Problem 1

Consider the following Lagrangian:

$$\mathcal{L} = -\frac{1}{2m}\partial_x\Psi^*\partial_x\Psi - V\Psi^*\Psi + \frac{1}{2}i(\Psi^*\partial_t\Psi - \Psi\partial_t\Psi^*), \tag{1}$$

where $\Psi(x,t)$ is a complex valued field, and $\Psi^*(x,t)$ denotes its complex conjugate. There are many ways to handle complex fields. Now we follow the most pedestrian way: we describe the field as a combination of two independent real fields.

(a) Consider the complex field as a real field with two-components (the real and the imaginary part.) Here $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are standard real fields. Rewrite the Lagrangian in the terms of these two real fields.

(b) Show that the Lagrangian is real (no complex factors are present).

(c) Write down the action using the real form of the Lagrangian.

(d) Derive the equations of motion for the two fields $\Psi_{1,2}$.

(e) Show that the two equations are the real and imaginary parts of the usual Schrödinger equation, (we use $\hbar = 1$ units.).

(f) Compute the energy density of the system!

(g) Express the energy density current and write down the continuity equation!

Solution:

(a) Easily let us separate the field and its complex conjugate as $\Psi = \Psi_1 + i\Psi_2,\ \Psi^* = \Psi_1 - i\Psi_2$ and write teh Lagrangian density:

$$\mathcal{L} = -\frac{1}{2m}\left[(\partial_x\Psi_1)^2 + (\partial_x\Psi_2)^2\right] - V\left[\Psi_1^2 + \Psi_2^2\right] + \left[\Psi_2\partial_t\Psi_1 - \Psi_1\partial_t\Psi_2\right] \tag{2}$$

(b) We can see, that indeed this Lagrangian is real.

(c) Equation of motion with a field having the two components of $\Psi_{1,2}$:

$$\frac{\partial\mathcal{L}}{\partial\Psi_{1,2}} = \partial_t\frac{\partial\mathcal{L}}{\partial\partial_t\Psi_{1,2}} + \partial_x\frac{\partial\mathcal{L}}{\partial\partial_x\Psi_{1,2}} \tag{3}$$

valid for both components as we consider them separately as independent componetns of a field! Now the relevant derivatives $\frac{\partial\mathcal{L}}{\partial\Psi_{1,2}} = -2V\Psi_{1,2} \mp \partial_t\Psi_{2,1}$, $\frac{\partial\mathcal{L}}{\partial\partial_t\Psi_{1,2}} = \pm\Psi_{2,1}$ and $\frac{\partial\mathcal{L}}{\partial\partial_x\Psi_{1,2}} = -\frac{1}{m}\partial_x\Psi_{1,2}$ from where the equation of motion for the two components:

$$-2V\Psi_{1,2} \mp \partial_t\Psi_{2,1} = \pm\partial_t\Psi_{2,1} - \frac{1}{m}\partial_x^2\Psi_{1,2} \Rightarrow -\frac{1}{2m}\partial_x^2\Psi_{1,2} + V\Psi_{1,2} = \pm\partial_t\Psi_{2,1} \tag{4}$$

Now writing $\Psi = \Psi_1 + i\Psi_2$ we obtain

$$-\frac{1}{2m}\partial_x^2(\Psi_1 + i\Psi_2) + V(\Psi_1 + i\Psi_2) = i\partial_t(\Psi_1 + i\Psi_2) = i\partial_t\Psi_1 - \partial_t\Psi_2 \tag{5}$$

(d) Eenrgy density by definition $\mathcal{H} = \partial_t\Psi_1\frac{\partial\mathcal{L}}{\partial\partial_t\Psi_1} + \partial_t\Psi_2\frac{\partial\mathcal{L}}{\partial\partial_t\Psi_2} - \mathcal{L} = \frac{1}{2m}\left[(\partial_x\Psi_1)^2 + (\partial_x\Psi_2)^2\right] + V\left[\Psi_1^2 + \Psi_2^2\right]$

