Problem:

Starting from the expression

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

work out the transformation rules of the fields $\vec{E}, \vec{B}$ under Lorentz-boost in the x direction. (or you could choose the z direction, as we discussed in class) Which components are invariant? Show that certain pairs of components transform as Lorentzian 2-vectors!

Solution:

The Lorentz transformation for speed v pointing in the x direction can be written as a concrete matrix

$$\Lambda^\mu_\nu = \begin{pmatrix} \cosh(\beta) & -\sinh(\beta) & & \\ -\sinh(\beta) & \cosh(\beta) & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad \tanh(\beta) = v \tag{1}$$

The Lorentz transformation of the electro-magnetic field tensor is

$$F'^{\mu\nu} = \Lambda^\mu_\delta \Lambda^\nu_\kappa F^{\mu\kappa} \tag{2}$$

As concrete matrices this can be written as

$$F' = \Lambda F \tilde{\Lambda} \tag{3}$$

So we need to multiply three matrices. After this is done, we obtain

$$F' = \begin{pmatrix} 0 & -E_x & -\cosh(\beta)E_y + \sinh(\beta)B_z & -\cosh(\beta)E_z - \sinh(\beta)B_y \\ E_x & 0 & \cosh(\beta)B_z + \sinh(\beta)E_y & \cosh(\beta)B_y + \sinh(\beta)E_z \\ \cosh(\beta)E_y - \sinh(\beta)B_z & \cosh(\beta)B_z - \sinh(\beta)E_y & 0 & -B_x \\ \cosh(\beta)E_z + \sinh(\beta)B_y & -\cosh(\beta)B_y - \sinh(\beta)E_z & B_x & 0 \end{pmatrix} \tag{4}$$