Problem 1

We have built a "light-clock" that works in the following way. A light source emits a very short pulse, which pulse travels to a mirror whose distance from the source is denoted by d_0. The reflected light is detected by a small detector that is placed near the light source (see the figure). When the pulse arrives to the detector, the counter steps by one and a new light pulse is emitted immediately.

a.) What is the time unit of the clock? (What time is needed for the pulse to reach the detector from the source?)

Our "light-clock" uses the velocity of light to measure time, therefore it is surely Lorentz-covariant. If we put it on a moving vehicle, it will measure the "time" of the moving frame of reference. Let's put the clock on a moving vehicle as shown in the middle panel of the figure. The velocity of the vehicle is v. The moving vehicle is observed by the stick man standing on the street.

b.) Draw the trajectory of the light-ray that is emitted in the source and after the reflecion it is received by the detector, according to the observers frame of reference.

c.) What is the length of the route of the pulse?

d.) How much time is needed for the light-clock, according to the observer, to step its counter by one?

Our clock is "Lorentz covariant", therefore its time unit cannot depend on the direction of its axis. Let's put the clock on the vehicle as shown in the right panel of the figure.

e.) What is the length of the route of the light in this case?

f.) What must be the distance between the source and the mirror in this case?

Solution:

a.) Time unit: the time it take for the light signal to travel the distance to the mirror and back to the detector:

$$T_0 = \frac{2d_0}{c} \tag{1}$$

b.) Drawing in the class.

c.) Length of the route the light travels when the vehicle moves with velocity v perpendicular to the direction of the light signal.

$$l = \sqrt{d_0^2 + \left(\frac{vT}{2}\right)^2} \tag{2}$$

d.) The time it takes for the counter according to the external observer:

$$T = \frac{2l}{c} \Rightarrow d_0^2 = \frac{c^2T^2}{4} - \frac{v^2T^2}{4} \Rightarrow T = \frac{2d_0}{\sqrt{c^2 - v^2}} = \frac{T_0}{\sqrt{1 - v^2/c^2}} \tag{3}$$

but the unit of time, the instant the insider measures between two counts is unchanged, so the time measured by the observer on the street gets longer than the time measured in the reference frame $\Rightarrow$ *time dilatation.*

e.) In the x direction the time it takes for the detector to count one can be divided into two parts, the way until the detector and the way back, $T = T_1 + T_2$. The observer on the street measurs these as

$$\begin{aligned} &cT_1 = d + vT_1 \rightarrow T_1 = \frac{d}{c - v}, \quad cT_2 = d + vT_2 \rightarrow \frac{d}{c + v} \\ &T = T_1 + T_2 = \frac{\frac{2d}{c}}{1 - v^2/c^2} \end{aligned} \tag{4}$$

where now d is the distance measured bythe observer on the street, but we know that $T = \frac{\frac{2d_0}{c}}{\sqrt{1-v^2/c^2}}$ leading to the relation of the Lorentz contraction of length:

$$d = d_0\sqrt{1 - v^2/c^2} \Rightarrow \frac{c^2t^2}{c^2T^2} - \frac{x^2}{c^2T^2} = 1 \tag{5}$$

which is hyperbole.

Problem 2

The one dimensional relativistic motions are often illustrated in the so called Minkowski-plane that is spanned by the (one-dimensional) space and time axes. For convenience we use the values of ct on the time axis, therefore the coordinate axes are x and ct.

a.) Draw the coordinate axes of the Minkowski-plane! For convenience let them be perpendicular. Show the scales on the axes (in units of 'light-years')

b.) At time $t = 0$ we send light signals from the $x = 0$ position in the $+x$ and $-x$ directions. Draw the world lines of these light signals.

c.) A spacecraft is traveling with velocity V in the $+x$ direction. Draw the world line of the spacecraft.

d.) There is a very accurate atomic clock in the spacecraft. The clock was set to 0 at start. Using the invariance of the Minkowski length, draw the event in the Minkowski-plane when the clock shows "1 year".

e.) According to d.) how long does it take on the Earth while "1 year" passes on the spacecraft?

f.) Generalize d.) for spacecrafts with different velocities V. Draw the "1 year" events on the spacecrafts' world lines. What kind of curve do these events define?

