Problem 1

Consider the following Lagrangian:

$$\mathcal{L} = -\frac{1}{2m}\partial_x\Psi^*\partial_x\Psi - V\Psi^*\Psi + \frac{1}{2}i(\Psi^*\partial_t\Psi - \Psi\partial_t\Psi^*), \tag{1}$$

where $\Psi(x,t)$ is a complex valued field, and $\Psi^*(x,t)$ denotes its complex conjugate. There are many ways to handle complex fields. Now we follow the most pedestrian way: we describe the field as a combination of two independent real fields.

(a) Consider the complex field as a real field with two-components (the real and the imaginary part.) Here $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are standard real fields. Rewrite the Lagrangian in the terms of these two real fields.

(b) Show that the Lagrangian is real (no complex factors are present).

(c) Write down the action using the real form of the Lagrangian.

(d) Derive the equations of motion for the two fields $\Psi_{1,2}$.

(e) Show that the two equations are the real and imaginary parts of the usual Schrödinger equation, (we use $\hbar = 1$ units.).

(f) Compute the energy density of the system!

(g) Express the energy density current and write down the continuity equation!

Problem 2

One of the simplest non-quadratic field theories is the so called φ^4 theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_t\varphi)^2 - \frac{1}{2}(\partial_x\varphi)^2 + \frac{1}{2}\varphi^2 - \frac{1}{4}\varphi^4 \tag{2}$$

(a) Write down the Euler-Lagrange equations of motion!

(b) Express the energy density in the system!

(c) Give the expression for the energy density current!

(d) First seek for the constant solutions of the system, φ_0!

(e) Now look for $\varphi(x)$ stationary solutions! What equations do they satisfy?

(f) We would like to get a result that brings us from one constant solution to the otheer as $\varphi(x \to \infty) = \varphi_1$ and $\varphi(x \to -\infty) = \varphi_2$ (This is the so called domain wall solution).
Show that the function $\varphi(x) = \tanh(x/\sqrt{2})$ is such a solution to the problem.

(g) Now look for the time-dependnet solution in form of $\varphi(x,t) = \tanh\left(\frac{x-vt}{\sqrt{2}\sqrt{1-v^2}}\right)$

(h) Give the expression for the energy density!

(i) Express the energy current density for the above domain wall solution!

Problem 3

The energy density of ferrmagnetic spin chain with one axes is approximated by

$$\varepsilon = \frac{1}{2}(\partial_x\mathbf{M})^2 - \frac{\lambda}{2}M_z^4 \tag{3}$$

where the first term lowers the energy for spins aligned parallel to each ther while the second when align along the z axis. We can always take $\mathbf{M}^2 = 1$.

(a) We take into account this cosntraint by the parametrization

$$\mathbf{M} = \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix} \tag{4}$$

Rewrite down the energy with θ and φ!

(b) Give the equation for the stationary configurations by minimizing the energy using the variational principle!

(c) Look for constant solutions!

(d) Look for stationary solutions which saturate from one constant to solution to another, these are called domain wall solutions.