Problem 1

The change of physical quantities in time characterized by their total time derivatives/substantial derivatives describe both their motion in the phase space spanned by the coordiante and momentum and their explicit time-evolutions. When written out the partial derivatives with respect to the coordinate and momentum unavoidably the Hamiltonian gets involved, leading us to the so called Poisson bracket of the physical quantity and the Hamiltonian.

(a) Repeat the steps at class to arrive at the Poissonian brackets expressing the total time derivative of an $F(\mathbf{r}, \mathbf{p})$ function representing a physical quantity.

(b) Derive the Poissonian bracket of the x, y, z coordinates and p_x, p_y, p_z momenta.

(c) Show that rotation around the z axis does not change the result obtained for the Poisson brackets.

Solution:

(a) Note that for sake of simplicity we have chosen F such that it does not have explicit time dependence, $\partial_t F = 0$, so the its total time derivative by definition for the time being in one dimension with x and p:

$$\frac{\mathrm{d}}{\mathrm{d}t}F = \frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial p}\dot{p} + \partial_t F = \frac{\partial F}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial x} \equiv \{F, H\} \tag{1}$$

where in the last step we used the Hamiltonian equations of motion, $\dot{x} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial x}$. Now in the general three dimensional case we have for the total derivative $\frac{\partial F}{\partial \mathbf{r}}\dot{\mathbf{r}} = \frac{\partial F}{\partial \mathbf{r}}\frac{\partial H}{\partial \mathbf{p}}$, $\frac{\partial F}{\partial \mathbf{p}}\dot{\mathbf{p}} = -\frac{\partial F}{\partial \mathbf{p}}\frac{\partial H}{\partial \mathbf{r}}$, understood as the scalar product of two gradient vectors, which gives in total:

$$\frac{\mathrm{d}}{\mathrm{d}t}F = \frac{\partial F}{\partial \mathbf{r}}\dot{\mathbf{r}} + \frac{\partial F}{\partial \mathbf{p}}\dot{\mathbf{p}} = \frac{\partial F}{\partial \mathbf{r}}\frac{\partial H}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}}\frac{\partial H}{\partial \mathbf{r}} \equiv \{F, H\} \tag{2}$$

Poisson brackets are linear in tis arguments, that is $\{F_1 + F_2, G\} = \{F_1, G\} + \{F_2, G\}$ as a consequence of the linearity of differentiation.

(b) Now the Poisson brackets for the coordinates and momenta

$$\{x, p\} = \frac{\partial x}{\partial x}\frac{\partial p}{\partial p} - \frac{\partial x}{\partial p}\frac{\partial p}{\partial x} = 1 \tag{3}$$

as x and p are independent variables and so the partial derivatives, $\frac{\partial p}{\partial x} = \frac{\partial x}{\partial p} = 0$ disappear! More generally for three component coordinates and momenta we have with summation over repeated indices:

$$\{r_i, p_j\} = \frac{\partial r_i}{\partial r_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial r_i}{\partial p_k}\frac{\partial p_j}{\partial r_k} = \delta_{ik}\delta_{jk} = \delta_{ij} \tag{4}$$

as now also different coordinate and momentum variables are independent, $\frac{\partial r_i}{\partial r_k} = \frac{\partial p_i}{\partial p_k} = \delta_{ik}$

(c) Perform a rotation around the z axis with angle φ and arrive at the new coordinates and momenta

$$x' = x\cos\varphi + y\sin\varphi \tag{5}$$
$$y' = -x\sin\varphi + y\cos\varphi \tag{6}$$
$$z' = z \tag{7}$$

and similar relations for p'_x, p'_y, p'_z. Now

$$\{x', p'_x\} = \cos^2\varphi\{x, p_x\} + \sin^2\varphi\{y, p_y\} + 2\sin\varphi\cos\varphi(\{x, p_y\} + \{y, p_x\}) = \cos^2\varphi + \sin^2\varphi = 1 \tag{8}$$
$$\{y', p'_y\} = \cos^2\varphi\{x, p_x\} + \sin^2\varphi\{y, p_y\} - 2\sin\varphi\cos\varphi(\{x, p_y\} + \{y, p_x\}) = \cos^2\varphi + \sin^2\varphi = 1 \tag{9}$$
$$\{z', p'_z\} = \{z, p_z\} = 1 \tag{10}$$
$$\{x', p'_y\} = -\sin\varphi\cos\varphi\{x, p_x\} + \sin\varphi\cos\varphi\{y, p_y\} - \sin^2\varphi\{y, p_x\} + \cos^2\varphi\{x, p_y\} = 0 \tag{11}$$
$$\{x', p'_z\} = \cos\varphi\{x, p_z\} + \sin\varphi\{y, p_z\} = 0 \tag{12}$$
$$\{y', p'_z\} = -\sin\varphi\{x, p_z\} + \cos\varphi\{y, p_z\} = 0 \tag{13}$$
$$\tag{14}$$

where we used the linearity of the Poisson brackets, $\{F_1 + F_2, G\} = \{F_2, G\} + \{F_1, G\}$.

