A20

A cilinder having moment of inertia Θ can easily rotate around a fixed axis. The Hamiltonian of the system is

$$H(\phi, p_\phi) = \frac{p_\phi^2}{2\Theta} \tag{1}$$

(a) Write down the Hamilton-Jacobi equation for the system.

(b) Following the usual separation $S(\phi, t) = S_0(\phi) - Et$ write down the abbreviated Hamilton Jacobi equation for S_0.

(c) Solve the abbreviated equation, i.e. express the solution $S_0(\phi, E)$.

(d) At $t = 0$ the cilinder is in the position $\phi = 0$, and has angular momentum $p_\phi = L$. Using this information determine the values of the constants of motion E and $\beta_E = \frac{\partial S}{\partial E}$.

(e) From the results of d.) determine the $\phi(t)$ solution of the equations of motion.

A21

The Hamiltonian of a free particle in special relativity is

$$H(x, p) = \sqrt{m^2c^4 + p^2c^2} \tag{2}$$

(a) Write down the Hamilton-Jacobi equation for the system.

(b) Following the usual separation of time, by introducing the constant E, write down the abbreviated Hamilton-Jacobi equation.

(c) Solve the abbreviated equation. Determine also the solution of the full Hamilton-Jacobi equation.

(d) Determine β_E , which is the canonical pair of E.

Solution:
Hamilton-Jacobi with the the Hamiltonain with the help of the genreating function is expressed as

$$H\left(x, \frac{\partial S}{\partial x}\right) = \sqrt{m^2c^4 + c^2\left(\frac{\partial S}{\partial x}\right)^2} \tag{3}$$

So the equation takes the form

$$\sqrt{m^2c^4 + c^2\left(\frac{\partial S}{\partial x}\right)^2} + \frac{\partial S}{\partial t} = 0 \tag{4}$$

again looking for S in terms of $S = S_x(x) - Et$

$$\sqrt{m^2c^4 + c^2\left(\frac{\partial S}{\partial x}\right)^2} = E \tag{5}$$

From here

$$\frac{\partial S}{\partial x} = \frac{1}{c}\sqrt{E^2 - m^2c^4} \to S = \frac{1}{c}\sqrt{E^2 - m^2c^4}x - Et \tag{6}$$

Now let us introduce for the new canonical and conserved momentum $\alpha = E$. Suppose initially the particle travels with momentum p_0 starting from position x_0, then

$$\partial_x S_x = \frac{1}{c}\sqrt{E^2 - m^2c^4} = \frac{1}{c}\sqrt{\alpha^2 - m^2c^4} \tag{7}$$

as already obtained above, as the derivative of the action is independent of the coordinate. Now for new coordinate we have

$$\beta = \frac{\partial S}{\partial \alpha} = -t + \frac{1}{c}\frac{\alpha}{\sqrt{\alpha^2 - m^2c^4}}x = \frac{1}{c}\frac{\alpha}{\sqrt{\alpha^2 - m^2c^4}}x_0 \to x(t) = \frac{\sqrt{\alpha^2 - m^2c^4}}{\alpha}ct + x_0\,. \tag{8}$$

A22

A particle of mass m can move along the x axis while a $V(x) = k|x|$ potential is also present. The Hamiltonian of the system is

$$H = \frac{p^2}{2m} + k|x| \; .$$

Our goal is to determine the the period T of the oscillations as a function of the energy of the particle.

(a) Knowing the energy E of the particle, determine amplitude of the oscillations, i.e. $x_{\max}(E)$.

(b) Determine the particles momentum as a function of its position and energy, $p(x, E)$.

(c) For a fixed value E draw the contour line of $H(p, x)$ in the $p - x$ phase plane.

(d) By calculating the “phase-area” surrounded by the contour line determine the action variable as a function of energy $I(E)$. We know that $\int_{-1}^{1} dx\sqrt{1-|x|} = \frac{4}{3}$.

(e) Determine the period of the oscillation as a function of the energy.

A23

A charged particle can move in the $x - y$ plane in the presence of a magnetic field parallel to the z axis. A possible Hamiltonian of the system is

$$H = \frac{p_x^2}{2m} + \frac{1}{2m}(p_y - eBx)^2 \; . \tag{9}$$

(a) Write down the Hamilton-Jacobi equation of the system.

(b) Apply the separation $S(x, y, t) = S_0(x, y, E) - Et$ and write down the differential equation for S_0.

(c) Separate further as $S_0(x, y, E) = S_y(x, E, \alpha) + S_y(y, E, \alpha)$ and substitute this form into the Hamilton-Jacobi equation of S_0.

(d) All the y dependences are encoded only in the $\frac{\partial S_y}{\partial y}$. Choose therefore the constant solution of $\frac{\partial S_y}{\partial y} = \alpha$.

