Problem 1

Consider the famous "twin-paradox". Let's call the twins Bobby and George. Bobby travels to one of the exo-planets of the Alpha Centauri system, whose distance is 4.5lightyears from the Earth, and it is almost in rest in the reference frame of the Earth. The maximal speed of the spacecraft is $0.75c$, and the accelaration and braking times are negligeble.

After reaching his destination, Bobby studies the exo-planet for 1year time and then he travels back to Earth.

a.) Draw the world lines of Bobby and George in the Minkowski plane.

b.) How many years does George age, who stayed on Earth, until Bobby gets back to the Earth?

c.) How many years does Bobby age at the same time?

A (wrong) explanation of the twin paradox states, that the different aging is caused by the acceleration of Bobby. Indeed, Bobby needs to have nonzero acceleration, if he wants to come home. However, using the following thought experiment we can exclude that explanation.

Let's suppose that Bobby and George both travel on the spacecraft, but at half distance George decides to stop – using the spacecrafts rescure cabin – and takes a long holiday at a space-motel that rests in the frame of Earth. When Bobby is traveling back, George accelerates his cabin to 0.75 c, joins Bobby in the spacecraft, and they arrive together back to Earth. We can see, that in this case George and Bobby can have exactly the same acceleration processes.

d.) Draw the modified world line of George in the figure!

e.) How much time does George spend in the motel?

f.) What is George's total aging during the travel?

Solution:

(a) George's world line is a vertical line at $x = 0$ in Earth's rest frame, while Bobby's world line with slope $c/v = 4/3$ until $x = 4.5$ lightyears, then a vertical line of length 1 year and finally back to $x = 0$ with negative slope of $-v/c = -4/3$.

(b) Now from Earth's frame in total 9 lightyears distance is travelled with velocity $v = 0.75c$ giving a time of $t = 9/0.75\,y = 12$ years spent travelling and a year at rest. The time spent travelling will be shorter in the moving frame, as $\tau = t\sqrt{1 - v^2/c^2} \approx 12\sqrt{7}/4 \approx 7.94$, so in total George got older only by 8.94 years while Bob by 13 years.

(c) In the modified picture George and Bobby's world line stays together until $x = 2.25$ lightyears, then George's transforms into vertical one going until the point where Boby's worldline again is at $x = 2.25$ lightyears on his way back from the Alpha Centauri then thye stay together back to $x = 0$ with slope $-c/v = -4/3$.

(d) From the motel's or Earth's rest frame they both spend 13 years travelling, however they age differnetely as George spends less time travelling with velocity $v = 0.75c$. George spends the half way to Alpha Centauri and half way back and plus one year while Bobby rests as well, at rest, in total 7 years, during which Bobby travels with $v = 0.75c$ and for him only $1 + 6\sqrt{7}/4 \approx 3.97 + 1 = 4.97$ years pass by. On the way back while in the same way as before, from the Earth's frame Bobby got older by 8.94 years, while George by 10.97 years. Note that to achive these travelling scheme, they had the same acceleration protocoll, as both of them needed to slow down and speed up twice during their journeys, so indeed it is not the acceleration causing different aging of the twins.

Problem 2

In the upper athmosphere μ particles or muons are produced by cosmic rays colliding with molecules, and then these unstable particles are moving with almost constant velocity towards the Earth's surface. The half time of decay for resting μ is $T_{1/2} = 2.2\mu\text{s} = 2.2 \cdot 10^{-6}s$.

a.) Assuming Newtonian mechanics to be correct, what distance would a μ travel (with having velocity $V_\mu \approx c$) until it is expected to decay?

b.) Assuming that muons are produced in an altitude of 10km, what fraction of them would reach the Earth's surface?

We know that Newtonian mechanics fails to describe the above questions. We want to measure the velocity of μ, therefore we perform the following experiment. We have created two identical μ detectors. One is attached to a wheather balloon and is lifted up to $h = 3$km altitude. The other one remains on the surface of Earth. We measure $n_b = 700$ counts at the balloon while only $n_s = 500$ counts on the surface in an hour.

c.) Assuming we know the V_μ velocity of the muons, what is the connection between n_b and n_s?

d.) According to the measured velues, determine the velocity of the μ particles.

