Problem 1

Consider the one-parameter subgroup of Lorentz transformations that containes the boosts in the x direction. In that case one can simply forget the y and z coodinates because these are not transformed. Consequently it is sufficient to consider only the top left 2×2 block of the Lorentz matrix. In the lecture it was shown that in this special case, the Lorentz matrix can be parametrized as

$$\Lambda(\theta) = \begin{pmatrix} \cosh(\theta) & -\sinh(\theta) \\ -\sinh(\theta) & \cosh(\theta) \end{pmatrix} \tag{1}$$

(a) What is the connection between the parameter θ (the rapidity) and the velocity v of the boost?

(b) Show that the above transformation has the following property

$$\Lambda(\theta_1)\Lambda(\theta_2) = \Lambda(\theta_1 + \theta_2) \tag{2}$$

(c) By the use of this property, derive the "rule of addition" for relativistic velocities. What is the meaning of this formula?

(d) Two relativisticly fast cars are traveling by $0.8c$ towards each other. According to one of the drivers, what is the velocity of the other car?

Solution:

(a) By construction

$$\Lambda^{\mu}_{\cdot\nu} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & \frac{-v/c}{\sqrt{1-v^2/c^2}} \\ \frac{-v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} \end{pmatrix} \tag{3}$$

from where $\cosh\theta = \frac{1}{\sqrt{1-v^2/c^2}}$, $\sinh\theta = \frac{v/c}{\sqrt{1-v^2/c^2}} \Rightarrow \tanh\theta = v/c \Rightarrow \theta = \text{atanh}\,(v/c)$

(b) Multiplication of two matrices:

$$\begin{aligned} \Lambda(\theta_1)\Lambda(\theta_2) &= \begin{pmatrix} \cosh\theta_1 & -\sinh\theta_1 \\ -\sinh\theta_1 & \cosh\theta_1 \end{pmatrix} \begin{pmatrix} \cosh\theta_2 & -\sinh\theta_2 \\ -\sinh\theta_2 & \cosh\theta_2 \end{pmatrix} \\ &= \begin{pmatrix} \cosh\theta_1\cosh\theta_2 + \sinh\theta_1\sinh\theta_2 & -\cosh\theta_1\sinh\theta_2 - \sinh\theta_1\cosh\theta_2 \\ -\cosh\theta_1\sinh\theta_2 - \sinh\theta_1\cosh\theta_2 & \cosh\theta_1\cosh\theta_2 + \sinh\theta_1\sinh\theta_2 \end{pmatrix} = \\ &\begin{pmatrix} \cosh(\theta_1+\theta_2) & -\sinh(\theta_1+\theta_2) \\ -\sinh(\theta_1+\theta_2) & \cosh(\theta_1+\theta_2) \end{pmatrix} = \Lambda(\theta_1+\theta_2). \end{aligned} \tag{4}$$

(c) Addition of velocities by Galilei transformation is simply $v_{\text{tot}} = v_1 + v_2$. But now consider a moving frame with velocity V and in the moving frame an object with velocity v (so the object's velocity is measured to be v from the frame moving with V), again in Galilei's picture the total velocity of the object in the rest frame would again yield $v_{\text{tot}} = v + V$, but according to the above discussion we need to transform into first the moving frame and then to the frame of the object with $\tanh\theta_1 = v/c$ and $\tanh\theta_2 = V/c$ giving a total Lorentz matrix with $\theta_1 + \theta_2 \Rightarrow v_{\text{tot}} = c\tanh(\theta_1+\theta_2) = c\frac{\tanh\theta_1 + \tanh\theta_2}{1+\tanh\theta_1\tanh\theta_2} \equiv \frac{v+V}{1+\frac{vV}{c^2}} < v + V$ and this always equals c if one of the velocities are the speed of light!