Solution:

a.) 2 dimensional Descartes coordiantes system, with rods on horizontal, x axis, denoting 1 lightyears.

b.) Light cones: 2 lines with slopes ± 1 starting from zero.

c.) A line of $x = Vt = \frac{V}{c} ct$, that with slope c/V.

d.) Invariace of the Minkowski length means that it is the same in all inertial systems. Namely in the moving frame it equals the proper time of the system. Let us denote with $x_e = Vt$ the position of the spacecraft from theframe of Earth and the time in ths frame by t_e. Simlarly with x'_2 and t'_e the position and time in the frame of the spacecraft, that is the position in the spacecraft's frame is trivially zero, $x' = 0$ and in umnits of years $t'_e = 1$:

$$c^2 t_e^2 - x_e^2 = c^2 \left(t'_e\right)^2 - \left(x'_e\right)^2 \Rightarrow c^2 t_e^2 - x_e^2 = c^2 \Rightarrow \frac{c^2 t^2}{c^2} - \frac{x^2}{c^2} = 1 \tag{6}$$

which are hyperbolas in the Minkowski space, where x and t is where and when we see/measure the spacecraft!

e.) Frame of Earth:

$$x_e = V t_e, \;\; t_e \tag{7}$$

Frame of the spacecraft:

$$x'_e = 0, \;\; t'_e = T = 1 \text{ year} \tag{8}$$

Using Minkowski equality as before

$$c^2 t_e^2 - V^2 t_e^2 = c^2 T^2 - 0 \Rightarrow t_e = \frac{T}{\sqrt{1 - v^2/c^2}} \tag{9}$$

f.) for general events we use again the Minkowski equality:

$$c^2 t_e^2 - x_e^2 = c^2 \left(t'_e\right)^2 - \left(x'_e\right)^2 \tag{10}$$

abd arrie for the same eresult as above, that the one-year events lie on a hyperbolae.

Problem 3

a.) A rod of length 1 meter rests in our frame of reference. Draw the world lines of its endpoints in the Minkowski plane.

b.) An observer having velocity V passes the rod. Draw the events in the Minkowski plane where the observer is at the endpoints of the rod.

c.) How long time does it take in the reference frame of the rod while the observer passes the rod?

d.) How much time does it take to pass the rod from the observers point of view? (Use the result of Problem 1.)

e.) The observer has calculated the length of the rod. What is the result?

Solution:

a.) The resting rod's endpoints are represented by just vertical lines at $x = 0$ nad $x = L$.

b.) This world line is of slope c/V and passes trhough $x = 0$ at $t = 0$ and at $x = L$ at $t = L/v$

c.) In the rod's reference frame the observer passes it in time $t = L/v$.

d.) From the observer's point of view we can use the results of Problem 1., namely time dilatation says that

$$t = \frac{\tau}{\sqrt{1 - v^2/c^2}} \Rightarrow \tau = t\sqrt{1 - v^2/c^2} \tag{11}$$

e.) The observer calculates its length, when the rod moves with velocity v backwards and it takes τ time for the observer while the rod passes it, so the measured length is

$$L' = v\tau = vt\sqrt{1 - v^2/c^2} = L\sqrt{1 - v^2/c^2} \tag{12}$$

leading again to Lorentz contraction, that is, from another point of view, the observer feels the rod coming in the opposite direction with the same velocity, that is why he measures a contracted length of the moving object, again as in Problem 1.

Problem 4

Einstein's famous train was hit by two lightnings at the two ends. According to the workers who worked on the fields nearby, the two events happened exactly at the same time. The velocity of the train is V.

a.) Draw the world lines of the trains endpoints in the Minkowski plane. Mark the two lightning hits.

b.) Draw the world lines of the lightnings' light in the figure.

c.) At the middle of the train an observer is traveling. Draw her/his world line in the figure!

d.) Which lightning happend earlier from the observers point of view?

e.) How large is this time difference, if the (resting) length of the train is 100meters, and its velocity is $V = 180\text{km/h}$?