Problem 2

Consider a particle in a central potential. The Hamiltonian of the system is:

$$H = \frac{p^2}{2m} + V(r) \tag{15}$$

(a) Write down the components of the angular momentum (L_x, L_y and L_z) using the canonical momentum $\mathbf{p}$ and the position $\mathbf{r}$.

(b) Show that the Poisson brackets for product of functions work in the same way as differentiation!

(c) Determine the Poisson brackets $\{L_x, x\}$, $\{L_x, y\}$, $\{L_x, p_x\}$ and $\{L_x, p_y\}$.

(d) Generalize the results of b.), so determine the Poisson brackets $\{L_i, r_j\}$ and $\{L_i, p_j\}$ for any i, j indices.

(e) Determine the Poisson bracktes $\{L_i, L_j\}$.

(f) Determine the Poisson brackets $\{L_i, H\}$ for any value of i. What does this tell about the angular momentum?

(g) As an extra exercise give the Poisson bracket $\left\{x^n, p^k\right\}$ with $n, k \in \mathbb{Z}$!

Solution:

(a) The ith component is easily expressed as

$$L_i = \varepsilon_{ijk} x_j p_k \tag{16}$$
$$L_x = y p_z - z p_y \tag{17}$$
$$L_y = z p_x - x p_z \tag{18}$$
$$L_z = x p_y - y p_x \tag{19}$$

(b) Poisson bracket for product of functions:

$$\{F_1 F_2, G\} = \frac{\partial}{\partial \mathbf{r}}(F_1 F_2)\frac{\partial G}{\partial \mathbf{p}} - \frac{\partial}{\partial \mathbf{p}}(F_1 F_2)\frac{\partial G}{\partial \mathbf{r}} = F_1 \frac{\partial F_2}{\partial \mathbf{r}}\frac{\partial G}{\partial \mathbf{p}} - F_1 \frac{\partial F_2}{\partial \mathbf{p}}\frac{\partial G}{\partial \mathbf{r}} + F_2 \frac{\partial F_1}{\partial \mathbf{r}}\frac{\partial G}{\partial \mathbf{p}} - F_2 \frac{\partial F_1}{\partial \mathbf{p}}\frac{\partial G}{\partial \mathbf{r}}$$
$$\equiv F_1\{F_2, G\} + F_2\{F_1, G\}\,, \tag{20}$$

indeed behaving in an analogous way as the derivative of product of functions.

(c) Poisson bracket for two quantities $A(\mathbf{r}, \mathbf{p})$, $B(\mathbf{r}, \mathbf{p})$

$$\{A, B\} = \frac{\partial A}{\partial \mathbf{r}}\frac{\partial B}{\partial \mathbf{p}} - \frac{\partial B}{\partial \mathbf{r}}\frac{\partial A}{\partial \mathbf{p}}. \tag{21}$$

For the angular momentum components we have:

$$\{L_x, x\} = \{y p_z - z p_y, x\} = 0 \tag{22}$$
$$\{L_x, y\} = \{y p_z - z p_y, y\} = \frac{\partial z p_y}{\partial p_y}\frac{\partial y}{\partial y} = z \tag{23}$$
$$\{L_x, p_x\} = \{y p_z - z p_y, p_x\} = 0 \tag{24}$$
$$\{L_x, p_y\} = \{y p_z - z p_y, p_y\} = \frac{\partial y p_z}{\partial y}\frac{\partial p_y}{\partial p_y} = p_z \tag{25}$$
$$\tag{26}$$

(d) Generalizing the above result for arbitrary x_j and L_i exploiting the property that $\{x_j, p_i\} = \delta_{ij}$

$$\{L_i, x_j\} = \{\varepsilon_{ikl} x_k p_l, x_j\} = \varepsilon_{ikl}\left[\frac{\partial(x_k p_l)}{\partial x_a}\frac{\partial x_j}{\partial p_a} - \frac{\partial(x_k p_l)}{\partial p_a}\frac{\partial x_j}{\partial x_a}\right] = -\varepsilon_{ikl}\delta_{ja}\delta_{la}x_k = \varepsilon_{ijk}x_k. \tag{27}$$