(e) Current density $J_E(x,t) = \partial_t\Psi_1\frac{\partial\mathcal{L}}{\partial\partial_x\Psi_1} + \partial_t\Psi_2\frac{\partial\mathcal{L}}{\partial\partial_x\Psi_2} = -\frac{1}{m}\left[\partial_t\Psi_1\partial_x\Psi_1 + \partial_t\Psi_2\partial_x\Psi_2\right]$.
One can check that indeed continuity rquationis satisfied:

$$\partial_t\mathcal{H} = -\partial_x J_E(x,t) \tag{6}$$

The right hand side gives simply $-\partial_x J_E(x,t) = \frac{1}{m}\left[\partial_t\Psi_1\partial_x^2\Psi_1 + \partial_x\partial_t\Psi_1\partial_x\Psi_1 + \partial_t\Psi_2\partial_x^2\Psi_2 + \partial_x\partial_t\Psi_2\partial_x\Psi_2\right]$.
Now rewrite according to the Schrödinger equation the $\frac{1}{m}\partial_x^2\Psi_1 = 2V\Psi_1 - 2\partial_t\Psi_2$, while for Ψ_2 they

give $\frac{1}{m}\partial_x^2\Psi_2 = 2V\Psi_2 + 2\partial_t\Psi_1$ which cancel out in the term as $\frac{1}{m}(\partial_t\Psi_1\partial_x^2\Psi_1 + \partial_t\Psi_1\partial_x^2\Psi_1) = 2V(\Psi_1\partial_t\Psi_1 + \Psi_2\partial_t\Psi_2)$. So the spatial derivative ofteh curent gives

$$\partial_x J_E(x,t) = 2V\left[\Psi_1\partial_t\Psi_1 + \Psi_2\partial_t\Psi_2\right] + \frac{1}{m}\left[\partial_x\partial_t\Psi_1\partial_x\Psi_1 + \partial_t\Psi_2\partial_x^2\Psi_2\right] \tag{7}$$

Now this is exactly the tiem derivative of the energy density as the second term is just the time derivative of $\partial_t\frac{1}{2m}\left[(\partial_x\Psi_1)^2 + (\partial_x\Psi_2)^2\right] = \frac{1}{m}\left[\partial_x\partial_t\Psi_1\partial_x\Psi_1 + \partial_t\Psi_2\partial_x^2\Psi_2\right]$ while the other tiem derivative just matches the first term in $\partial_x J_E(x,t)$ as $\partial_t V\left[\Psi_1^2 + \Psi_2^2\right] = 2V\left[\Psi_1\partial_t\Psi_1 + \Psi_2\partial_t\Psi_2\right]$.

Problem 2

One of the simplest non-quadratic field theories is the so called φ^4 theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_t\varphi)^2 - \frac{1}{2}(\partial_x\varphi)^2 + \frac{1}{2}\varphi^2 - \frac{1}{4}\varphi^4 \tag{8}$$

(a) Write down the Euler-Lagrange equations of motion!

(b) Express the energy density in the system!

(c) Give the expression for the energy density current!

(d) First seek for the constant solutions of the system, φ_0!

(e) Now look for $\varphi(x)$ stationary solutions! What equations do they satisfy?

(f) We would like to get a result that brings us from one constant solution to the otheer as $\varphi(x \to \infty) = \varphi_1$ and $\varphi(x \to -\infty) = \varphi_2$ (This is the so called domain wall solution).
Show that the function $\varphi(x) = \tanh(x/\sqrt{2})$ is such a solution to the problem.

(g) Now look for the time-dependnet solution in form of $\varphi(x,t) = \tanh\left(\frac{x-vt}{\sqrt{2}\sqrt{1-v^2}}\right)$

(h) Give the expression for the energy density!

(i) Express the energy current density for the above domain wall solution!

Solution:

(a) The relevant derivatives for the problem: $\frac{\partial\mathcal{L}}{\partial\varphi} = \varphi - \varphi^3$, $\partial_t\frac{\partial\mathcal{L}}{\partial\partial_t\varphi} = \partial_t^2\varphi$ and $\partial_x\frac{\partial\mathcal{L}}{\partial\partial_x\varphi} = -\partial_x^2\varphi$ by which the equation of motion reads

$$\varphi - \varphi^3 - \partial_t^2\varphi + \partial_x^2\varphi = 0 \tag{9}$$

Now as an extra exercise let us calculate it using the variational principle:
Action:

$$S = \int \mathrm{d}t\,\mathrm{d}x\,\frac{1}{2}(\partial_t\varphi)^2 - \frac{1}{2}(\partial_x\varphi)^2 + \frac{1}{2}\varphi^2 - \frac{1}{4}\varphi^4 \tag{10}$$

