Problem 1

Second type of canonical transformations. Repeat the steps of the lecture and derive the differential equations for second type of generator functions.
Solution:

(a) Generalized least action principle (the original one but with the Hamiltonian)

$$S = \int \mathbf{p}\,\mathrm{d}\mathbf{q} - H\,\mathrm{d}t \to \delta S = 0 \leftrightarrow \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}},\ \ \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} \tag{1}$$

(b) Taking the transformed coordinates and momenta $\mathbf{Q}$, $\mathbf{P}$ and the correspondig Hamiltonian $K(\mathbf{Q},\mathbf{P})$ that sastisfy again the Hamiltonian equations of motion:

$$\dot{\mathbf{P}} = -\frac{\partial K}{\partial \mathbf{Q}},\ \ \dot{\mathbf{Q}} = \frac{\partial K}{\partial \mathbf{P}}\,. \tag{2}$$

We can again write up the action and equal it to the original action with $H(\mathbf{p},\mathbf{q})$:

$$S = \int \mathbf{P}\,\mathrm{d}\mathbf{Q} - K\,\mathrm{d}t = \int \mathbf{p}\,\mathrm{d}\mathbf{q} - H\,\mathrm{d}t \Leftrightarrow \mathbf{P}\,\mathrm{d}\mathbf{Q} - K\,\mathrm{d}t + \frac{\mathrm{F}}{\mathrm{t}}\,\mathrm{d}t = \mathbf{p}\,\mathrm{d}\mathbf{q} - H\,\mathrm{d}t \tag{3}$$

Now instead of considering tarnsformations which transform constant $\mathbf{q}$ to constant lines of $\mathbf{Q}$ as was the case for type 1 generators, $W_1 = W_1(\mathbf{q},\mathbf{Q})$, we would like to have new canonical variables with *constant new momenta* associated to the $\mathbf{q}$ coordinates held fixed. This is achieved by expanding the generator function's total time derivative, $F \equiv W_2(\mathbf{q},\mathbf{P})$, according to their variables, $\frac{\mathrm{d}W_2}{\mathrm{d}t} = \frac{\partial W_2}{\partial \mathbf{q}}\mathrm{d}\mathbf{q} + \frac{\partial W_2}{\partial \mathbf{P}}\mathrm{d}\mathbf{P} + \frac{\partial W_2}{\partial t}\mathrm{d}t$. This leads to the following in the integrands of the actions:

$$H = K + \frac{\partial W_2}{\partial t},\ \mathbf{p} = \frac{\partial W_2}{\partial \mathbf{q}},\ \mathbf{Q} = \frac{\partial W_2}{\partial \mathbf{P}} \tag{4}$$

where we have completed difference $\mathbf{P}\,\mathrm{d}\mathbf{Q} = \mathrm{d}(\mathbf{PQ}) - \mathrm{d}\mathbf{P}\,\mathbf{Q}$ to obtain an expression proportional to $\mathrm{d}\mathbf{P}$.

(c) Remember if we perform a symmetry transformation meaning that the infinitesimal change obeys the rule

$$\boldsymbol{\eta} \to \boldsymbol{\eta}' = \boldsymbol{\eta} + \delta\boldsymbol{\theta}\{\boldsymbol{\eta}, F\}\,,\ \{F, H\} = 0\,, \tag{5}$$

then $K(\boldsymbol{\eta}') = H(\boldsymbol{\eta})$ and the equations of motions do not change.

Problem 2

Consider a linear harmonic oscillator whose Hamiltonian reads

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) \tag{6}$$

Consider the following generator functions and try to derive transformation rules.

(a) $W_2(q,P) = q + P$

(b) $W_2(q,P) = (q+P)^2$

(c) $W_2(q,P) = \frac{q}{P}$

(d) Which generator function describes indeed a transformation? Perform the transformation and determine the „new" Hamiltonian $K(Q,P)$.

(e) Determine the canonical equations using the new form of the Hamiltonian. Solve the equations!

Solution:

(a) For the first generator

$$p = \frac{\partial W_2}{\partial q} = 1,\ Q = \frac{\partial W_2}{\partial P} = 1 \tag{7}$$

seemingly wrong results are obtained as we have lost all dependences.

(b) For the second one we have

$$p = \frac{\partial W_2}{\partial q} = 2(q+P),\ Q = \frac{\partial W_2}{\partial P} = 2(q+P) \Rightarrow q = \frac{Q}{2} - P,\ p = Q\,. \tag{8}$$

From here the Hamiltonian takes the form of

$$K = \frac{Q^2}{2m} + \frac{1}{2}m\omega^2(Q/2-P)^2. \tag{9}$$

(c) Now the equations of motion:

$$\dot{Q} = \frac{\partial K}{\partial P} = -m\omega^2(Q/2-P),\ \dot{P} = -\frac{\partial K}{\partial Q} = -\frac{Q}{m} + \frac{1}{2}m\omega^2(Q/2-P) \tag{10}$$

So we immediately have that

$$2\dot{P} + \dot{Q} = -2\dot{q} = -\frac{2Q}{m} \equiv \frac{-2p}{m}\,, \tag{11}$$

$$\dot{Q} = -m\omega^2 q \equiv \dot{p}\,, \tag{12}$$

indeed, implying that the above transformation is canonical as the new variables' Hamiltonian equations of motion are equivalent to the ooriginal ones.