Similarly one can handle the Poisson bracket with the angular momenta and momenta:

$$\{L_i, p_j\} = \varepsilon_{ikl}\left[\frac{\partial(x_k p_l)}{\partial x_a}\frac{\partial p_j}{\partial p_a} - \frac{\partial(x_k p_l)}{\partial p_a}\frac{\partial p_j}{\partial x_a}\right] = \varepsilon_{ikl}\delta_{ja}\delta_{ka}p_l = \varepsilon_{ijk}p_k. \tag{28}$$

A much more simple derivation is provided using the product rules:

$$\{L_i, x_j\} = \varepsilon_{ikl}\{x_k p_l, x_j\} = \varepsilon_{ikl}x_k\{p_l, x_j\} + \varepsilon_{ikl}p_l\{x_k, x_j\} = -\varepsilon_{ikl}\delta_{lj}x_k = \varepsilon_{ijk}x_k \tag{29}$$

and also for the other bracket:

$$\{L_i, p_j\} = \varepsilon_{ikl}\{x_k p_l, p_j\} = \varepsilon_{ikl}x_k\{p_l, p_j\} + \varepsilon_{ikl}p_l\{x_k, p_j\} = \varepsilon_{ikl}\delta_{kj}p_l = \varepsilon_{ijl}p_l \tag{30}$$

(e) Now a little bit trickier: $\{L_i, L_j\}$:

$$\begin{aligned}
&\varepsilon_{ikl}\varepsilon_{jmn}\{x_k p_l, x_m p_n\} = \varepsilon_{ikl}\varepsilon_{jmn}\left[\frac{\partial}{\partial x_a}(x_k p_l)\frac{\partial}{\partial p_a}(x_m p_n) - \frac{\partial}{\partial p_a}(x_k p_l)\frac{\partial}{\partial x_a}(x_m p_n)\right]\\
&= \varepsilon_{ikl}\varepsilon_{jmn}[\delta_{ak}p_l\delta_{na}x_m - \delta_{al}x_k\delta_{ma}p_n] = [\varepsilon_{ikl}\varepsilon_{jmk}p_l x_m - \varepsilon_{ikl}\varepsilon_{jln}x_k p_n]\\
&= [(\delta_{lj}\delta_{im} - \delta_{lm}\delta_{ij})p_l x_m - (\delta_{in}\delta_{kj} - \delta_{ij}\delta_{kn})x_k p_n] = p_j x_i - x_m p_m\delta_{ij} - x_j p_i + \delta_{ij}x_n p_n\\
&= p_j x_i - x_j p_i \equiv \varepsilon_{ijk}\varepsilon_{klm}x_l p_m \equiv \varepsilon_{ijk}L_k
\end{aligned} \tag{31}$$

(f) Now we need to calculate two Poisson brackets, namely

$$\{L_i, p_j p_j\} = 2p_j\{L_i, p_j\}\, 2p_j\varepsilon_{ijk}p_k = 2(\mathbf{p}\times\mathbf{p})_i = 0 \tag{32}$$

and the second one, where we use that $\frac{\partial V(r)}{\partial x_a} = \frac{\partial V}{\partial r}\frac{x_a}{r}$ and $\frac{\partial V}{\partial p_a} = 0$, as the potential does not depend on p_a:

$$\{L_i, V(r)\} = -\varepsilon_{ijk}\frac{\partial}{\partial p_a}(x_j p_k)\frac{\partial V}{\partial x_a} = -\varepsilon_{ijk}\delta_{ka}x_j\frac{x_a}{r}\frac{\partial V}{\partial r} = \frac{1}{r}\frac{\partial V}{\partial r}(\mathbf{r}\times\mathbf{r})_k = 0. \tag{33}$$

So in total we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}L_i \equiv \{L_i, H\} = 0 \tag{34}$$

as we expected. Indeed for central potentials angular momentum is conserved!

