Problem 1

Consider a two-dimensional anisotropic oscillator. The Hamiltonian of the system is

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2) \tag{1}$$

(a) Write down the full (time dependent) Hamilton-Jacobi equation for the system.

(b) The Hamiltonian does not depend on time, therefore the Hamilton-Jacobi equation can be separated in the form $S(x, y, t) = S_0(x, y, E) - Et$. Write down the abbreviated Hamilton-Jacobi equation for S_0.

(c) Separate further the function S_0, i.e. look for the solution in the form

$$S_0(x, y, E) = S_x(x, \alpha_x) + S_y(y, \alpha_y) \tag{2}$$

Write down the equations for S_x and S_y. Denote the new constants by $\alpha_{x,y}$.

(d) Determine the functions S_x, S_y, and express the full solution $S(x, y, \alpha_x, \alpha_y, t)$ of the Hamilton-Jacobi equation.

(e) The particle is initially ($t = 0$) at the position $x = x_0$ and $y = y_0$, and has zero momentum. Determine the values of the constants.

(f) How can one get the $x(t)$, $y(t)$ solutions of the equations of motion, using $S(x, y, t)$? (Don't calculate it! It's a lengthy calculation.)

Solution

(a) Hamilton Jacobi equation with second type generator unction $S(x, y, P_x, P_y)$ where the old momenta again can be expressed as $p_x = \partial_x S$ and $p_y = \partial_y S$.

$$H(x, y, \partial_x S, \partial_y S) + \partial_t S = 0 \tag{3}$$

(b) Looking for the solution in form of $S(x, y, t) = S_0(x, y) - Et$ we get

$$H(x, y, \partial_x S_0, \partial_y S_0) = E \tag{4}$$

(c) Now separate the action as $S(x, y, E) = S_x(x, \alpha_x) + S_y(y, \alpha_y)$ with $\alpha_{x,y}$ denoting the new momenta, we also write the Hamiltonaina as

$$H = H_x + H_y \tag{5}$$

and then

$$H_x(x, \partial_x S_x) = \alpha_x \qquad H_x(y, \partial_y S_y) = \alpha_y \qquad E = \alpha_x + \alpha_y \tag{6}$$

(d) Writing in the the above two equations the S_x and S_y parts of the action we get

$$\frac{(S_x')^2}{2m} + \frac{m}{2}\omega_x^2 x^2 = \alpha_x \tag{7}$$

Denoting

$$\alpha_x = \frac{m}{2} A_x^2 \omega_x^2 \tag{8}$$

we get

$$S_x' = m\omega_x \sqrt{A_x^2 - x^2} \tag{9}$$

so

$$S_x = \frac{m\omega_x}{2}\left[A_x^2 \arcsin\left(\frac{x}{A_x}\right) + x\sqrt{A_x^2 - x^2}\right] \tag{10}$$

The full solution is of course

$$\begin{aligned} S &= S_x + S_y - Et = \\ &= \frac{m\omega_x}{2}\left[A_x^2 \arcsin\left(\frac{x}{A_x}\right) + x\sqrt{A_x^2 - x^2}\right] + \frac{m\omega_y}{2}\left[A_y^2 \arcsin\left(\frac{y}{A_y}\right) + x\sqrt{A_y^2 - y^2}\right] - (\alpha_x + \alpha_y)t \\ &= \frac{m\omega_x}{2}\left[A_x^2 \arcsin\left(\frac{x}{A_x}\right) + x\sqrt{A_x^2 - x^2} - A_x^2\omega_x t\right] + \frac{m\omega_y}{2}\left[A_y^2 \arcsin\left(\frac{y}{A_y}\right) + x\sqrt{A_y^2 - y^2} - A_y^2\omega_y t\right] \end{aligned} \tag{11}$$

(e) With initially zero momentum we get $\partial_x S_x = \partial_y S_y = 0$, so $A_x = x_0$ and $A_y = y_0$.

(f) The orbits would be given simply by the second equation for the new coordinates according to the rule for the second type of generating functions, $\beta_{x,y} = \partial_{\alpha_x,\alpha_y} S = \text{const.}$ from where we can trace back the time-evolution of $x(t)$ and $y(t)$.

$$\partial_{A_x,A_y} S = \text{const.} \tag{12}$$

Note that it does not matter which variable we use here, so we were free to take derivative with respect to A_x, A_y, etc.