(d) Now for this case the situation is a bit more complicated, as the new variables

$$p = \frac{\partial W_2}{\partial q} = \frac{1}{P},\ Q = \frac{\partial W_2}{\partial P} = -\frac{q}{P^2} \Rightarrow q = -QP^2\,. \tag{13}$$

From here the new Hamiltonian reads:

$$K(Q,P) = \frac{1}{2mP^2} + \frac{1}{2}m\omega^2 Q^2 P^4\,, \tag{14}$$

from where the equations of motion

$$\dot{P} = -\frac{\partial K}{\partial Q} = -m\omega^2 QP^4,\ \dot{Q} = \frac{\partial K}{\partial P} = -\frac{1}{mP^3} + 2m\omega^2 Q^2 P^3\,. \tag{15}$$

Now one can easily check that new equations of moiton can be traced back to the original one

$$\dot{p} = -\frac{\dot{P}}{P^2} = m\omega^2 QP^2 \equiv -m\omega^2 q,\ \dot{q} = -\dot{Q}P^2 - 2QP\dot{P} = \frac{1}{mP} + 2m\omega^2Q^2P^5 - 2m\omega^2Q^2P^5 = \frac{p}{m}\,. \tag{16}$$

again implying that if the transformed coordinates and momenta follow the Hamiltonian equations of motion with $K(Q,P)$ we get back the original equations of motion.

Problem 3

Consider the following (2nd type) generator function

$$W_2(q,P) = \sum_l f_l(q)P_l \tag{17}$$

that leads to the "point-transformation"

$$Q_l = f_l, \qquad p_l = \frac{\partial f_m}{\partial q_l}P_m \tag{18}$$

(a) Consider the transformation from Descartes- to polar coordinates:

$$r = \sqrt{x^2+y^2} \qquad \phi = \arctan(x/y) \tag{19}$$

Determine the generator function for this transformation.

(b) Express the "old" momenta p_x and p_y in terms of P_r and P_ϕ.

(c) Invert the expressions, and express the "new" momenta P_r and P_ϕ in terms of the "old" momenta.

(d) Test the results for a particle moving in gravitational potential!

Solution:

(a) The new coordinates are r and ϕ. Now according to the rule above we can construct the following generator function

$$W_2(r,\phi,P_r,P_\phi) = rP_r + \phi P_\phi\,. \tag{20}$$

(b) According to this the new momenta read

$$p_x = \frac{\partial r}{\partial x}P_r + \frac{\partial \phi}{\partial x}P_\phi = \frac{x}{r}P_r + \frac{y}{r^2}P_\phi\,, \tag{21}$$

$$p_y = \frac{\partial r}{\partial y}P_r + \frac{\partial \phi}{\partial y}P_\phi = \frac{y}{r}P_r - \frac{x}{r^2}P_\phi\,. \tag{22}$$

(c) From here one can express the new momenta as

$$P_r = \frac{xp_x + yp_y}{r},\ \ P_\phi = yp_x - xp_y\,. \tag{23}$$

(d) Now the 'Cartesian' Hamiltonian for the particle in gravitational potential reads

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + mgy\,. \tag{24}$$

Now the 'Cartesian' equations of motion

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0,\ \dot{p}_y = -\frac{\partial H}{\partial y} = -mg,\ \dot{x} = \frac{p_x}{m} \to \ddot{x} = 0,\ \ddot{y} = -g\,. \tag{25}$$

Now with the new variables the Hamiltonian

$$K = \frac{P_r^2}{2m} + \frac{P_\phi^2}{2mr^2} + mgr\sin\phi\,. \tag{26}$$

Now the equations fo motion

$$\dot{P}_r = -\frac{\partial K}{\partial r} = -mg\sin\phi + \frac{P_\phi^2}{mr^3},\ \dot{P}_\phi = -\frac{\partial K}{\partial \phi} = -mgr\cos\phi,\ \dot{r} = \frac{\partial K}{\partial P_r} = \frac{P_r}{m},\ \dot{\phi} = \frac{\partial K}{\partial P_\phi} = \frac{P_\phi}{m} \tag{27}$$

It can be checked that original equations of motion are recovered. Again the lesson is the following, the above highly complicated systems of differential equations can be traced back to very simple ones by the above canonical transformation!