Taking its variation with some dispalcement field $\delta\varphi$ disappearing at the boundaries we get:

$$\delta S = \int \mathrm{d}t\,\mathrm{d}x\,\partial_t\varphi\partial_t\delta\varphi - \partial_x\varphi\partial_x\delta\varphi + \varphi\delta\varphi - \varphi^3\delta\varphi = \int \mathrm{d}t\,\mathrm{d}x\, -\partial_t^2\varphi\delta\varphi - \partial_x^2\varphi\delta\varphi + \varphi\delta\varphi - \varphi^3\delta\varphi = 0 \tag{11}$$

where in the last step we just transferred the ∂_t, ∂_t differentiations from the displacements to the fields and which equality should hold for all $\delta\varphi$ displacements giving the equation:

$$\varphi - \varphi^3 - \partial_t^2\varphi + \partial_x^2\varphi = 0 \tag{12}$$

(b) The energy density by definition

$$\mathcal{H} = \partial_t\varphi\frac{\partial\mathcal{L}}{\partial\partial_t\varphi} - \mathcal{L} = \frac{1}{2}(\partial_t\varphi)^2 + \frac{1}{2}(\partial_x\varphi)^2 - \frac{1}{2}\varphi^2 + \frac{1}{4}\varphi^4 \tag{13}$$

(c) By definition

$$J_E(x,t) = \partial_t\varphi \frac{\partial \mathcal{L}}{\partial \partial_x \varphi} = -\partial_x\varphi\partial_t\varphi \tag{14}$$

(d) Constant solutions are given by the equation:

$$\varphi = \varphi^3 \Rightarrow \varphi_0 = \pm 1 \tag{15}$$

(e) For the stationary solutions we have the equation:

$$\varphi - \varphi^3 + \partial_x^2\varphi = 0 \tag{16}$$

Our guess is $\varphi(x) = \tanh(x/\sqrt{2})$, let us check it, by knowing that $\partial_x^2 \tanh(x/\sqrt{2}) = \frac{1}{\sqrt{2}}\partial_x(1 - \tanh^2(x/\sqrt{2})) = \tanh(x/\sqrt{2})\left[\tanh^2(x/\sqrt{2}) - 1\right]$

$$\tanh(x/\sqrt{2}) - \tanh^3(x/\sqrt{2}) + \tanh(x/\sqrt{2})\left[\tanh^2(x/\sqrt{2}) - 1\right] = 0 \tag{17}$$

which indeed satureates as $\lim_{x\to\pm\infty}\tanh(x) = \pm 1$ to the constant solutions, thus behaves as a domain wall.

(f) Time dependent solution in form of $\varphi(x,t) = \tanh\left(\frac{1}{\sqrt{2}}\frac{x-vt}{\sqrt{1-v^2}}\right)$. Now the space differentiation gives an additional factor of $\frac{1}{\sqrt{1-v^2}}$, while the time differentiation brings in a $\frac{-v}{\sqrt{1-v^2}}$ so the term $(\partial_x^2 - \partial_t^2)\tanh\left(\frac{1}{\sqrt{2}}\frac{x-vt}{\sqrt{1-v^2}}\right)\frac{1}{1-v^2} - \frac{v^2}{1-v^2}\partial_y^2\tanh(y)\Big|_{y=\frac{1}{\sqrt{2}}\frac{x-vt}{\sqrt{1-v^2}}} = \partial_y^2\tanh(y)\Big|_{y=\frac{1}{\sqrt{2}}\frac{x-vt}{\sqrt{1-v^2}}}$ by which the whole derivation above applies! We can see that time evolution describes the propagation of the domain wall!

(g) Energy density

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\left[\partial_t \tanh\left(\frac{1}{\sqrt{2}}\frac{x-vt}{\sqrt{1-v^2}}\right)\right]^2 + \frac{1}{2}\left[\partial_x \tanh\left(\frac{1}{\sqrt{2}}\frac{x-vt}{\sqrt{1-v^2}}\right)\right]^2 - \frac{1}{2}\tanh^2\left(\frac{1}{\sqrt{2}}\frac{x-vt}{\sqrt{1-v^2}}\right) \\ &+ \frac{1}{4}\tanh^4\left(\frac{1}{\sqrt{2}}\frac{x-vt}{\sqrt{1-v^2}}\right) = \frac{1}{4}\frac{1+v^2}{1-v^2}\left[1 - \tanh^2\left(\frac{1}{\sqrt{2}}\frac{x-vt}{\sqrt{1-v^2}}\right)\right]^2 - \frac{1}{2}\tanh^2\left(\frac{1}{\sqrt{2}}\frac{x-vt}{\sqrt{1-v^2}}\right) \\ &+ \frac{1}{4}\tanh^4\left(\frac{1}{\sqrt{2}}\frac{x-vt}{\sqrt{1-v^2}}\right)\end{aligned} \tag{18}$$

