A14

Two bricks (masses: m_1 and m_2) are connected by a spring (strength: D). The Hamiltonian of the system is

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{D}{2}(x_2 - x_1)^2 \ .$$

Consider the following quantities:

$$P = p_1 + p_2 \ ,$$

$$F = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} - t\frac{p_1 + p_2}{m_1 + m_2} \ .$$

(a) By determining the $\{P, H\}$ Poisson bracket show that P is a conserved quantity.

(b) By determining the $\{F, H\}$ Poisson bracket show that F is a conserved quantity. Be careful, F depends explicitly on time (Include also the partial time derivative into the total time derivative of F).

A15

The Hamiltonian of a system with two degrees of freedom reads

$$H = q_1 p_1 - q_2 p_2 - A q_1^2 + B q_2^2, \tag{1}$$

where A and B are real parameters.

(a) Write down the equations of motion (Hamilton's canonical equations) for the system..

(b) Consider the following quantities:

$$F_1 = \frac{p_1 - A q_1}{q_2} \qquad F_2 = q_1 q_2 \tag{2}$$

Calculate the Poisson brackets $\{F_1, H\}$ and $\{F_2, H\}$. Show that both quantities are constants of motion.

(c) Calculate the Poisson bracket $\{F_1, F_2\}$. Let $F_3 = \{F_1, F_2\}$. Is it a constant of motion?

B11

Consider the two dimensional harmonic oscillator, given by the Hamiltonian:

$$H = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2}m\omega^2(x_1^2 + x_2^2) \ .$$

Let us construct the 2×2 matrix A_{jk} through

$$A_{jk} = \frac{1}{2}\left(\frac{1}{m}p_i p_j + m\omega^2 x_i x_j\right) \ ,$$

Show that the A_{jk} matrix elements are conserved quantities. **Solution:**
We again use the trivial rules that $\{p_i, p_j\} = \{x_i, x_j\} = 0$, $\{x_i, p_j\} = \delta_{ij}$. So in general the first term in A_{ij}

$$\begin{aligned}
&\frac{1}{2m}\{p_i p_j, H\} = \frac{\omega^2}{4}\{p_i p_j, x_1^2 + x_2^2\} = \frac{\omega^2}{2}\left(x_1\{p_i p_j, x_1\} + x_2\{p_i p_j, x_2\}\right) \\
&= \frac{\omega^2}{2}\left(x_1 p_i\{p_j, x_1\} + x_2 p_i\{p_j, x_2\} + x_1 p_j\{p_i, x_1\} + x_2 p_j\{p_i, x_2\}\right) \\
&= -\frac{\omega^2}{2}\left(x_1 p_i \delta_{j2} + x_1 p_j \delta_{i2} + x_2 p_i \delta_{j1} + x_2 p_j \delta_{i1}\right)
\end{aligned} \tag{3}$$

Now for the second term we have

$$\frac{\omega^2}{4}\{x_i x_j, p_1^2+p_2^2\} = \frac{\omega^2}{2}\left(2p_1\{x_i x_j, p_1\} + 2p_2\{x_i x_j, p_2\}\right)$$
$$= \frac{\omega^2}{2}\left(2p_1 x_i\{x_j, p_1\} + 2p_1 x_j\{x_i, p_1\} + 2p_2 x_i\{x_j, p_2\} + 2p_2 x_j\{x_i, p_2\}\right) = \frac{\omega^2}{2}\left(x_1 p_i \delta_{j2} + x_1 p_j \delta_{i2} + x_2 p_i \delta_{j1} + x_2 p_j \delta_{i1}\right) \tag{4}$$

which is exactly the same as the first term but with different sign, giving in total a zero Poisson bracket!

B12

Consider again the two dimensional harmonic oscillator as defined in the previous exercise. In particular let us construct the following linear combinations of the A_{jk} matrix elements:

$$S_1 = \frac{A_{12}}{\omega} \qquad S_2 = \frac{A_{22}-A_{11}}{2\omega} \qquad S_3 = \frac{L}{2} = \frac{1}{2}(xp_y - yp_x)\ .$$

(a) Show that the Poisson brackets of the S_i quantities are

$$\{S_1, S_2\} = S_3\ , \qquad \{S_3, S_1\} = S_2\ , \qquad \{S_2, S_3\} = S_1$$

(b) Show that $H^2 = 4\omega^2(S_1^2 + S_2^2 + S_3^2)$.