Solution:

a.) Decay law can be written in terms of distances, $n(t) = n(0)e^{-t/T_{1/2}} = n(0)e^{-x/x_{1/2}}$, when the particles have veloctiy v with $x_{1/2} = vT_{1/2}$. The approximate value of $x_{1/2} \approx 660\,m$.

b.) This implies that the number of muons reaching the surface is $n(x_{\text{Earth}}) \approx n(0)e^{-10000/660} \approx \frac{n_0}{30000}$, which is not the correct measured values. The reason for the discrepancy is that decay happens in the moving frame of the muons, so the number of muons is halved when their *proper time* is equal to the decay time. The error is coming from the fact, that we did our predictions from our rest/laboratory frame!

c.) Strategy: knowing the number of muons at altitude of $h = 3$ km and on the surface we calculate the decay time measured in the muons' frame:

$$n_s = n_b e^{-\tau/T_{1/2}} \Rightarrow \tau = T_{1/2} \ln(n_b/n_s) \approx 10^{-6}s \tag{1}$$

where every calculation was performed with the proper time of the muons and from where one can calculate the muons' velocities, that is the time passed by in the laboratory frame is simply h/v, from where

$$\tau = \frac{h}{v}\sqrt{1 - v^2/c^2} \Rightarrow v = \frac{1}{\sqrt{(\tau/h)^2 + 1/c^2}} = \frac{c}{\sqrt{1 + 9 \times 10^{16}\frac{10^{-12}}{9\times10^6}}} \approx \frac{c}{1 + 5 \times 10^{-3}} \approx 0.995c \tag{2}$$

Problem 3

At time $t = 0$ two spacecrafts depart from Earth in perpendicular directions with velocities $3/5c$.

a.) Determine the position vectors $r_1(t)$ and $r_2(t)$ of the two spacecrafts as a function of time. (Use a convenient coordinate-system in the reference frame of Earth.)

b.) Let's sit in the reference frame of the spacraft "1". Determine the position vector $r_2'(t')$ of the spacraft "2" in this reference frame.

c.) What is the velocity vector of the 2nd spacecraft in that reference frame? Determine also the direction of this velocity vector.

Solution:

a.) The vectors of the spacecrafts: $\vec{r}_1(t) = \begin{pmatrix} vt \\ 0 \\ 0 \end{pmatrix}$, $\vec{r}_2(t) = \begin{pmatrix} 0 \\ vt \\ 0 \end{pmatrix}$. The value of the contraction factor is $\sqrt{1 - v^2/c^2} = 4/5$.

b.) Writing up the second spacecraft's coordinates from the frame of the first we have the Lorentz transformations:

$$ct' = \frac{ct - \frac{v}{c}x}{\sqrt{1 - v^2/c^2}} = \frac{ct}{\sqrt{1 - v^2/c^2}} = \frac{5}{4}ct \tag{3}$$

$$x' = \frac{x - \frac{v}{c}ct}{\sqrt{1 - v^2/c^2}} = \frac{-vt}{\sqrt{1 - v^2/c^2}} = -\frac{5}{4}vt = -vt' \tag{4}$$

$$y' = vt = vt'\sqrt{1 - v^2/c^2} = \frac{4}{5}vt'. \tag{5}$$

$$z' = 0 \tag{6}$$

The last two rows need a bit more explanation. First in the direction of z nothing happens none of the spacecrafts moves in that way. Nevertheless Lorentz transformation does not change the $y = y'$ component of the spacecraft, as it is time dependent and we are writing up everything according to the coordinates of the first spacecraft's frame we simply substitute the expression obtained from the first row, $t = \frac{4}{5}t'$. The first two rows follow easily form the "usual" $1+1$ dimensional Lorentz transformation with $x = 0$ as the position from rest frame of the second spacecraft.

