2}, \frac{1}{3}$ and $\frac{2}{3}$ exist. We see that there are narrow frequency-locked tongues of rational R which extend down to $k = 0$. For higher periods (i.e., larger $\tilde{q}$ in $R = \tilde{p}/\tilde{q}$) the frequency-locked intervals becomes extremely small for small k. (This qualitative type of frequency-locking behavior occurring in tongues in parameter space has been found in numerical solutions of ordinary different equations, as well as in physical experiments.)

Figure 6.7 Arnold tongues for the circle map



For $k < 1$ and a w-value yielding an irrational value of R, the orbit points on the resulting quasiperiodic orbit generate a smooth invariant density $\rho(\theta)$. In this case, by a smooth change of variables $\phi = f(\theta)$ the circle map can be transformed to the pure rotation

$$\phi_{n+1} = \phi_n + 2\pi R(w, k)$$

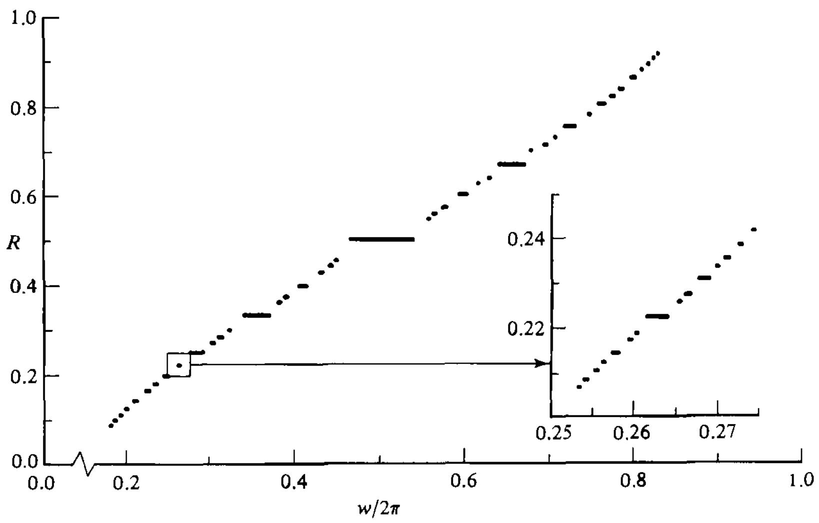
Since the pure rotation generates a uniform density $\tilde{\rho}(\phi) = 1/2\pi$, and the circle map is invertible for $k < 1$, we see by $\tilde{\rho}(\phi)\,\mathrm{d}\phi = \rho(\theta)\,\mathrm{d}\theta$ that the change of variables is

$$\phi = f(\theta) \equiv 2\pi \int_0^\theta \rho(\theta')\,\mathrm{d}\theta'. \tag{6.16}$$

As k approaches 1 from below, the widths $\Delta w_{p/q}$ increase, and the sum $\Sigma\,\Delta w_r$, approaches 2π. That is, at $k = 1$, the entire Lebesgue measure in w is occupied by frequency-locked periodic orbits, and the quasiperiodic orbits occupy zero Lebesgue measure in w. Figure 6.8 shows a numerical plot of R versus w at $k = 1$. We see that R increases monotonically with w. The set of w values on which R increases is the Cantor set of zero Lebesgue measure on which R is irrational (i.e., the motion is quasiperiodic). The function R versus w at $k = 1$ is called a *complete devil's staircase*. At lower k we again obtain a monotonic function which increases only on the Cantor set of w-values where R is irrational, but now the Cantor set has positive Lebesgue measure (it is a fat fractal (Section 3.9)). We consequently say that R versus w is an *incomplete devil's staircase* for $1 > k > 0$.

The box-counting dimension of the set on which R increases for $k = 1$ (the complete devil's staircase case) has been calculated by Jensen, Bak

Figure 6.8 Complete devil's staircase at $k = 1$ (Jensen *et al.*, 1984).



and Bohr (1983). They obtain a dimension value of $D_0 \simeq 0.87$. Furthermore, they claim that this value is universal in that it applies to a broad class of systems, not just the circle map. This contention is supported by the renormalization group theory of Cvitanović *et al.* (1985).

For $k > 1$, the circle map is noninvertible ($\mathrm{d}\theta_{n+1}/\mathrm{d}\theta_n$ changes sign as θ_n varies when $k > 1$). As a consequence of this, typical initial conditions can yield chaotic orbits but do not yield quasiperiodic orbits for $k > 1$. To see why quasiperiodic orbits do not result from typical initial conditions, we note that we have previously seen that a smooth change of variables Eq. (6.16) transforms the circle map to the pure rotation if there is a quasiperiodic orbit with a smooth invariant density $\rho(\theta)$. Since it is not possible to transform a noninvertible map to an invertible one (i.e., the pure rotation), we conclude that there can be no quasiperiodic orbits generating smooth invariant densities[1] for $k > 1$.