Problem 2

Two identical bodies can move along the x axis in a box. The two bodies are attached to the walls through two springs with spring constant D, and there is also a spring between the two bodies. The Hamiltonian of the system is

$$H(x_1, p_1, x_2, p_2) = \frac{p_1^2 + p_2^2}{2m} + \frac{D}{2}(x_1^2 + x_2^2) + \frac{D}{2}(x_1 - x_2)^2 \tag{13}$$

(a) Write down the full (time dependent) Hamilton-Jacobi equation for the system.

(b) The Hamiltonian does not depend on time, therefore the Hamilton-Jacobi equation can be separated int the form $S = S_0 - Et$. Write down the abbreviated Hamilton-Jacobi equation for S_0.

(c) Further separation cannot be done using the coordinates x_1 ,x_2. Transform the equation to the new variables $X = (x_1 + x_2)/2$ and $y = x_1 - x_2$ and rewrite the equation of b.) using these variables.

(d) Separate the S_0 function as $S_0(x_1, x_2, E) = S_y(y, \alpha_y) + S_X(X, \alpha_X)$. Write down the equations for S_X and S_y! Denote the new constants by α_y, α_X.

(e) Determine the functions S_X and S_y.

(f) Knowing the initial conditions $(x_{1,0}, x_{2,0}, p_{1,0}, p_{2,0})$ determine the values of the $\alpha_{X,y}$ parameters.

Solution:

(a) Hamilton-Jacobi equation with new momenta α_{x_1}, α_{x_2}

$$H(x, y, \partial_x S, \partial_y S, t) + \partial_t S(x, y, \alpha_{x_1}, \alpha_{x_2}, t) = 0 \tag{14}$$

(b) Again looking for the action as $S(x, y, \alpha_{x_1}, \alpha_{x_2}, t) = S_0(x, y, \alpha_{x_1}, \alpha_{x_2}) - Et$

$$E = \frac{(\partial_{x_1} S_0)^2 + (\partial_{x_2} S_0)^2}{2m} + \frac{D}{2}(x_1 - x_2)^2 + \frac{D}{2}(x_1^2 + x_2^2) \tag{15}$$

(c) With the new coordinates $X = (x_1 + x_2)/2$, $y = x_1 - x_2$ and $x_1 = X + y/2$, $x_2 = X - y/2$ the derivatives become

$$\partial_{x_1} S_0 = \partial_y S_0 \frac{\partial y}{\partial x_1} + \partial_X S_0 \frac{\partial X}{\partial x_1} = \partial_y S_0 + \frac{\partial_X S_0}{2} \tag{16}$$

and

$$\partial_{x_2} S_0 = -\partial_y S_0 + \frac{\partial_X S_0}{2} \tag{17}$$

Summing the squares:

$$(\partial_{x_1} S_0)^2 + (\partial_{x_2} S_0)^2 = 2(\partial_y S_0)^2 + \frac{1}{2}(\partial_X S_0)^2 \tag{18}$$

So

$$E = \frac{(\partial_y S_0)^2}{m} + \frac{(\partial_X S_0)^2}{4m} + \frac{D}{2}y^2 + \frac{D}{2}(2X^2 + y^2/2). \tag{19}$$

(d) Now separating the action as $S_0(x_1, x_2, \alpha_{x_1}, \alpha_{x_2}, t) = S_y(y, \alpha_y) + S_X(X, \alpha_X)$ we can write:

$$\alpha_y = \frac{(S_y')^2}{m} + \frac{3D}{4}y^2, \qquad \alpha_X = \frac{(S_X')^2}{4m} + DX^2, \qquad E = \alpha_X + \alpha_y \tag{20}$$

(e) We can easily express now S_y and S_X as

$$S'_y = \sqrt{m\alpha_y - 3Dy^2/4} \to S_y = \arcsin\left(\sqrt{\frac{3D}{4m\alpha_y}}y\right)\frac{m\alpha_y}{\sqrt{3D}} + \frac{y\sqrt{m\alpha_y - 3Dy^2/4}}{2} \tag{21}$$

$$S'_X = \sqrt{16m\alpha_X - DX^2} \to S_X = \arcsin\left(\sqrt{\frac{D}{4m\alpha_X}}X\right)\frac{2m\alpha_X}{\sqrt{D}} + \frac{X\sqrt{4m\alpha_X - DX^2}}{2} \tag{22}$$

$$\tag{23}$$

(f) Again being of a cumbersome task to determine the new constant cooridnates β_X and β_y according to the initial conditions. First we should determine the values of α_y, α_X as

$$\partial_X S = p_{X,0} = (p_{10} + p_{20})/2,\ \partial_y S = p_{y,0} = p_{1,0} - p_{2,0} \tag{24}$$