(e) Write down then the equation for $S_x(x, y, E, t)$ and solve it!

Solution:
Hamilton-Jacobi with $p_{x,y} = \partial_{x,y} S$:

$$\frac{(\partial_x S)^2}{2m} + \frac{1}{2m}(\partial_y S - eBx)^2 + \partial_t S = 0 \tag{10}$$

Again separating the generator as $S(x, y, E, \alpha, t) = S_x(x, E, \alpha) + S_y(y, E, \alpha) - Et$ we have

$$\frac{(\partial_x S_x)^2}{2m} + \frac{1}{2m}(\partial_y S_y - eBx)^2 = E \tag{11}$$

Now as only $\partial_y S_y$ dependson y it will be a constatn, $\partial_y S_y = \alpha$, rewriting the equation we get

$$\frac{(\partial_x S_x)^2}{2m} + \frac{1}{2m}(\alpha - eBx)^2 = E \tag{12}$$

fro mhere expressing $\partial_x S_x$ we get

$$\begin{aligned} \partial_x S_x &= \sqrt{2mE - (\alpha - eBx)^2} \to S_x = \sqrt{2mE}\int \mathrm{d}x \sqrt{1 - \left(\frac{\alpha}{\sqrt{2mE}} - \frac{eBx}{\sqrt{2mE}}\right)^2} \\ &= \frac{mE}{eB}\left(\left(\frac{\alpha}{\sqrt{2mE}} - \frac{eBx}{\sqrt{2mE}}\right)\sqrt{1 - \left(\frac{\alpha}{\sqrt{2mE}} - \frac{eBx}{\sqrt{2mE}}\right)^2} + \arcsin\left(\frac{\alpha}{\sqrt{2mE}} - \frac{eBx}{\sqrt{2mE}}\right)\right) \end{aligned} \tag{13}$$

now supposing to start initially with momentum $p_x(0) = p_0$ and $p_y(0) = 0$ and positions $x_0 = y_0 = 0$, we can write that

$$\partial_x S_x = p_0 = \sqrt{2mE - (\alpha - eBx_0)^2} \to \alpha = \sqrt{2mE - p_0^2} \tag{14}$$

Now the time-evolution of the coordinates can obtained again by the initial conditions

$$\partial_E S = \text{const.} = \beta_E \equiv \partial_E S|_{t=0}, \;\; \partial_\alpha S = \text{const.} = \beta_\alpha \equiv \partial_\alpha S|_{t=0} \tag{15}$$

fro which knwoing the initial conditions one can trae back the time-evolution of the coordinates!

B19

A particle can move along the x axis. An external force that increases linearly in time acts on the particle, therefore the Hamiltonian of the system is

$$H(x,p) = \frac{p^2}{2m} - Axt \tag{16}$$

(a) Write down the Hamilton-Jacobi equation for the system.

(b) The Hamilton function depends explicitly on time, therefore we cannot follow the usual separation technique. Try the following form instead:

$$S(x,t) = F(t)x + G(t) \tag{17}$$

Substitute this to the Hamilton-Jacobi equation.

(c) Collect the terms according to the powers of x.

(d) In order to find a solution to the equation, both the first order and zero'th order terms in x must be zero. Use this to determine the functions $F(t)$ and $G(t)$. If your calculation is correct, an integration constant appears. Denote it by α.

(e) Write down the solution $S(x,\alpha,t)$. At $t = 0$ the particle is at $x_0 = 0$ and has momentum $p_0 = 0$. Using this information determine the value of α.

(f) According to the Hamilton-Jacobi theory, $\beta = \frac{\partial S}{\partial \alpha}$ is also a constant of motion. Express its value using the initial conditions.

(g) Use the equation for β to determine the $x(t)$ solution of the equation of motion.

B20

In class we considered the particle in gravitational field. The Hamiltonian is

$$H(x,p) = \frac{p^2}{2m} + mgx \tag{18}$$

In the usual Hamilton-Jacobi approach we search a 2nd type generator function $S(x,\alpha,t)$ that makes the new Hamiltonian zero. Then we exploit the fact that the new momentum α is a constant of motion. However, this is not the only possible choice.

In this problem we consider a 3rd type generator function $S(p,\alpha)$. In this case $x = -\frac{\partial S}{\partial p}$.

(a) Write down the Hamilton-Jacobi equation.

(b) Separate the time using $S(p,E,t) = S_0(p,E) - Et$. Write down the abbreviated Hamilton-Jacobi equation for S_0.

(c) Solve the equation, and express the function $S_0(p,E)$.

(d) At the moment $t = 0$ the particle starts from the position $x = 0$ having momentum p_0. Express the value of the constant E.

(e) The quantity $\beta = \frac{\partial S}{\partial E}$ is also a constant of motion. Express its value using the initial conditions.

(f) Determine the $x(t)$ function.