(h) Energy current density: By our expression from above

$$J_E(x,t) = \partial_t\varphi\partial_x\varphi = \frac{v}{1-v^2}\left[1 - \tanh\left(\frac{x-vt}{\sqrt{1-v^2}}\right)\right]^2 \tag{19}$$

Problem 3

The energy density of ferrmagnetic spin chain with one axes is approximated by

$$\varepsilon = \frac{1}{2}(\partial_x \mathbf{M})^2 + \frac{\lambda}{4}M_z^4 \tag{20}$$

where the first term lowers the energy for spins aligned parallel to each ther while the second when align along the z axis. We can always take $\mathbf{M}^2 = 1$.

(a) We take into account this cosntraint by the parametrization

$$\mathbf{M} = \begin{bmatrix}\sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta\end{bmatrix} \tag{21}$$

Rewrite down the energy with θ and φ!

(b) Give the equation for the stationary configurations by minimizing the energy using the variational principle!

(c) Look for constant solutions!

(d) Look for stationary solutions which saturate from one constant to solution to another, these are called domain wall solutions.

Solution:

(a) Spatial derivative $\partial_x \mathbf{M} = \partial_x \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix} = \partial_x\theta\,\mathbf{e}_\theta + \partial_x\varphi\,\mathbf{e}_\varphi$ and $(\partial_x \mathbf{M})^2 = (\partial_x\theta)^2 + (\partial_x\varphi)^2$. While for the z dependent part we simply have $-\frac{\lambda}{2}\cos^2\theta$ by whic the total energy density is given by

$$\varepsilon = \frac{1}{2}\left[(\partial_x\theta)^2 + (\partial_x\varphi)^2\right] + \frac{\lambda}{4}\cos^4\theta \tag{22}$$

(b) Energy integral

$$\begin{aligned} E &= \int_0^\infty \mathrm{d}x\,\varepsilon \Rightarrow \delta E = \int_0^\infty \mathrm{d}x\,\partial_x\theta\partial_x\delta\theta + \partial_x\varphi\partial_x\delta\varphi + 2\lambda\cos^3(\theta)\sin(\theta)\delta\theta \\ &\Rightarrow \int_0^\infty \mathrm{d}x\, -\partial_x^2\theta\delta\theta - \partial_x^2\varphi\delta\varphi + \frac{\lambda}{2}\sin(2\theta)\delta\theta = 0 \Rightarrow -\partial_x^2\theta + 2\lambda\cos^3(\theta)\sin(\theta) = 0,\ \partial_x^2\varphi = 0 \end{aligned} \tag{23}$$

(c) Now constant solutions are given by $\theta = -\pi,\ -\pi/2,\ 0,\ \pi/2$, where either $\sin(\theta) = 0$ or $\cos(\theta) = 0$ and $\varphi = \varphi_0$, corresponding to spins aligned either perpendicular or parallel to the z axis and they all point in the same direction in the $x - y$ plane parametrized by angle φ_0.
Stability conditions: derivative of the sine at the θ values is only positive for $\theta = 0 \Rightarrow$stable solution.

(d) Now for the spatial solution we need to consider the equation:

$$\partial_x^2\theta = -\lambda\sin\theta\cos^3\theta \tag{24}$$

which is satisfied by $\theta(x) = \mathrm{arctg}\left(\sqrt{\lambda/2}x\right)$, as $\sin^3(\mathrm{arctg}(x)) = \frac{x^3}{(1+x^2)^{3/2}}$, $\cos(\mathrm{arctg}(x)) = \frac{1}{\sqrt{1+x^2}}$. The real solution which connects 2 constant solutions is given by either $\theta = \pi + \mathrm{arctg}\left(\sqrt{\lambda/2}x\right)$ or $\theta = \mathrm{arctg}\left(\sqrt{2\lambda}x\right) + \pi/2$. That is the first connects the domain walls of all spins pointing upwards or downwards while for the second domains of spins align along the $\pm x$ direction.