(g) First let us calcualte $\{x^n, p\}$:

$$\begin{aligned}
&\{x^n, p\} = \{x x^{n-1}, p^k\} = x\{x^{n-1}, p\} + x^{n-1} = x^{n-1} + x^2\{x^{n-2}, p\} + x^{n-1} = \dots\\
&\dots = (n-1)x^{n-1} + x^{n-1}\{x, p\} = n x^{n-1}.
\end{aligned} \tag{35}$$

Now knowing this we can express the general expression as

$$\begin{aligned}
&\{x^n, p^k\} = p\{x^n, p^{k-1}\} + p^{k-1}\{x^n, p\} = p\{x^n, p^{k-1}\} + n x^{n-1}\\
&= (p+1)n x^{n-1} + p^2\{x^n, p^{k-2}\} = \dots = (1 + p + \dots + p^{k-1})n x^{n-1} = \frac{1-p^k}{1-p}n x^{n-1}.
\end{aligned} \tag{36}$$

Problem 3

Introduce in two dimensions polar coordinates according to the usual relations:

$$x = r\cos\varphi, \tag{37}$$

$$y = r\sin\varphi \tag{38}$$

and the associated momentum in case of the Lagrangian $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$ with central potential $U(r)$ depending only on the radial variable.

(a) Find the associated radial and tangent momenta!

(b) Derive the results for the Poisson brackets for the canonical coordiantes and momenta!

(c) Consider a pendulum and derive its equation of motion using Poisson brackets!

Solution:

(a) The associated momenta are $p_\varphi = \frac{\partial L}{\partial \dot\varphi} = mr^2\dot\varphi$, $p_r = \frac{\partial L}{\partial \dot r} = m\dot r$.

(b) Now as p_φ, p_r, r, φ are the canonical coordinates and momenta Poisson brackets are defiend with the help of them as well.

$$\{r, p_r\} = \frac{\partial r}{\partial r}\frac{\partial p_r}{\partial p_r} + \frac{\partial r}{\partial \varphi}\frac{\partial p_r}{\partial p_\varphi} - \frac{\partial r}{\partial p_r}\frac{\partial p_r}{\partial r} - \frac{\partial r}{\partial p_\varphi}\frac{\partial p_r}{\partial \varphi} = 1 \tag{39}$$

$$\{r, p_\varphi\} = \frac{\partial r}{\partial r}\frac{\partial p_\varphi}{\partial p_r} + \frac{\partial r}{\partial \varphi}\frac{\partial p_\varphi}{\partial p_\varphi} - \frac{\partial r}{\partial p_r}\frac{\partial p_\varphi}{\partial r} - \frac{\partial r}{\partial p_\varphi}\frac{\partial p_\varphi}{\partial \varphi} = 0 \tag{40}$$

$$\{\varphi, p_r\} = \frac{\partial \varphi}{\partial r}\frac{\partial p_r}{\partial p_r} + \frac{\partial \varphi}{\partial \varphi}\frac{\partial p_r}{\partial p_\varphi} - \frac{\partial \varphi}{\partial p_r}\frac{\partial p_r}{\partial r} - \frac{\partial \varphi}{\partial p_\varphi}\frac{\partial p_r}{\partial \varphi} = 0 \tag{41}$$

as all derivatives with different variables give zero because they are considered now independent ones $\frac{\partial r}{\partial p_r} = \frac{\partial r}{\partial p_\varphi} = \frac{\partial r}{\partial \varphi} = 0$ and similar relations hold for $\{p_\varphi, p_r\} = \{\varphi, r\} = 0$.

(c) Consider an ordinary penulum with length l and derive the equations of motion for it. The corresponding Lagrangian $L = \frac{1}{2}ml^2\dot\varphi^2 - mgl\cos\varphi$ and this system has only one degree of freedom, φ and the associated momentum is $p_\varphi = \frac{\partial L}{\partial \dot\varphi} = ml^2\dot\varphi$ and Hamiltonian $H = \dot\varphi p_\varphi - L = \frac{p_\varphi^2}{2ml^2} + mgl\cos\varphi$. Now the Poisson brackets for φ and p_φ give

$$\dot\varphi = \{\varphi, H\} = \frac{\partial \varphi}{\partial \varphi}\frac{\partial H}{\partial p_\varphi} - \frac{\partial \varphi}{\partial p_\varphi}\frac{\partial H}{\partial \varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{ml^2}, \tag{42}$$

$$\dot p_\varphi = \{p_\varphi, H\} = \frac{\partial p_\varphi}{\partial \varphi}\frac{\partial H}{\partial p_\varphi} - \frac{\partial p_\varphi}{\partial p_\varphi}\frac{\partial H}{\partial \varphi} = -\frac{\partial H}{\partial \varphi} = mgl\sin\varphi \Rightarrow \ddot\varphi = \frac{mg}{l}\sin\varphi. \tag{43}$$

Problem 4

Consider a harmonic oscillator with the Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$ in one dimension and a solution of $x(t) = \cos(\omega t + \varphi)$. The goal of this exercise is to check explicitly the kinetic energy of the system.