This leads to

$$\sqrt{m\alpha_y - 3Dy_0^2/4} = p_{y,0} \to \alpha_y = \frac{(p_{1,0} - p_{2,0})^2 + 3D(x_{1,0} - x_{2,0})^2/4}{m} \tag{25}$$

$$\sqrt{16m\alpha_X - DX_0^2} = p_{X,0} \to \alpha_X = \frac{(p_{1,0} + p_{2,0})^2 + D(x_{1,0} + x_{2,0})^2}{64m} \tag{26}$$

From this the original momenta

$$\alpha_{x_1} = \alpha_X + \alpha_y/2,\ \alpha_{x_2} = \alpha_X - \alpha_y/2 \tag{27}$$

Then from it we can write the condition that

$$\partial_{\alpha_y} S_y = \text{const.} \equiv \beta_y,\ \partial_{\alpha_X} S_X = \text{const.} \equiv \beta_X \tag{28}$$

which can in theory be obtained i na closed formula.

Problem 3

Consider the following generalized oscillator, that is described by a power-law potential as

$$H(p, x) = \frac{p^2}{2m} + k|x|^4 \tag{29}$$

(a) Draw the contour lines $H(p, x) = E$ on the $p - x$ plane.

(b) Determine the integral that equals the phase-surface bounded by the contour-lines. Denote it by $2\pi I$.

(c) In the generic case the integral cannot be analytically determined. The best we can do is to determine the (power-law) dependence on the parameters E, m, and k. Performing appropriate variable transformations make the integral dimensionless, i.e. collect all the dependence on the parameters outside the integral. In this case the value of the dimensionless integral is only a number, that can be calculated numerically.

(d) Using the derivation of $I(E)$ determine the period of the oscillation as a function of the parameters.

Solution:

(a) Contours of $H(p, x) = E$ are determined by the relation $p = \sqrt{2mE}\sqrt{1 - k|x|^4}$

(b) The integral is given by

$$I(E) = \frac{1}{2\pi}\oint \mathrm{d}q\, p \tag{30}$$

Now consider a small segment d between the contours related to energies $H(p, x) = E$ and $H(p, x) = E + \Delta E$. Then $d = \frac{\Delta E}{|\text{grad}H|}$ with $|\text{grad}H| \equiv v_{\text{ph}} = \sqrt{\left(\frac{\partial H}{\partial p}\right)^2 + \left(\frac{\partial H}{\partial x}\right)^2}$. The the phase volume is given by $dv\,\mathrm{d}t = \Delta E\,\mathrm{d}t$. In other words the period can be expressed as the change of the phase space volume divided by the change in the energy, $T = \frac{\mathrm{dphase\ space\ volume}}{\mathrm{d}E}$:

$$T = 2\pi \frac{\mathrm{d}I}{\mathrm{d}E}\,. \tag{31}$$

which also imply a canonical transformation as

$$x,\, p \to \Phi,\, I\,. \tag{32}$$

As te action only depends on the energy one can consider it as a new momentum and try to introduce a second type generator function as $W_2 = W_2(q, J)$. Introducing the new coordinate accordign to the "rule"

$$\Phi = \frac{\partial W_2}{\partial I} \Rightarrow \dot{\Phi} = \frac{\partial H(J)}{\partial J} = \omega(J) \tag{33}$$

which is time-independent, where we wrote for the new Hamiltonian also H but now only depending on the actio nas we are considering constant energies and there is a one-to-one correspondence between the energy and the action! Let us consider how it changes druing a complete cyle of the motion:

$$\Delta\Phi = \oint \mathrm{d}q \frac{\partial \Phi}{\partial q} = \oint \mathrm{d}q \frac{\partial^2 W}{\partial q \partial\partial J} = \frac{\partial}{\partial J} \oint \mathrm{d}qp = 2\pi. \tag{34}$$

that is Φ evolves linear in time with cosntant *frequency*, depending only on E and so on J and takes the multiplets of 2π per cylce!

(c) Now let us calculate the action at energy E

$$p = \pm\sqrt{2mE - 2mk|x|^4} \Rightarrow I = \frac{1}{\pi}\int_{x_{\min}}^{x_{\max}} \sqrt{2mE - 2mk|x|^4} = \frac{\sqrt{2mE}}{\pi}\left(\frac{E}{k}\right)^{1/4}\int_{-1}^{1}\sqrt{1-|y|^4}\mathrm{d}y$$
$$\approx 1.748 \frac{\sqrt{2mE}}{\pi}\left(\frac{E}{k}\right)^{1/4} \tag{35}$$

From here we can express the energy dependence of the period

$$T = 2\pi \frac{\mathrm{d}I}{\mathrm{d}E} = 1.748 \frac{3\sqrt{m}}{4\pi}\left(\frac{E}{k}\right)^{1/4} \tag{36}$$

where we exploited that at $x_{\min}$a and $x_{\max}$ momentum is zero and the energy equals the potential term.