c.) So we get from the above discussion:

$$\vec{r}_2'(t') = \begin{pmatrix} -vt' \\ v\sqrt{1-v^2/c^2}t' \\ 0 \end{pmatrix} \Rightarrow \vec{v}' = \frac{\mathrm{d}\vec{r}_2'(t')}{\mathrm{d}t'} = \begin{pmatrix} -v \\ v\sqrt{1-v^2/c^2} \\ 0 \end{pmatrix} \tag{7}$$

From the point of view of the 2. spacecraft we just swap the coordinates $x \leftrightarrow y$ and write the previous results:

$$\vec{r}_2''(t'') = \begin{pmatrix} v\sqrt{1-v^2/c^2}t'' \\ -vt'' \\ 0 \end{pmatrix} \Rightarrow \vec{v}'' = \frac{\mathrm{d}\vec{r}_2(t'')}{\mathrm{d}t''} = \begin{pmatrix} v\sqrt{1-v^2/c^2} \\ -v \\ 0 \end{pmatrix} \tag{8}$$

Problem 4

Revision of four vectors and tensors:

a.) Using Einstein's convention, and the metric tensor, express the Minkowski length square of a^μ and the Minkowskian scalar product of a^μ and b^μ.

b.) How we define the covariant coordinates of these 4-vectors? Determine the $a_\mu = (a_0\, a_1\, a_2\, a_3)$ and $b_\mu = (b_0\, b_1\, b_2\, b_3)$ "lower index" coordinates. With the help of these covariant coordinates, express again the Minkowski length square of a_μ and the Minkowskian scalar product of a_μ and b_μ.

c.) As we see, the indices can be lowered by multiplication with the $g_{\mu\nu}$ tensor. The inverse of this manipulation is the "raising" of indices. What tensor $g^{\mu\nu}$ can be used to raise the indices?

Solution:

a.) In relativity laws of nature should be invariant under Lorentz transformation $\Rightarrow$ most convenient way to formulate laws of physics is using four vectors and four tensors:
A general n component tensor, for $n = 1$ we have the familiar four vectors, while for $n = 2$ four matrices, is called four tensor if it transforms according to some Lorentz transformation:

$$(Q')^{\mu_1\mu_2\ldots\mu_n} = \Lambda^{\mu_1}_{\cdot\nu_1}\Lambda^{\mu_2}_{\cdot\nu_2}\ldots\Lambda^{\mu_n}_{\cdot\nu_n}Q^{\nu_1\nu_2\ldots\nu_n} \tag{9}$$

this is the transformation of a tensor of n contravariant indices. More generally tensors with n covariant and m contravariant indices transform as

$$(Q')^{\mu_1\mu_2\ldots\mu_n}_{\nu_1\nu_2\ldots\nu_n} = \Lambda^{\mu_1}_{\cdot\rho_1}\Lambda^{\mu_2}_{\cdot\rho_2}\ldots\Lambda^{\mu_n}_{\cdot\rho_n}\Lambda^{\cdot\sigma_1}_{\nu_1}\Lambda^{\cdot\sigma_2}_{\nu_2}\ldots\Lambda^{\cdot\sigma_n}_{\nu_n}Q^{\rho_1\rho_2\ldots\rho_n}_{\sigma_1\sigma 2\ldots\sigma_n} \tag{10}$$

b.) Convention for *contravariant* four vectors, the ones that transform according to the contravariant Lorentz transformations, like four coordinates transformed according to the familiar Lorentz boosts:

$$x^\mu = \left(x^0, x^1, x^2, x^3\right) = (ct, x, y, z) = (ct, \mathbf{x}) \tag{11}$$

$$a^\mu = \left(a^0, a^1, a^2, a^3\right) = \left(a^0, \mathbf{a}\right) \to (a')^\mu = \Lambda^\mu_{\cdot\nu}a^\nu \tag{12}$$

$$\text{e.g.: for one-dimensional motion along the x axis:} \tag{13}$$

$$(a')^0 = \frac{a^0 - \frac{v}{c}a^1}{\sqrt{1-v^2/c^2}} \tag{14}$$

$$(a')^1 = \frac{a^1 - \frac{v}{c}a^0}{\sqrt{1-v^2/c^2}} \tag{15}$$

for an x directed boost.
For two contravariant four vectors we have the "scalar product" or "Minkowski product", and the "Minkowski length" (in reality a form):