(d) In the driver's frame the rest/road's frame moves with velocity $v_2 = 0.8c$ and above that in the rest/road's frame the other car moves with velocity $v_2 = 0.8c$. So again we first need to transform into the frame of the road (originally the rest frame) and then inside the moving frame of the road into the moving frame of the other car (which is moving with velocity v in the road's/ moving frame). So we have rapidities $\theta_1 = \theta_2 = \text{atanh}(0.8) \Rightarrow \tilde{\theta} = 2\text{atanh}(0.8) \Rightarrow \tilde{v} \approx 0.975c$.

Problem 2

In the lecture the 4-velocity vector $u^\mu = \frac{dx^\mu}{d\tau}$ has been introduced, and it has been shown that this is a proper 4-vector.

(a) Write down the connection between the 4-velocity and the usual (3-)velocity vector.

(b) Let's suppose, that watching the sky, we see two spacecrafts that are flying towards each other, and both have velocity 0.6c. We use a coordinate system, where the trajectories of the spacecrafts lie on the x-axis.

Determine the 4-velocities of the spacecrafts.

(c) Write down a Lorentz-transformation that transforms into the frame of one of the spacecrafts.

(d) Express the 4-velocities of the spacecrafts in that frame of reference.

(e) What is the usual 3-velocity of the spacecrafts in that frame?

Let suppose now, that – as we see from the Earth – the two spacecrafts travel in perpendicular directions, x and y.

(f) Determine the modified 4-velocities of the spacecrafts

(g) Transform to the frame of the spacecraft travelling in the x direction. What is the 4-velocity of the other spacecraft in this frame?

(h) What is the usual 3-velocity of the other spacecraft in this frame?

Solution:

(a) Four velcoity is defined as the change in position, but with respect to the proper time, implying that usual time derivative is multiplied by the familiar contraction factor $\mathrm{d}\tau = \mathrm{d}t\sqrt{1 - v^2/c^2}$.

$$u^\mu = \frac{\mathrm{x}^\mu}{\mathrm{d}\tau} = \begin{pmatrix} \frac{c}{\sqrt{1-v^2/c^2}} \\ \frac{\mathrm{d}x^1/\mathrm{d}t}{\sqrt{1-v^2/c^2}} \\ \frac{\mathrm{d}x^2/\mathrm{d}t}{\sqrt{1-v^2/c^2}} \\ \frac{\mathrm{d}x^3/\mathrm{d}t}{\sqrt{1-v^2/c^2}} \end{pmatrix} = \frac{1}{\sqrt{1 - v^2/c^2}} \begin{pmatrix} c \\ \mathbf{v} \end{pmatrix} \tag{5}$$

with $\mathbf{v}$ denoting the usual 3 velocity, derivative with respect to time, $\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$. The four velocity Minkowski length is Lorentz invariant, as $u^\mu u_\mu = \frac{1}{1-\mathbf{v}^2/c^2}(c^2 - \mathbf{v}^2) = c^2$

(b) Two space crafts with velocities along x and $v_{1,2} = \pm 0.6c$

$$u^\mu_{1,2} = \frac{1}{\sqrt{1 - v^2/c^2}} \begin{pmatrix} c \\ \pm\mathbf{v} \end{pmatrix} = \frac{c}{\sqrt{1 - v^2/c^2}} \begin{pmatrix} 1 \\ \pm 0.6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5/4 \\ \pm 3/4 \\ 0 \\ 0 \end{pmatrix} c \tag{6}$$

The length of them are easily calculated and gives c^2.