As an example of circle map type dynamics appearing in an experiment, we mention the paper of Brandstater and Swinney (1987) on Couette–Taylor flow (see Section 3.7). Under particular conditions the authors observe two frequency quasiperiodic motion on a two-dimensional toroidal surface. Figure 6.9 (*a*) shows a delay coordinate plot of the orbit $V(t)$ versus $V(t - \tau)$ where $V(t)$ is the radial velocity component measured at a particular point in the flow. Taking a surface of section along the dashed line in Figure 6.9 (*a*) one obtains a closed curve indicating that the orbit in Figure 6.9 (*a*) lies on a two-dimensional torus. Figure 6.9 (*b*) shows such a surface of section plot (for slightly different conditions from those in Figure 6.9 (*a*)). Brandstater and Swinney then parameterize the location of orbit points in the surface of section by an angle θ measured from a point inside the closed curve. In Figure 6.9(*c*) they plot the value of θ at the $(n + 1)$th piercing of the surface of section versus its value at the nth piercing of the surface of section. We see that this map is indeed of a similar form to the circle map of Arnold:[2] it is invertible and is close to a pure rotation with an added nonlinear piece, $\theta_{n+1} = [\theta_n + w + P(\theta_n)]$ modulo 2π, where $P(\theta)$ is the periodic nonlinear piece, $P(\theta) = P(\theta + 2\pi)$. (In the absence of $P(\theta)$ the map would be two parallel straight lines at 45° (pure rotation), which Figure 6.9 (*c*) would resemble if the wiggles due to $P(\theta)$ were absent.)

As an example of how circle map type phenomena can appear in a differential equation consider the equation,

$$\mathrm{d}\theta/\mathrm{d}t + \delta \sin\theta = V + W\cos(\Omega t), \qquad (6.17)$$

which may be viewed as a highly damped slowly forced pendulum such that the inertia term $\mathrm{d}^2\theta/\mathrm{d}t^2$ is negligible.[3] For small δ one may expand the solution as a power series in δ retaining only the first two terms,

$$\theta(t) = \theta^{(0)}(t) + \delta\theta^{(1)}(t) + O(\delta^2). \quad (6.18)$$

The lowest order solution is obtained by setting $\delta = 0$ in (6.17),

$$\theta^{(0)}(t) = \theta(t') + V(t - t') + (W/\Omega)[\sin(\Omega t) - \sin(\Omega t')], \quad (6.19)$$

and the next order solution is

$$\theta^{(1)}(t) = -\int_{t'}^{t} \sin\theta^{(0)}(t)\,\mathrm{d}t. \quad (6.20)$$

Letting $\theta_n = \theta(t_n)$ modulo 2π, where $t_n = 2\pi n/\Omega$, we obtain a map from (6.18)–(6.20) by setting $t = t_{n+1}$ and $t' = t_n$,

$$\theta_{n+1} = \{\theta_n + w + kf(w,u)\sin[\theta_n + \phi(w,u) + \pi]\} \text{ modulo } 2\pi, \quad (6.21)$$

where (Problem 4)

$$f(w,u) = \left| \int_0^{2\pi} \exp\left[\mathrm{i}\left(\frac{w\tau}{2\pi} + u\sin\tau \right) \right] \mathrm{d}\tau \right|,$$

$w = 2\pi V/\Omega$, $k = \delta/\Omega$, $u = W/\Omega$, and $\phi(w,u)$ is the angle of the complex quantity whose magnitude appears above. Equation (6.21) with u fixed is, aside from the more complicated dependence on w, the same as the circle map. (Note, however, that Eq. (6.21) is only valid for small δ.) Thus frequency locking and Arnold tongues also occur here although the

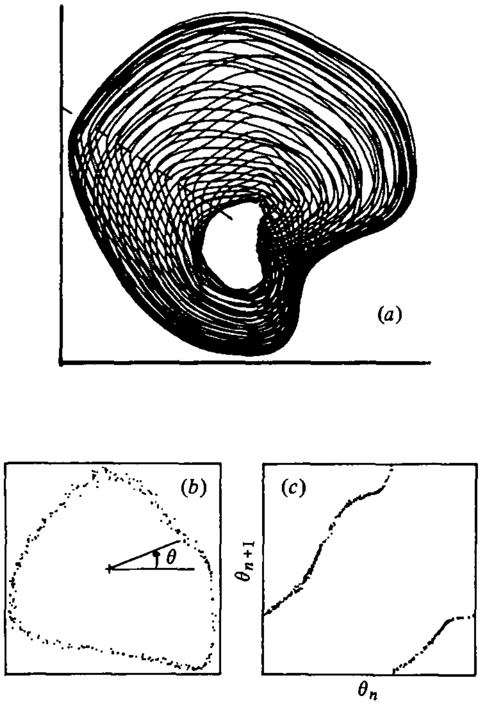


Figure 6.9(*a*) Projection of the orbit onto the delay coordinate plane $V(t)$ versus $V(t - \tau)$. (*b*) The surface of section given by the dashed line in (*a*). (*c*) Experimental circle map obtained from (*b*) (Brandstater and Swinney, 1987).

picture, Figure 6.7, will be distorted by the different parameter dependence of the map on w.