$$a \cdot a = a^\mu g_{\mu\nu} a^\nu = a^\mu a_\mu = a^0 a^0 - a^1 a^1 - a^2 a^2 - a^3 a^3 \tag{16}$$

where $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is the metric tensor for contravariant four vectors, which in special relativity takes this simple form and in general it lower indices, that is it transforms *contravariant* vectors to its *covariant* versions in a symmetric way, as can be seen from the invariance of scalar product, $a^\mu g_{\mu\nu} = a_\nu, \ g_{\mu\nu} a^\nu = a_\mu$

c.) Covariant four vectors, the ones that transfrom according to the covariant Lorentz transformations

$$x_\mu = (x_0, x_1, x_2, x_3) = \left(x^0, -x^1, -x^2, -x^3\right) = (ct, x, y, z) = (ct, -\mathbf{x}) \tag{17}$$

$$a_\mu = (a_0, a_1, a_2, a_3) = \left(a^0, -a^1, -a^2, -a^3\right) = \left(a^0, -\mathbf{a}\right) \Rightarrow a'_\mu = \Lambda_\mu^{\cdot\nu} a_\nu \tag{18}$$

The Minkowski length and scalar product then can be expressed analogously with the help of covariant vectors:

$$a \cdot a = a_\mu g^{\mu\nu} a_\nu = a_0 a_0 - a_1 a_1 - a_2 a_2 - a_3 a_3, \quad a \cdot b = a_\mu g^{\mu\nu} b_\nu = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 \tag{19}$$

which are the same as for contravariant vectors, as the minus signs cancel in the spatial parts and where we introduced the $g^{\mu\nu}$ metric tensor for *covariant* vectors raising indices, agian in a symmetric way! We see that this is nothing else but "scalar product" between the same vectors *covariant* and *contravariant* representations, between which the metric tensor provides connection by "lowering" and "raising" the indices, now written out for both cases

$$a^\mu = g^{\mu\nu} a_\nu, \quad a_\mu = g_{\mu\nu} a^\nu. \tag{20}$$

The natural question arises, what is the connection between the two metric tensors?

d.) The answer is that they are each other's inverses: abstract way:

$$a \cdot b = a^\mu b_\mu = a^\mu g_{\mu\nu} b^\nu = a_\nu b^\nu = a_\nu g^{\nu\mu} b_\mu = a_\nu g^{\nu\mu} g_{\mu\epsilon} b^\epsilon = a_\nu \delta^\nu_\epsilon b^\epsilon \Rightarrow g^{-1}_{\mu\nu} = g^{\mu\nu} \tag{21}$$

and vice versa, $(g^{\mu\nu})^{-1} = g_{\mu\nu}$.

Problem 5

Consider the following transformation,

$$\Lambda^\mu_{\cdot\nu} = \begin{pmatrix} 5/3 & 0 & -4/3 & 0 \\ -4/3 & 0 & 5/3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

a.) Consider the 4-vector $a^\mu = (1, 1, 0, 0)$. What is its Minkowski length square? Apply the above transformation on this vector. Show that its Minkowski length square is invariant.

b.) Consider the 4-vector $b^\mu = (6, 1, 3, 1)$, and show that its Minkowski lenght square is also invariant.

c.) Show in general, that the transformation $\Lambda^\mu_{\cdot\nu}$ is a Lorentz transformation.

d.) Express the components b_μ. Express also the transformed b'_μ components.

e.) Determine the appropriate form of Λ that transforms the lower-index vectors, $b'_\mu = \Lambda_\mu^{\cdot\nu} b_\nu$.

f.) Show that $a^\mu b_\mu$ remains invariant.

g.) Show directly that $\Lambda_{\rho}^{\cdot\mu}\Lambda^{\rho}_{\cdot\nu} = \delta^{\mu}_{\cdot\nu}$, where $\delta^{\mu}_{\cdot\nu}$ stands for the Kronecker-delta.

