Problem 1

Revisit the key ideas of canonical transformations and enlist the most important conserved quantitiies generating the most frequently used symmetries, further investigate in detail the canonical transformation generated by L_z.
As these simple considerations already signal changing transforming coordinates and momenta such that Hamiltonian equations of motion remain invariant. The above trivial case is the example of the so called infinitesimal generator method while the most general one ist the *generating function* method. REvisit the ky steps of type 1 generating functions.

(a) Consider a general physical quantity depending on $\mathbf{q}$ and $\mathbf{p}$, $F = F(\mathbf{q}, \mathbf{p})$. Following the steps in the lecture if F is a conserved quantity, i.e.: $\frac{\mathrm{d}F}{\mathrm{d}t}F = \{F, H\} = 0$ (for sake of simplicity we supposed that $\frac{\partial F}{\partial t} = 0$) then it generates a symmetry. In particular for the generalized coordinates and momenta we have that

$$\mathbf{q}' = \mathbf{q} + \delta\theta\{\mathbf{q}, F\} \tag{1}$$
$$\mathbf{p}' = \mathbf{p} + \delta\theta\{\mathbf{p}, F\} \tag{2}$$

Then the calim is that the Hamiltonian function remains invariant with the transformed variables, $H(\mathbf{q}, \mathbf{p}) = H(\mathbf{q}', \mathbf{p}')$. Correspondingly, Hamiltonian equations of moitons remain invariant for the transformed cordinates:

$$\dot{\mathbf{q}}' = \frac{\partial H}{\partial \mathbf{p}'} \equiv \{\mathbf{q}', H\}\,,\ \dot{\mathbf{p}}' = -\frac{\partial H}{\partial \mathbf{q}'} \equiv \{\mathbf{p}', H\} \tag{3}$$

(b) The most well known examples are p_x generator of translational symmetry along the x axis, L_z generator rotational symmetry around the axis z and H itself generator of time-translational invariance.

(c) Let us know turn to the particular case of L_z, where we consider an infinitesimal change in a physical quantity $A(\mathbf{q}, \mathbf{p})$

$$A'(\mathbf{q}', \mathbf{p}') = A(\mathbf{q}, \mathbf{p}) + \delta\varphi\{A, L_z\} \equiv A + \delta A \tag{4}$$

Now the infinetaismal change can be expressed as

$$\delta A = \frac{\partial A}{\partial \mathbf{q}}\delta\mathbf{q} + \frac{\partial A}{\partial \mathbf{p}}\delta\mathbf{p} \tag{5}$$

For this we intriduce the compact notation $\boldsymbol{\eta} = (\mathbf{q}, \mathbf{p}) \Rightarrow \delta\boldsymbol{\eta} = \delta\varphi\{\boldsymbol{\eta}, L_z\} = \delta\varphi J \frac{\partial L_z}{\partial \boldsymbol{\eta}}$ with $J = \begin{pmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{pmatrix}$. From this we can observe that

$$\delta A = \frac{\partial A}{\partial \boldsymbol{\eta}}\delta\boldsymbol{\eta} = \delta\varphi\frac{\partial A}{\partial \boldsymbol{\eta}}J\frac{\partial L_z}{\partial \boldsymbol{\eta}} = \delta\{A, L_z\} \tag{6}$$

as expected. Now indeed for 'large' transformations we apply the small ones $N \gg 1$ times for infinitesimal angle $\delta\varphi = \varphi/N$ and arrive at

$$A'(\mathbf{q}', \mathbf{p}') = (A + \frac{\varphi}{N}\{A, L_z\})^N \to e^{\varphi\{A, L_z\}} \tag{7}$$

while for the coordinates and momenta we obtain

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = (1 - i\frac{\varphi}{N}\sigma^y)^N \to e^{-i\varphi\sigma^y} = \begin{pmatrix} \cos(\lambda) & -\sin(\lambda) \\ \sin(\lambda) & \cos(\lambda) \end{pmatrix} \tag{8}$$

indeed! Again the trick was that we applied consequetively the infinitesimal changes

$$\boldsymbol{\eta}(\varphi_0) \to \boldsymbol{\eta}(\varphi_0 + \delta\varphi) \to \boldsymbol{\eta}(\varphi_0 + 2\delta\varphi) \to \cdots \to \boldsymbol{\eta}(\varphi_0 + N\delta\varphi) \equiv \boldsymbol{\eta}(\varphi_0 + \varphi) \tag{9}$$