Equation (6.17) can be considered as a two-dimensional dynamical system in the two variables $\theta^{(1)} = \theta$ and $\theta^{(2)} = \Omega t$ which are both angle variables. Hence (6.17) describes a flow on a two-dimensional toroidal surface. On this surface we can either have a quasiperiodic orbit, or an attracting periodic orbit, the latter corresponding to a frequency-locked situation. The attraction of orbits on the torus to a periodic orbit is illustrated in Figure 6.10. As mentioned already, this behavior is displayed by higher-dimensional systems. What happens in these higher-dimensional systems is that there is an invariant two-dimensional torus embedded in the phase space flow. On the torus, the flow can be either quasiperiodic or else it can have an attracting periodic orbit (Figure 6.10). When the flow is quasiperiodic, a surface of section yields a picture of the attractor cross section which is either a closed curve, or several closed curves, resulting from the intersection of the surface of section with the attracting invariant torus. When the attractor is periodic, the surface of section intersection with the attractor reveals a finite number of discrete points (note, however, that there can still be an invariant torus on which the attractor lies). We can think of the flow in the higher-dimensional phase space as being attracted to a lower-dimensional (two-dimensional) flow on the torus, on which, in turn, there can be quasiperiodic motion or a periodic attractor.

A fairly common way that one sees chaos appearing as a system parameter is varied is that first two frequency quasiperiodicity is seen, then frequency locking to a periodic attractor, and then a chaotic attractor. Since chaos is not possible for a two-dimensional flow, in order for the chaos to appear, the orbit can no longer be on a two-dimensional torus. Typically, as the parameter is increased toward the value yielding chaos, the invariant two-dimensional torus is destroyed. When this happens, it does so while in the parameter range in which the periodic

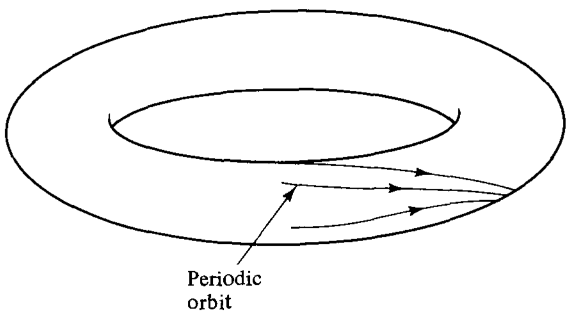


Figure 6.10 Attraction of initial conditions on a two-dimensional torus to a periodic orbit.

attractor exists. In terms of the circle map, we can think of the destruction of the torus as analogous to the map becoming noninvertible as k increases through 1 (quasiperiodic orbits do not occur for typical initial conditions for $k > 1$). If we were to fix w and increase k, we might expect to see quasiperiodicity and then frequency locking as k is increased toward one, since the frequency locked regions have Lebesgue measure 2π in w at $k = 1$. The periodic solutions at $k = 1$ typically remain stable as k is increased past 1 into the region where chaos becomes possible. These periodic solutions can then become chaotic, for example, by going through a period doubling cascade.

In our discussion above of the onset of chaos for the circle map, we imagined a typically chosen variation along a path in parameter space; specifically, we imagined choosing a typical w and then increasing k. Another possibility is carefully to choose a path in parameter space such that we maintain the rotation number to be constant and irrational. Thus, as we increase k, we adjust w to keep $R(k, w)$ the same. Such a path threads between the frequency-locked Arnold tongues all the way up to $k = 1$. The same can be done in an experiment on a higher-dimensional system, in which case $k = 1$ corresponds to the point at which the torus is destroyed. Studies of this type of variation have revealed that there is a universal phenomonology in the behavior of systems approaching torus destruction along such a path in parameter space. The behavior depends on the rotation number R chosen but is essentially system-independent. Extensive work demonstrating this has been done for the case of the path on which the rotation number is held constant at the value given by the golden mean, $R = (\sqrt{5} - 1)/2 \equiv R_g$ (Shenker, 1982; Feigenbaum *et al.*, 1982; Ostlund *et al.*, 1983; Umberger *et al.*, 1986). This number is of particular significance because of its number theoretic properties. Specifically, in some sense (see Section 7.3.2), R_g is the most irrational of all irrational numbers in that it is most difficult to approximate by rational numbers of limited denominator size. These results for $R = R_g$ are obtained using the renormalization group technique, the same technique used to analyze the universal properties of the period doubling cascade (cf. Chapter 8). Perhaps the most striking of these results is that for the low frequency power spectrum of a process which is quasiperiodic with rotation number equal to the golden mean and parameters corresponding to the critical point at which the torus is about to be destroyed ($k = 1$ for the circle map). As illustrated in Figure 6.11, if one plots the frequency power spectrum $\hat{P}(\omega)$ divided by ω^2 versus the frequency ω on a log–log plot, then the result is predicted to be universal and periodic in $\log \omega$ for small ω. Furthermore, the periodicity length in $\log \omega$ is just the logarithm of the golden mean.