(c) The spacecrafts go in opposite direction so we need to consider again the negative velocity and rapidity in case of one of them, which have the same magnitude. Then we just easily express the velocity of one of the spacecrafts measured from the other's frame by which we can immediately tell the corresponding Lorentz transformation's matrix

$$\tilde{v} = \frac{2v}{1 + v^2/c^2} = \frac{1.2}{1.36}c = 15/17c, \tag{7}$$

$$\Lambda = \begin{pmatrix} \frac{1}{\sqrt{1-\tilde{v}^2/c^2}} & \frac{-\tilde{v}/c}{\sqrt{1-\tilde{v}^2/c^2}} \\ \frac{-\tilde{v}/c}{\sqrt{1-\tilde{v}^2/c^2}} & \frac{1}{\sqrt{1-\tilde{v}^2/c^2}} \end{pmatrix} = \begin{pmatrix} 2.125 & -1.875 \\ -1.875 & 2.125 \end{pmatrix}. \tag{8}$$

(d) The spacecraft in its own rest frame stays, but the other's four velocity is easily given by the composite Lorentz transformation:

$$u_2' = \begin{pmatrix} 2.125 & -1.875 \\ -1.875 & 2.125 \end{pmatrix} u_2 = \begin{pmatrix} 34/16c \\ -30/16c \end{pmatrix} \tag{9}$$

the length of which is indeed $\frac{34^2-30^2}{16^2}c^2 = c^2$. Here we could have just plug in the result for $\tilde{v}$ and write

$$u_2' = \frac{1}{\sqrt{1-\tilde{v}^2/c^2}}\begin{pmatrix} c \\ \tilde{v} \\ 0 \\ 0 \end{pmatrix}. \tag{10}$$

(e) The 3 velocities take the form

$$\mathbf{v}_2' = \begin{pmatrix} -30/16 \\ 0 \\ 0 \end{pmatrix} c \tag{11}$$

$$\mathbf{v}_1' = \mathbf{0} \tag{12}$$

(f) Now with the second spacecraft going in y direction we can again calculate the modified velocity in the frame of the first spacecraft by just acting with an x dieected boost on the y dierected four velocity:

$$u_2' = c\begin{pmatrix} 5/4 & -3/3 & 0 & 0 \\ -3/4 & 5/4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 5/4 \\ 0 \\ 3/4 \\ 0 \end{pmatrix} = c\begin{pmatrix} 25/16 \\ -15/16 \\ 3/4 \\ 0 \end{pmatrix} \tag{13}$$

Note that the acquired velocity along the x axis has a different magnitude as the other spacecraft's original velocity!

(g) One can now extract the transformed 3 velocity from this result via the prefactor of the zeroth component, $\sqrt{1-(v_2')^2/c^2} = 16/25$, from where we have for the velocity part:

$$\mathbf{v}_2' = \frac{16c}{25}\begin{pmatrix} -15/16 \\ 3/4 \\ 0 \end{pmatrix} = c\begin{pmatrix} -3/5 \\ 12/25 \\ 0 \end{pmatrix} \tag{14}$$

Problem 3

The Compton effect (Artur Holly Compton 1892 – 1962. Nobel-prize: 1927) was one of the important experimental results that led to the birth of quantum mechanics. This experiment showed that a photon of energy $\hbar\omega$ has also a momentum $\hbar\omega/c$. Here ω is the frequency of the photon.

In the experiment a photon of frequency ω_0 collides with an electron that is initially in rest (mass m). After the collision the electron has a momentum p while the photon loses from its energy, and its trajectory distorts by an angle of ϑ. After the collision we detect the scattered photon.

(a) Define a convenient coordinate system. Sketch a figure about the process.

(b) Write down the total 4-momentum of the system before and after the collision.

(c) Determine the frequency ω' of the scattered photon as a function of the distortion angle ϑ. Exploit the conservation of 4-momentum.

Solution:

(a) Four momentum generally, with $p^\mu = mu^\mu$, m is the rest mass

$$p^\mu = \begin{pmatrix} E/c \\ \mathbf{p} \end{pmatrix} = \frac{1}{\sqrt{1-v^2/c^2}}\begin{pmatrix} \frac{mc}{\sqrt{1-v^2/c^2}} \\ \frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}} \end{pmatrix} \tag{15}$$

and trivially we have that $p^\mu p_\mu = m^2u^\mu u_\mu = m^2c^2$. For zero rest mass particles we have $p^\mu p_\mu = 0$ that is $|\mathbf{p}| = E/c$. For a photon it yields $E = \hbar\omega,\ |\mathbf{p}| = \frac{\hbar\omega}{c}$.