Problem 4

Consider a pendulum with Hmailtonian with length l

$$H = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta \tag{37}$$

(a) Sketch the phase space trajectory for a given energy $H(p_\theta, \theta) = E$.
(b) Write down the energy expression for the action.
(c) Give the expression for the period/frequency of the periodic motion in the phase-space.
(d) For small amplitudes, $\theta \ll 1$ determine the frequency of the motion.

Solution:

(a) Trajectories are given by the equation $p_\theta = \pm\sqrt{2ml^2E + 2m^2gl^2\cos\theta}$.

(b) The action is then exrpessed as comolicated integral

$$2\pi I = 2\int_{\theta_{\min}}^{\theta_{\max}} \mathrm{d}\theta\, \sqrt{2ml^2E + 2m^2gl^2\cos\theta} \tag{38}$$

which cannot be determined in general, nevertheless we can again apply our knwoledge for the period of time and furthermore no power law behavior can be obtained as generally for all angles we may have non-zero momentum!

$$T = \frac{\mathrm{d}I}{\mathrm{d}E} = \frac{\sqrt{2m}l}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \frac{1}{\sqrt{E + gl\cos\theta}} \tag{39}$$

(c) Now with small amplitudes we can write with redefined energy $E \to E + gl$.

$$T = \frac{\sqrt{2m}l}{4\pi}\sqrt{\frac{E}{gl}}\frac{2}{\sqrt{2mE}}\int_{-1}^{1} \mathrm{d}y\, \frac{1}{\sqrt{1-y^2}} = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \tag{40}$$

where we again used that in the small amplitude case we have $E = mgl\theta_{min,max}/2$.

Extra exercise

Consider the problem of the vertical motion in a homogeneous gravitational field. The Hamiltonian of the system is

$$H(p,x) = \frac{p^2}{2m} + mgx \tag{41}$$

- Write down the full (time dependent) Hamilton-Jacobi equation for the system.
- Because the Hamiltonian does not depend on time explicitly, we can look for the function $S(x,t)$ in the form $S(x,t) = S_0(x,E) - Et$, where E is a constant. Express the abbreviated Hamilton-Jacobi equation for the function S_0.
- Solve the equation for S_0.
- Knowing the function $S(x,E,t)$, determine the canonical transformation that it generates. Express the canonical coordinate β_E, that is the canonical pair of E.
- The particle is initially in the position $x = 0$ and has momentum p_0. Using this information determine the values of E and β_E.
- Express the $x(t)$ solution of the equation of motion.

Solutions

Full:

$$H\left(x, \frac{\partial S}{\partial x}, t\right) = -\frac{\partial S(x,t)}{\partial t} \tag{42}$$

Now

$$S(x,t) = S_0(x) - E(t) \tag{43}$$

and

$$E = H\left(x, \frac{\partial S}{\partial x}\right) \tag{44}$$

From this:

$$S_0' = \sqrt{2mE - 2m^2gx} \tag{45}$$

and

$$S_0 = -(2mE - 2m^2gx)^{3/2}\frac{2}{3}\frac{1}{2m^2g} + c \tag{46}$$

as a canonical transformation:

$$W_2(x,P,t) = S_0(x,E) - Et \tag{47}$$

it will give

$$H(E,\beta_E) = 0 \tag{48}$$

and

$$p = \frac{\partial W_2}{\partial x} \qquad \beta_E = \frac{\partial W_2}{\partial E} = \frac{-\sqrt{2mE - 2m^2 gx}}{mg} - t \tag{49}$$

We know that $\beta_E = 0$, therefore

$$\beta_E(t = 0) = -\frac{\sqrt{2mE}}{mg} = -\frac{p_0}{mg} \tag{50}$$

and

$$\frac{p_0}{mg} - t = \frac{\sqrt{2mE - 2m^2 gx}}{mg} \tag{51}$$

or

$$(p_0 - mgt)^2 = 2mE - 2m^2 gx = p_0^2 - 2m^2 gx \tag{52}$$

and

$$2m^2 gx = -m^2 g^2 t^2 + 2p_0 mgt \tag{53}$$

and

$$x = -\frac{g}{2}t^2 + \frac{p_0}{m}t \tag{54}$$