(d) Now we consider new coordinate and momentum variables as $\boldsymbol{\eta} \rightarrow \boldsymbol{\xi}$ with a new Hamiltonian $K(\boldsymbol{\xi})$ satisfying

$$\dot{\boldsymbol{\xi}} = J\frac{\partial K}{\partial \boldsymbol{\xi}} \tag{10}$$

The iff relation is provieded by $MJM^{\mathrm{T}} = J$ with $M_{ij} = \frac{\partial \xi_i}{\partial \eta_j}$, that is if and only if this realtion is satisfied $\boldsymbol{\eta} \rightarrow \boldsymbol{\xi}$ is a canoncial transformation.
So the natural question arises how to find these transformations, well a simple scae is provided by symmetry transformations by conserved quantities as discussed above with L_z but hte more genral one is provided by requiring the invariance of the action in the general Hamitlonian picture

$$S = \int \mathbf{p}\mathrm{d}\mathbf{q} - H\mathrm{d}t = \int \mathcal{P}\mathrm{d}Q - K\mathrm{d}t + \mathrm{d}F \tag{11}$$

with $\mathrm{d}F$ integrating to a trivial constant and not modifying the action, usually referred to as the generating function. Now type one generating functions are the ones when

$$\mathbf{p} = \frac{\partial F}{\partial \mathbf{q}} \tag{12}$$

$$\mathcal{P} = -\frac{\partial F}{\partial \mathbf{Q}} \tag{13}$$

$$H = K - \frac{\partial F}{\partial t} \tag{14}$$

Note that this allows one to handle also explicit time dependent Hamiltonians and to find a substuting time- independnent dyanmics!

(e) Let us check how the above formalism works for L_z with canoncial transformations

$$\begin{aligned} x &= X\cos(\varphi) - Y\sin(\varphi) \\ y &= X\sin(\varphi) + Y\cos(\varphi) \\ p_x &= P_x\cos(\varphi) - P_x\sin(\varphi) \\ p_y &= P_x\sin(\varphi) + P_x\cos(\varphi) \end{aligned} \tag{15}$$

Teh Jacobian, nevertheless, is a 4×4 matrix is decomposes into 2 2×2 blocks as

$$M = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 & 0 \\ 0 & 0 & \cos(\varphi) & \sin(\varphi) \\ 0 & 0 & -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \tag{16}$$

Now we need to check that the relation, $J = MJM^T$, holds, with $J = \begin{pmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{pmatrix}$ Well, let us see

$$MJ = \begin{pmatrix} 0 & 0 & -\cos(\varphi) & -\sin(\varphi) \\ 0 & 0 & \sin(\varphi) & -\cos(\varphi) \\ \cos(\varphi) & \sin(\varphi) & 0 & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 & 0 \end{pmatrix} \tag{17}$$

Then this is jsut multiplied by the M^T matrix coresponding just to the same M matrix but with rotations with $-\varphi$. Then the lower block of M^T meets the upper of MJ and gives $\mathbb{I}_2$ and they are 2×2 rotation matrices of angles with opposite signs! Furthermore the upper block of M^T meets the lower one of MJ giving agian the multiplication of rotations with opposite nagles and an additional minus sing as well!
So indeed rotations constitute caonincal tranformations. One could have proved it even with infinitesimal angles, $\delta\varphi$, as well dropping all $\sim o(\delta\varphi^2)$ terms.

Problem 2

A particle of mass m can move in the $x - y$ plane where a conservative, isotropic $V(r)$ potential is also present ($r = \sqrt{x^2 + y^2}$). The goal of this exercise is to show that indeed Hamiltonians in central potentials are indeed rotationally invariant and so a canonical transformation is given by rotatin coordinates and momenta around the z axis.

(a) Write down the Lagrangian of the system and determine the Hamiltonian as a function of p_x, p_y, x and y.

(b) We would like to transform to the rotating frame. The transformation is described by

$$x(t) = X(t)\cos(\omega t) - Y(t)\sin(\omega t)$$
$$y(t) = X(t)\sin(\omega t) + Y(t)\cos(\omega t) \tag{18}$$

Write down the Lagrangian in the X, Y variables.

(c) Determine the "new" Hamiltonian K as a function of X,Y,P_X and P_Y. What is the connection between the "new" and the "old" Hamiltonian?

(d) Show that the Poisson brackets between the variables x, y, p_x, p_y don't change if we calculate them using the new canonical coordinates X, Y, P_X, P_Y.