(b) Four momentum is Lorentz invariant, so it must be conserved during elastic processes, that is we write up its conservation in the rest frame of the electron with q^μ and p^μ denoting the photon's and electron's four momenta

$$q^\mu = \begin{pmatrix} \hbar\omega_0/c \\ \hbar\omega_0/c \\ 0 \\ 0 \end{pmatrix}, \quad p^\mu = \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{16}$$

$$P^\mu = q^\mu + p^\mu = \begin{pmatrix} \hbar\omega_0/c + m \\ \hbar\omega_0/c \\ 0 \\ 0 \end{pmatrix} \tag{17}$$

which is conserved and is equal to the final sum, which we look for in case of scattering angle of the photon, θ, with new frequency ω and some momenta and energy p_x, p_y, E, which gives

$$(q')^\mu = \begin{pmatrix} \hbar\omega/c \\ \hbar\omega/c\cos\theta \\ \hbar\omega/c\sin\theta \end{pmatrix} \tag{18}$$

$$p^\mu = \begin{pmatrix} E/c \\ p_x \\ p_y \end{pmatrix} \tag{19}$$

$$\tag{20}$$

Note that this is the usual energy and momentum conservation for elastic scatterings but with relativistic masses. Now writing up the three equations belonging to the given coordinates:

$$E + \hbar\omega = mc^2 + \hbar\omega_0 \tag{21}$$

$$\hbar\omega_0 = \hbar\omega\cos\theta + cp_x \tag{22}$$

$$0 = \hbar\omega\sin\theta + cp_y \tag{23}$$

$$\tag{24}$$

Our aim is to express the new frequency in terms of the scattering angle. First exploit the fact that $E^2 = c^2p_x^2 + c^2p_y^2 + m^2c^4$ and express the momenta as $p_x = \frac{\hbar}{c}(\omega_0 - \omega\cos\theta)$, $p_y = -\frac{\hbar\omega}{c}\sin\theta$. We write this back into the first equation and square everything:

$$\begin{aligned} E^2 &= \left(mc^2 + \hbar(\omega - \omega_0)\right)^2 = c^2p_x^2 + c^2p_y^2 + m^2c^4 \\ &= m^2c^4 + \hbar^2(\omega-\omega_0)^2 + 2mc^2\hbar(\omega-\omega_0) = m^2c^4 + \hbar^2\omega^2\sin^2\theta + \hbar^2(\omega\cos\theta - \omega_0)^2 \end{aligned} \tag{25}$$

which after sime trivial simplifications results in

$$2mc^2\hbar(\omega-\omega_0) - 2\hbar^2\omega\omega_0 = -2\hbar^2\omega\omega_0\cos\theta \Rightarrow \omega = \frac{mc^2}{mc^2 - 2\hbar\omega_0\sin^2(\theta/2)}\omega_0 \tag{26}$$

Problem 4

Let's consider the elastic collision of two particles. The particles move on a common, straight trajectory. One has rest mass m_1 and (usual) velocity v_1 while the other has rest mass m_2 and velocity v_2. Their common trajectory defines the x-axis.

(a) Write down the 4-momenta p_1^μ and p_2^μ of the two particles. What is the meaning of their components?

(b) Write down the equation for the 4-momentum conservation. It's scary, isn't it?

(c) In non-relativistic collision problems it is a neat trick to transform of the frame of the "center of mass". In this frame, the 4-momentum conservation gives a much simpler equation, and one can immediately write down the momenta after the collision. Let's try to generalize this trick for the relativistic case.

(d) Write down the total 4-momentum of the system before the collision.

(e) Write down the matrix of a Lorentz boost with some arbitrary velocity V, and transform the 4-momentum with this transformation.