Solution:

1. The trains' endpoints go along lienswith slope c/v startin gat $\pm L/2$.
2. The lightnings, are sources of light emission, that is two light cones starting at $\pm L/2$.
3. The traveler$\frac{100\text{m}\times 50\text{m/s}}{9\times 10^{16}\text{m}^2/\text{s}^2 - 2500\text{m}^2/\text{s}^2}$'s world line goes with slope c/v and starts at the origin.
4. According to the outside observer the two lightinings happenedexactly at the same time, while for the traveler the lightning at the $L/2$ end of the train happened earlier.

5. The times to reach the observer are $T_{1,2} = \frac{L}{2(c \pm v)}$ in the rest frame, where $L = L_0\sqrt{1 - v^2/c^2}$ the Lorentz contracted length, but it must then be trasformed to the train's frame

$$\Delta\tau = \Delta T\sqrt{1 - v^2/c^2} \approx 16.5\mu\, s \tag{13}$$

by the data $L_0 = 100m,\ v = 50\, m/s$.
The boring calcualtion with numbers looks like:

$$\begin{aligned} \Delta\tau = \Delta T\sqrt{1 - v^2/c^2} &= \frac{100\text{m} \times 50\text{m/s}}{9 \times 10^{16}\text{m}^2/\text{s}^2 - 2500\text{m}^2/\text{s}^2}\sqrt{1 - 2500/9 \times 10^{-16}} \\ \approx \frac{100\text{m} \times 50\text{m/s}}{9 \times 10^{16}\text{m}^2/\text{s}^2 - 2500\text{m}^2/\text{s}^2} &\approx \frac{5 \times 10^3}{9 \times 10^{16}} \approx 5,5 \times 10^{-19}\text{s} \end{aligned} \tag{14}$$

Problem 5

In a reference frame it may be an important issue to synchronize clocks at different positions. However, we know that "moving" clocks can change their running speed, therefore we want to solve the problem without moving the clocks. We have figured out the following: we put a tape measure in the direction x, and at every $L_0 = 1$km we install a clock. However, the clocks are not synchronized yet. Then at $t = 0$ from the $x = 0$ a light pulse is emitted. The clocks at different positions are synchronized by setting them to $t = x/c$ at the moment when the light pulse arrives. After that setting all the clocks are synchronized to the one in the origin.

a.) Draw the world lines of the clocks in the Minkowski plane, and also the world line of the synchronizing pulse. Demonstrate that the clocks are properly synchronized.

Now let's suppose that the tape measure and the clocks are moving with velocity V in the x direction. (Let for example $V = 0.6c$)

b.) Using the result of Problem 3.) what is the measured distance between the moving clocks the resting reference frame? Using this result draw the world lines of the moving clocks in the Minkowski plane.

c.) A synchronizing pulse is now emitted from $x' = 0$ at time $t' = 0$. Draw the world line of this pulse, and mark the events where the moving clocks are synchronized.

d.) Following the result of Problem 2d) draw the space-time points in the Minkowski plane, where $t' = 0$. What is the meaning of this line?

Solution:

a.) Clocks' world lines are vertical spaced at $x = L_0,\ 2L_0, \ldots$

b.) Moving clocks' have contraced distance, $L = L_0\sqrt{1 - v^2/c^2}$ and theirworld lines now go with slope c/v.

c.) The events in the moving frame are the same, $t' = L_0/c,\ 2L_0/c \ldots$, but in the rest frame these times need to be transformed back

$$L = L_0\sqrt{1 - v^2/c^2} \Rightarrow \Delta t = \Delta t'/\sqrt{1 - v^2/c^2} \tag{15}$$

d.) For this we need to draw the lines of the new coordinate system of the moving frame, that is a ct' and a x' axis, with slopes c/v and v/c respectively, the new x' are the one where in the moving frame $t' = 0$! And we can see that indeed we have different times in the rest frames for the clocks simultaneous $t' = 0$ times!