Solution:

(a) Hamiltonian:

$$H = \frac{p_x^2 + p_y^2}{2m} + V(r) \tag{19}$$

(b) For shorter notaiton let us introduce

$$(x, y) = R(X, Y) \tag{20}$$

and so $(\dot{x}, \dot{y}) = \dot{R}(X, Y) + R(\dot{X}, \dot{Y})$. while $r^2 = x^2 + y^2 = X^2 + Y^2$ remains invariant at all instants. We need to calculate $(\dot{x}, \dot{y})^T(\dot{x}, \dot{y})$, where $\dot{R} = \omega \begin{pmatrix} -\sin(\varphi) & \cos(\varphi) \\ -\cos(\varphi) & -\sin(\varphi) \end{pmatrix}$ and so $\dot{R}^T R = \omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $R^T\dot{R} = -\dot{R}^T R$, while $\dot{R}^T\dot{R} = \omega^2 \mathbb{I}_2$

$$(\dot{X}, \dot{X})^T(\dot{X}, \dot{Y}) = (\dot{x}, \dot{y})^T R^T R(\dot{x}, \dot{y}) + (x, y)^T \dot{R}^T \dot{R}(x, y) = \frac{m}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}m(\omega\dot{Y}X - \omega\dot{X}Y + \omega\omega\dot{Y}X - \omega\dot{X}Y)$$
$$+ \frac{1}{2}m\omega^2(X^2 + Y^2) - V(r) \tag{21}$$

The new canonical momenta then are $P_x = \frac{\partial \mathcal{L}}{\partial \dot{X}} = m\dot{X} - m\omega Y$, $P_y = \frac{\partial \mathcal{L}}{\partial \dot{Y}} = m\dot{Y} + m\omega X$

(c) Fro mhere the new Hamiltonian

$$K = P_x\dot{X} + P_y\dot{Y} - L = \frac{(P_x + m\omega Y)^2}{2m} + \frac{(P_y - m\omega X)^2}{2m} - \frac{1}{2}m\omega^2 r^2 + V(r) \tag{22}$$

(d) Now the Poisson brackets

$$\{X, Y\} = \{X, P_y\} = \{Y, P_x\} = 0 \tag{23}$$

the first is trivial, while the last two is obtained by observing that $P_x(P_y)$ does not contain $P_y(P_x)$. Now

$$\{X, P_x\} = \{x(t)\cos(\omega t) + y(t)\sin(\omega t),\ p_x\cos(\omega t) + p_y\sin(\omega t) - m\omega(-x(t)\sin(\omega t) + y(t)\cos(\omega t))\} = 1 \tag{24}$$

$$\{Y, P_y\} = \{-x(t)\sin(\omega t) + y(t)\cos(\omega t),\ -p_x\sin(\omega t) + p_y\cos(\omega t) + m\omega(x(t)\cos(\omega t) + y(t)\sin(\omega t))\} = 1 \tag{25}$$

indeed.

Problem 3

Let η be a 2 component vector constructed from a pair of canonical coordinates $\{q, p\}$. Consider the following transformation

$$\xi = \begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \log(\sin(p)/q) \\ q\cot(p) \end{pmatrix} \tag{26}$$

(a) Compute the Jacobi matrix $M_{ij} = \frac{\partial \xi_i}{\partial \eta_j}$.

(b) Show that the transformation is canonical, i.e. it keeps the simplectic structure J_{ij} invariant:

$$J_{ij} = M_{ik} J_{kl} M_{jl} \tag{27}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{28}$$

(c) Test the results for a free particle moving gravitational potential, $H = \frac{p^2}{2m} + mgq$

Solution:

(a) Showing that a $\boldsymbol{\eta} = (q, p) \to \boldsymbol{\xi} = (Q, P)$ is canonical is satisfied if and only if the relation

$$J_{ij} = M_{ik} J_{kl} M_{jl} \tag{29}$$

is satisfied! First let us determine the Jacobian

$$M_{ij} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} = \begin{pmatrix} -\frac{1}{q} & \cot(p) \\ \cot(p) & -\frac{q}{\sin^2(p)} \end{pmatrix} \tag{30}$$

(b) Now the straightforward matrix multiplications imply

$$MJ = \begin{pmatrix} -\cot(p) & -\frac{1}{q} \\ \frac{q}{\sin^2(p)} & \cot(p) \end{pmatrix} \tag{31}$$

Now this is multipliead by $M^T = \begin{pmatrix} -\frac{1}{q} & \cot(p) \\ \cot(p) & -\frac{q}{\sin^2(p)} \end{pmatrix}$ giving

$$MJM^T = \begin{pmatrix} -\cot(p) & -\frac{1}{q} \\ \frac{q}{\sin^2(p)} & \cot(p) \end{pmatrix} \begin{pmatrix} -\frac{1}{q} & \cot(p) \\ \cot(p) & -\frac{q}{\sin^2(p)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{32}$$

where we used that $-\frac{1}{\sin^2(q)} = 1 + \cot^2(x)$.