(f) What should be V, if we want the total (3-)momentum to vanish in the moving frame? Let's define this velocity as the velocity of the "center of mass".

(g) Transform to the frame of the center of mass. Express the 4-momenta of the particles in that frame before and after the collision.

(h) Transform back to the original frame, and express the 4-momenta of the particles after the collision.

Solution:

(a) The four momenta with rest masses m_1 and m_2:

$$p_1^\mu = \frac{1}{\sqrt{1-v_1^2/c^2}} \begin{pmatrix} m_1 c \\ m_1 v_1 \end{pmatrix}, \; p_2^\mu = \frac{1}{\sqrt{1-v_2^2/c^2}} \begin{pmatrix} m_2 c \\ m_2 v_2 \end{pmatrix} \tag{27}$$

and similar expressions for their momenta after the collisions happened

$$(p_1')^\mu = \frac{1}{\sqrt{1-(v_1')^2/c^2}} \begin{pmatrix} m_1 c \\ m_1 v_1' \end{pmatrix}, \; (p_2')^\mu = \frac{1}{\sqrt{1-(v_2')^2/c^2}} \begin{pmatrix} m_2 c \\ m_2 v_2' \end{pmatrix}. \tag{28}$$

(b) Now the conservation laws:

$$\frac{m_1 c}{\sqrt{1-v_1^2/c^2}} + \frac{m_2 c}{\sqrt{1-v_2^2/c^2}} = \frac{m_1 c}{\sqrt{1-(v_1')^2/c^2}} + \frac{m_2 c}{\sqrt{1-(v_2')^2/c^2}}, \tag{29}$$

$$\frac{m_1 v_1}{\sqrt{1-v_1^2/c^2}} + \frac{m_2 v_2}{\sqrt{1-v_2^2/c^2}} = \frac{m_1 v_1'}{\sqrt{1-(v_1')^2/c^2}} + \frac{m_2 v_2'}{\sqrt{1-(v_2')^2/c^2}}, \tag{30}$$

which is, indeed, scary, I guess...

(c) Total four momentum before the collision happened:

$$P^\mu = \begin{pmatrix} \frac{m_1 c}{\sqrt{1-v_1^2/c^2}} + \frac{m_2 c}{\sqrt{1-v_2^2/c^2}} \\ \frac{m_1 v_1}{\sqrt{1-v_1^2/c^2}} + \frac{m_2 v_2}{\sqrt{1-v_2^2/c^2}} \end{pmatrix}. \tag{31}$$

(d) Lorentz boost with some undetermined velocity V:

$$\Lambda^\mu_{\cdot\nu} = \frac{1}{\sqrt{1-V^2/c^2}} \begin{pmatrix} 1 & -V/c \\ -V/c & 1 \end{pmatrix}. \tag{32}$$

Now the transformed four momentum looks a bit nasty as well:

$$(P')^\mu = \frac{1}{\sqrt{1-V^2/c^2}} \begin{pmatrix} \frac{m_1(c-v_1V/c)}{\sqrt{1-v_1^2/c^2}} + \frac{m_2(c-v_2V/c)}{\sqrt{1-v_2^2/c^2}} \\ \frac{m_1(v_1-V)}{\sqrt{1-v_1^2/c^2}} + \frac{m_2(v_2-V)}{\sqrt{1-v_2^2/c^2}} \end{pmatrix}. \tag{33}$$

(e) To make vanish the spatial component part we need to choose:

$$V = \frac{\frac{m_2 v_2}{\sqrt{1-v_2^2/c^2}} + \frac{m_1 v_1}{\sqrt{1-v_1^2/c^2}}}{\frac{m_2}{\sqrt{1-v_2^2/c^2}} + \frac{m_1}{\sqrt{1-v_1^2/c^2}}} \tag{34}$$

called center of mass velocity! Somehow trivial, isn't it? Indeed, it is just the average momentum, but with relativistic masses!