(a) Write out the kinetic energy's time derivative using Poisson brackets!

(b) Express it via the known form of $x(t)$.

(c) Express the total energy of the system as a function of time!

(d) Repeat the exercise for an anharmonic oscillator with $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \alpha x^4$

Soltuion:

(a) Using Poisson brackets we have for the $K = \frac{p^2}{2m}$ kinetic energy, exploiting that $\frac{\partial K}{\partial p} = \frac{p}{m}$, $\frac{\partial K}{\partial x} = 0$ and $\frac{\partial H}{\partial x} = -\dot p$, $\frac{\partial H}{\partial p} = \dot x$

$$\dot K = \{K, H\} = \frac{\partial K}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial K}{\partial p}\frac{\partial H}{\partial x} = -\frac{\partial K}{\partial p}\frac{\partial H}{\partial x} = \frac{p}{m}\dot p. \tag{44}$$

Now we still need the expression for p

$$\dot p = \{p, H\} = -\frac{\partial p}{\partial p}\frac{\partial H}{\partial x} = -m\omega^2 x = -m\omega^2\cos(\omega t + \varphi) \Rightarrow p = -m\omega\sin(\omega t + \varphi) \tag{45}$$

(b) Now substituing that $p = m\dot x = -m\omega\sin(\omega t + \varphi)$ we get

$$\dot K = m\omega^3\sin(2\omega t + 2\varphi) \Rightarrow K = \frac{1}{2}m\omega^2\sin^2(\omega t + \varphi) \tag{46}$$

(c) The total energy is then simply

$$E = \frac{1}{2}m\omega^2 \sin^2(\omega t + \varphi) + \frac{1}{2}m\omega^2 \cos^2(\omega t + \varphi) = \frac{1}{2}m\omega^2 \tag{47}$$

(d) Now for the anharmonic oscillator the only modifications are that $\frac{\partial H}{\partial x} = m\omega x + 4\alpha x^3$ and so the time derivative of K becomes

$$\dot{K} = -\frac{p}{m}\frac{\partial H}{\partial x} = -\omega^2 px - 4\alpha p x^3/m \tag{48}$$

Again calculating the derivative of the momentum:

$$\begin{aligned} \dot{p} =& \{p, H\} = -\frac{\partial p}{\partial p}\frac{\partial H}{\partial x} = -m\omega^2 x - 4\alpha x^3 = -m\omega^2 \cos(\omega t + \varphi) - 4\alpha \cos^3(\omega t + \varphi) \\ \Rightarrow& \; p = -m\omega \sin(\omega t + \varphi) - \frac{4\alpha}{\omega}\sin(\omega t + \varphi) + \frac{4\alpha}{3\omega}\sin^3(\omega t + \varphi) \end{aligned} \tag{49}$$

Problem 5

Consider two particles connected by a spring moving in one dimension, whose Lagrangian is given by $L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - D(x_1 - x_2)^2$. Prove that the total momentum is conserved. **Solution:** First we give the Hamiltonian by $p_1 = \frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1$, $p_2 = \frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2$. From here the Hamiltonian $H = \dot{x}_1 p_1 + \dot{x}_1 p_1 - L = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + D(x_1 - x_2)^2$

We show that the Poissonian bracket of $p_1 + p_2$ is zero:

$$\begin{aligned} &\dot{P} \equiv \dot{p}_1 + \dot{p}_2 = \{p_1 + p_2, H\} = \frac{\partial(p_1 + p_2)}{\partial x_1}\frac{\partial H}{\partial p_1} + \frac{\partial(p_1 + p_2)}{\partial x_2}\frac{\partial H}{\partial p_2} - \frac{\partial(p_1 + p_2)}{\partial p_1}\frac{\partial H}{\partial x_1} - \frac{\partial(p_1 + p_2)}{\partial p_2}\frac{\partial H}{\partial x_2} \\ &= -\frac{\partial H}{\partial x_1} - \frac{\partial H}{\partial x_2} = D(x_1 - x_2) - D(x_1 - x_2) = 0, \end{aligned} \tag{50}$$

indeed!