A18

The Hamiltonian of a system with one degree of freedom reads as

$$H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right) \tag{1}$$

Consider a canoncial transformation that is generated by a 2nd type generator function

$$W_2(q, P) = \frac{P}{q} \tag{2}$$

(a) Using the derivatives of the generator function determine the $p(q, P)$ and $Q(q, P)$ relations.

(b) Using the results of a.) express the "old" variables in terms of the "new" ones, i.e. find the $q(Q, P)$ and $p(Q, P)$ functions.

(c) Determine the new form $K(Q, P)$ of the Hamiltonian.

(d) Starting from the new Hamiltonian determine the canonical equations for the new coordinate and momentum.

(e) Determine the solutions $Q(t)$ and $P(t)$.

A19

Consider the following transformation that rotates the coordinate axes of the phase-space (α is a real parameter):

$$Q = q\cos(\alpha) - p\sin(\alpha) \qquad P = q\sin(\alpha) + p\cos(\alpha) \tag{3}$$

(a) By calculating the Poisson bracket $\{Q, P\}$ show that the transformation is canonical.

(b) We would like to find a $W_2(q, P)$ that generates the trasformation defined above. As a first step transform the relations above, and find the mixed $p(q, P)$ and $Q(q, P)$ functions.

(c) Using the results of b.) determine the derivatives $\frac{\partial W_2}{\partial q}$ and $\frac{\partial W_2}{\partial P}$.

(d) Solve the differential equations of c.), i.e. give an appropriate function $W_2(q, P)$.

B17

The Hamiltonian of a one-dimensional Harmonic oscillator reads as

$$H = \frac{1}{2}q^2 + \frac{1}{2}p^2 \tag{4}$$

(We arrived to this special form ($m = 1$, $\omega = 1$) by rescaling time and energy units.)

(a) Consider the complex transformation

$$Q = \frac{x + ip}{\sqrt{2}} \qquad P = \frac{ix + p}{\sqrt{2}} \tag{5}$$

Using Poisson brackets show, that the transformation is canonical.

(b) Construct a 2nd type generator function that generates the above defined transformation.

(c) Determine the new form $K(Q, P)$ of the Hamiltonian. Write down and solve the canonical equations of motion.

(d) You can see, that the new Hamiltonian is complex valued, and the solutions of the canonical equations are also complex functions. However, the original p and x variables are real. Show that for real x and p the relation $P = iQ^*$ holds. Show that during the time evolution of Q and P this condition is conserved.

B18

Consider the following transformation,

$$Q = q^\alpha \cos(\beta p), \qquad P = q^\alpha \sin(\beta p) \tag{6}$$

where α and β are real parameters.

(a) Calculate the Poisson bracket $\{Q, P\}$ for generic α, β.

(b) What should be the relation between α and β to get a canonical transformation?

(c) Divide the two equations with each other, and determine the $Q(p, P)$ relation.

(d) Search for an appropriate 4th type $W_4(p, P)$ generator function. Use the result of c.).