(f) Now the momenta of the individual particles read as

$$\tilde{p}_1^\mu = \frac{m_1}{\sqrt{1-v_1^2/c^2}\sqrt{1-V^2/c^2}}\begin{pmatrix} c - v_1 V/c \\ v_1 - V\end{pmatrix}, \; \tilde{p}_2^\mu = \frac{m_2}{\sqrt{1-v_2^2/c^2}\sqrt{1-V^2/c^2}}\begin{pmatrix} c - v_2 V/c \\ v_2 - V\end{pmatrix}$$
$$(\tilde{p}_1')^\mu = \frac{m_1}{\sqrt{1-(v_1')^2/c^2}\sqrt{1-V^2/c^2}}\begin{pmatrix} c - v_1' V/c \\ v_1' - V\end{pmatrix}, \; (\tilde{p}_2')^\mu = \frac{m_2}{\sqrt{1-(v_2')^2/c^2}\sqrt{1-V^2/c^2}}\begin{pmatrix} c - v_2' V/c \\ v_2' - V\end{pmatrix} \tag{35}$$

But we know that in center of mass coordinate the system's total spatial momentum is zero and as particles have the same mass we simply have that for velocities in COM frame satisfy taht $\tilde{v}_1 = -\tilde{v}_1'$ and $\tilde{v}_2 = -\tilde{v}_2'$. However it would include a tedious calculation to bring to the form the final momenta as $\tilde{p} = \frac{1}{\sqrt{1-\tilde{v}^2/c^2}}\begin{pmatrix} mc \\ m\tilde{v}\end{pmatrix}$. So we rather work with rapidities $\tanh\theta_{1,2} = \frac{v_{1,2}}{c}$, $\tanh\tilde{\theta}_{1,2} = \frac{\tilde{v}_{1,2}}{c}$ and $\tanh\Theta = \frac{V}{c}$ with $V = \frac{m_1 c\sinh\theta_1 + m_2 c\sinh\theta_2}{m_1 c\cosh\theta_1 + m_2 c\cosh\theta_2}$. After the transformation we have $\tilde{\theta}_{1,2} = \theta_{1,2} - \Theta$ and $\tilde{\theta}_{1,2}' = -\tilde{\theta}_{1,2} = \Theta - \theta_{1,2}$ giving for the fina momenta in COM frame

$$\tilde{p}_{1,2}' = c\begin{pmatrix} m_1\cosh\tilde{\theta}_{1,2}' \\ m_1\sinh\tilde{\theta}_{1,2}'\end{pmatrix} \tag{36}$$

(g) Now the final momenta are obtained by transforming back to the original system with the Lorentz transformation

$$\Lambda^\mu_{\cdot\nu} = \begin{pmatrix} \cosh\Theta & \sinh\Theta \\ \sinh\Theta & \cosh\Theta\end{pmatrix} \tag{37}$$

giving in the end momenta parametrized with rapidities $\theta_{1,2}' = 2\Theta - \theta_{1,2}$ rsulting in, indeed, a scary final result:

$$p_{1,2}' = m_{1,2}c\begin{pmatrix} \cosh\Theta\cosh(\Theta-\theta_{1,2}) + \sinh\Theta\sinh(\Theta-\theta_{1,2}) \\ \sinh\Theta\cosh(\Theta-\theta_{1,2}) + \cosh\Theta\sinh(\Theta-\theta_{1,2})\end{pmatrix} = m_{1,2}c\begin{pmatrix} \cosh(2\Theta-\theta_{1,2}) \\ \sinh(2\Theta-\theta_{1,2})\end{pmatrix}$$
$$= \frac{m_{1,2}}{(c^2-V^2)\sqrt{1-v_{1,2}^2/c^2}}\begin{pmatrix} V^2 v_{1,2} - Vcv_{1,2} + c^2 v_{1,2} - Vc^2 \\ cVv_{1,2} - c^2 v_{1,2} + Vcv_{1,2} - V^2 v_{1,2}\end{pmatrix}. \tag{38}$$