Solution:

a.) The length is simply $\sqrt{a \cdot a} = 1^2 - 1^2 - 0^2 - 0^2 = 0$. Applying the above transformation we have

$$(a')^{\mu} = \begin{pmatrix} 5/3 \\ -4/3 \\ -1 \\ 0 \end{pmatrix} \tag{22}$$

Now the length is invariant as

$$a' \cdot a' = 15/9 - 16/9 - 1 = 0 \tag{23}$$

b.) For vector $b^{\mu} = (6, 1, 3, 1)$ we have Minkowski length square, $b \cdot b = 36 - 1 - 9 - 1 = 25$, now the transformed vector has Minkowski length

$$(b')^{\mu} = \Lambda^{\mu}_{\cdot\nu} b^{\nu} = \begin{pmatrix} 6 \\ -3 \\ -1 \\ 1 \end{pmatrix} \Rightarrow b' \cdot b' = 36 - 9 - 1 - 1 = 25 \tag{24}$$

the same as for b^{μ}, as it should be the case.

c.) If $\Lambda^{\mu}_{\cdot\nu}$ is a Lorentz transformation then it need to preserve all four vectors scalar products, that is the scalar products are *Lorentz invariant.*

$$a^{\mu} g_{\mu\nu} b^{\nu} = (a')^{\mu} g_{\mu\nu} (b')^{\nu} = a^{\sigma} \Lambda^{\mu}_{\cdot\sigma} g_{\mu\nu} \Lambda^{\nu}_{\cdot\epsilon} b^{\epsilon} = a^{\sigma} g_{\sigma\epsilon} b^{\epsilon} \Rightarrow g_{\sigma\epsilon} = \Lambda^{\mu}_{\cdot\sigma} g_{\mu\nu} \Lambda^{\nu}_{\cdot\epsilon}, \;\; g = \Lambda^T g \Lambda \tag{25}$$

We check it explicitly below

$$\begin{aligned} &\begin{pmatrix} 5/3 & -4/3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -4/3 & 5/3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 5/3 & 0 & -4/3 & 0 \\ -4/3 & 0 & 5/3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 5/3 & 4/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -4/3 & -5/3 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 5/3 & 0 & -4/3 & 0 \\ -4/3 & 0 & 5/3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned} \tag{26}$$

d.) Covariant b_{μ} and a_{μ} components:

$$b_{\mu} = g_{\mu\nu} b^{\nu} = \begin{pmatrix} 6 \\ -1 \\ -3 \\ -1 \end{pmatrix}, \;\; (b')_{\mu} = g_{\mu\nu} (b')^{\nu} = \begin{pmatrix} 6 \\ 3 \\ 1 \\ -1 \end{pmatrix} \tag{27}$$

$$a_{\mu} = g_{\mu\nu} a^{\nu} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \;\; (a')_{\mu} = g_{\mu\nu} (a')^{\nu} = \begin{pmatrix} 5/3 \\ 4/3 \\ 1 \\ 0 \end{pmatrix} \tag{28}$$

e.) According to the above formulas let us determine how the covariant components transform:

$$(b')_{\mu} = g_{\mu\nu} (b')^{\nu} = g_{\mu\nu} \Lambda^{\nu}_{\cdot\sigma} b^{\sigma} = g_{\mu\nu} \Lambda^{\nu}_{\cdot\sigma} g^{\sigma\epsilon} b_{\epsilon} \equiv \Lambda_{\mu}^{\cdot\epsilon} b_{\epsilon} \tag{29}$$

where the Lorentz transformation's matrix transforming covariant vectors is defined as

$$\Lambda_{\mu}^{\cdot\nu} = g_{\mu\sigma} \Lambda^{\sigma}_{\cdot\epsilon} g^{\epsilon\nu} \tag{30}$$

f.) Now specifically it is really easy to check that the Minkowski scalar product remains invariant:

$$a \cdot b = 1 \cdot 6 - 1 \cdot 1 - 0 \cdot 3 - 0 \cdot 1 = 5 \tag{31}$$

$$a' \cdot b' = 5/3 \cdot 6 - 4/3 \cdot 3 - 1 \cdot 1 - 0 \cdot 1 = 5 \tag{32}$$

g.) The lesson is that the covariant form of Λ, denoted from now on by $\overline{\Lambda}$, can be expressed with the Lorentz matrices transforming contravariant four vectors

$$\overline{\Lambda} = g\Lambda g^{-1} = g\Lambda\Lambda^{-1}g^{-1}\left(\Lambda^{-1}\right)^T = \left(\Lambda^{-1}\right)^T \tag{33}$$

From where we have that

$$\Lambda_{\rho}^{\cdot\mu}\Lambda^{\rho}_{\cdot\nu} = \left(\overline{\Lambda}^T\right)^{\mu}_{\cdot\rho}\Lambda^{\rho}_{\cdot\nu} = \delta^{\mu}_{\cdot\nu} \tag{34}$$

where we exploited that $\overline{\Lambda} = \left(\Lambda^{-1}\right)^T \Rightarrow \overline{\Lambda}^T = \Lambda^{-1}$.

h.) Supplementary, extra exercise for tensors:
Four tensors with two indices: Show that the trace of a four tensor that is Lorentz invariant:

$$\mathrm{Tr}Q = Q^{\mu}_{\mu} \equiv g_{\mu\nu}Q^{\mu\nu} \tag{35}$$

Use the fact that $\Lambda^T g\Lambda = g$ or with indices again $g_{\mu\nu} = \Lambda^{\sigma}_{\cdot\mu}g_{\sigma\rho}\Lambda^{\rho}_{\cdot\nu}$ and introduce the Lorentz transformed tensor $(Q')^{\mu\nu} = \Lambda^{\mu}_{\cdot\sigma}\Lambda^{\nu}_{\cdot\rho}Q^{\sigma\rho}$

$$(Q')^{\mu\nu} = \Lambda^{\mu}_{\cdot\sigma}\Lambda^{\nu}_{\cdot\rho}Q^{\sigma\rho} \Rightarrow \mathrm{Tr}\,Q' = (Q')^{\mu}_{\mu} \equiv g_{\mu\nu}\,(Q')^{\mu\nu} = g_{\mu\nu}\Lambda^{\mu}_{\cdot\sigma}\Lambda^{\nu}_{\cdot\rho}Q^{\sigma\rho} = g_{\sigma\rho}Q^{\sigma\rho}. \tag{36}$$

Let Q be a four tensor and A a four vector, then $b^{\mu} = Q^{\mu\nu}a_{\nu}$ is also a four vector.
Proof:
Let the Lorentz transformations be $(a')^{\mu} = \Lambda^{\mu}_{\cdot\nu}a^{\nu}$ and $(Q')^{\mu\nu} = \Lambda^{\mu}_{\cdot\sigma}\Lambda^{\nu}_{\cdot\rho}Q^{\sigma\rho}$, then we have

$$(b')^{\mu} = (Q')^{\mu\nu}\,(a')_{\nu} = \Lambda^{\mu}_{\cdot\sigma}\Lambda^{\nu}_{\cdot\rho}Q^{\sigma\rho}\Lambda_{\nu}^{\cdot\nu'}a_{\nu'} = \delta^{\nu'}_{\cdot\rho}\Lambda^{\mu}_{\cdot\sigma}Q^{\sigma\rho}a_{\nu'} = \Lambda^{\mu}_{\cdot\sigma}Q^{\sigma\rho}a_{\rho} = \Lambda^{\mu}_{\sigma}b^{\sigma} \tag{37}$$

where we used the identity, derived above, $\Lambda^{\nu}_{\cdot\rho}\Lambda_{\nu}^{\cdot\nu'} = \delta^{\nu'}_{\cdot\rho}$ and indeed we recoverd the transformation rule $(b')^{\mu} = \Lambda^{\mu}_{\cdot\nu}b^{\nu}$ as a contravariant four vector should transform!