(c) First we need to express teh old coordinates in the new ones:

$$q = \frac{P}{\cot(p)} \Rightarrow \ln(sin(p)) - \ln(P) + \ln(\cot(p)) = Q \Rightarrow \ln(\cos(p)) = Q + \ln(P) \to p = \arccos(e^Q P) \tag{33}$$

$$q = \frac{P}{\cot(p)} = \frac{P\sqrt{1 - e^{2Q}P^2}}{e^Q P} = \sqrt{e^{-2Q} - P^2} \tag{34}$$

Now the new Hamiltonian:

$$K = \frac{\arccos^2(e^Q P)}{2m} + mg\sqrt{e^{-2Q} - P^2} \tag{35}$$

Equations of moiton:

$$\frac{\partial K}{\partial Q} = -\frac{\arccos(e^Q P)}{m} \frac{1}{\sqrt{e^{-2Q} - P^2}} - \frac{mg}{\sqrt{1 - e^{2Q}P^2}} \tag{36}$$

$$\frac{\partial K}{\partial P} = -\frac{\arccos(e^Q P)}{m} \frac{1}{\sqrt{1 - e^{2Q}P^2}} - \frac{mgP}{\sqrt{1 - e^{2Q}P^2}}. \tag{37}$$

Now if now one substituted the inverse transformation the original equations of motion would be recovered.

Problem 4

Consider a linear harmonic oscillator whose Hamiltonian reads

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) \tag{38}$$

Consider the following generator functions and try to derive transformation rules.

(a) $W_1(q, Q) = q + Q$

(b) $W_1(q, P) = (q + Q)^2$

(c) $W_1(q, P) = (qQ$

(d) Which generator function describes indeed a transformation? Perform the transformation and determine the ,,new" Hamiltonian $K(Q, P)$.

(e) Determine the canonical equations using the new form of the Hamiltonian. Solve the equations!

Solution:

(a) Now we use the iff relation for the first generator

$$p = \frac{\partial W_1}{\partial q} = 1,\; P = -\frac{\partial W_1}{\partial Q} = 1 \tag{39}$$

seemingly wrong results is obtained as we lost all dependences

(b)

$$p = \frac{\partial W_1}{\partial q} = 2(q + Q),\; P = -\frac{\partial W_1}{\partial Q} = -2(q + Q) \Rightarrow q = -\frac{P}{2} - Q,\; p = -P \tag{40}$$

From here the Hamiltonian takes the form of

$$K = \frac{P^2}{2m} + \frac{1}{2}m\omega^2(P/2 + Q)^2 \tag{41}$$

(c) Now the equations of motion:

$$\dot{Q} = \frac{\partial K}{\partial P} = \frac{P}{m} + \frac{m}{2}\omega^2(P/2 + Q),\; \dot{P} = -\frac{\partial K}{\partial Q} = -m\omega^2(P/2 + Q) \tag{42}$$

So we immediately have that

$$2\dot{Q} + \dot{P} = -2\dot{q} = \frac{2P}{m} \equiv \frac{-2p}{m} \tag{43}$$

$$\dot{P} = m\omega^2 q \equiv -\dot{p} \tag{44}$$

indeed!

(d) Now for the third generator we have that

$$p = \frac{\partial W_1}{\partial q} = Q,\; P = -\frac{\partial W_1}{\partial Q} = -q \tag{45}$$

From here again the Hamiltonian takes the form of

$$K = \frac{Q^2}{2m} + \frac{1}{2}m\omega^2 P^2 \tag{46}$$

Illustratively it is another harmonic oscillator but with 'new mass' $M = \frac{1}{m\omega^2}$. So the equatios of motion are again trivially

$$\dot{Q} = \frac{\partial K}{\partial P} = Pm\omega^2 = -qm\omega^2 \equiv \dot{p} \tag{47}$$

$$\dot{P} = \frac{\partial K}{\partial Q} = \frac{Q}{m} = \frac{p}{m} \equiv \dot{q} \tag{48}$$