Problem 4

Consider the following transformation

$$Q = q/p,\ \ P = \beta p^2 \tag{28}$$

(a) Determine β suh that the above transformation describes a type 2 canonical transformation

(b) Next considre the Hamiltonian $H = \frac{P}{m} + m\omega^2 Q^2 P$. Derive the Hamiltonian equations of motion for this system.

Solution:

(a) We express the relevant $p = \sqrt{P/\beta}$ from where one can apply the equations of the canonical transformations

$$p = \frac{\partial W_2}{\partial q} = \sqrt{P/\beta} \to W_2 = q\sqrt{P/\beta}\,. \tag{29}$$

Now the second equation for the new coordinate, Q

$$Q = \frac{\partial W_2}{\partial P} = q/p = \sqrt{\beta}q/\sqrt{P} \to W_2 = 2\sqrt{\beta P}q\,. \tag{30}$$

Now this should be equal to the previous expression obtained for W_2 as $1/\sqrt{\beta} = \sqrt{\beta} \to \beta = 1/2$.

(b) The equations of moiton read

$$\dot{P} = -\frac{\partial H}{\partial Q} = -2m\omega^2 QP,\ \dot{Q} = \frac{\partial H}{\partial P} = \frac{1}{m} + m\omega^2 Q^2\,. \tag{31}$$

(c) Now the new equations of motion with the above canonical transformations read easily for the new Hamiltonian $K = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$ being just the 'usual' Hamiltonian of a harmonic oscillator!

$$\dot{p} = -m\omega^2 q,\ \dot{q} = \frac{p}{m}\,. \tag{32}$$

Problem 5

Consider the following transformations

$$Q = q^\alpha \cos(\beta p),\ P = q^\alpha \sin(\beta p)\,. \tag{33}$$

(a) Calculate the Poisson brackets of Q and P and according to it determine the possible values of α and β such that the above transformation is a canonical one.

(b) Determine the relations for p and Q in terms of P and q!

(c) Construct the W_2 function that generates the above transformation!

Solution:

(a) The Poisson bracket with the derivatives $\frac{\partial Q}{\partial q} = \alpha q^{\alpha-1}\cos(\beta p)$, $\frac{\partial Q}{\partial p} = -\beta q^\alpha \sin(\beta p)$, $\frac{\partial P}{\partial q} = \alpha q^{\alpha-1}\sin(\beta p)$, $\frac{\partial P}{\partial p} = \beta q^\alpha \cos(\beta p)$

$$\{Q, P\} = \alpha q^{\alpha-1}\cos(\beta p)\beta q^\alpha \cos(\beta p) + \beta q^\alpha \sin(\beta p)\alpha q^{\alpha-1}\sin(\beta p) = \beta\alpha q^{2\alpha-1} = 1 \Rightarrow \beta = 2,\ \alpha = 1/2 \tag{34}$$

So $Q = \sqrt{q}\cos(2p),\ P = \sqrt{q}\sin(2p)$.

(b) From the second equation:

$$p = \frac{1}{2}\arcsin(P/\sqrt{q})\,, \tag{35}$$

while from the first we have

$$Q = \sqrt{q}\cos(\arcsin(P/\sqrt{q})) = \sqrt{q}\sqrt{1 - P^2/q}\,. \tag{36}$$

(c) Now construct the W_2 from the second equation by knowing that $\int \mathrm{d}x\,\sqrt{1-x^2} = \frac{\arcsin(x) + x\sqrt{1-x^2}}{2}$:

$$\frac{\partial W_2}{\partial P} = \sqrt{q}\sqrt{1 - P^2/q} \to W_2 = \frac{q}{2}\left(\arcsin(P/\sqrt{q}) + P/\sqrt{q}\sqrt{1 - P^2/q}\right) = \frac{q}{2}\arcsin(P/\sqrt{q}) + \frac{P}{2}\sqrt{q - P^2}\,. \tag{37}$$

One can easily check that

$$\frac{\partial W_2}{\partial q} = \frac{\arcsin(P/\sqrt{q})}{2} = P \to \sqrt{q}\sin(2p)\,, \tag{38}$$

indeed.

Problem 6

Hamiltonian Jacobi equations. Let us consider such a generating function that annulates the Hamiltonian, that is

$$K = H + \frac{\partial F}{\partial t} = 0\,. \tag{39}$$

So the new coordinates and momenta are all constants of motion, $\mathbf{Q}, \mathbf{P} \equiv$ const. that is the resulting equation with such transformations with generator functions of the first or second type $\mathbf{p} = \frac{\partial F}{\partial \mathbf{q}}$ resulting in

$$H(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t) + \frac{\partial S}{\partial t} = 0 \tag{40}$$

where S actually turn out to be the action of the motion but with quantities, generlaized momenta and coordinates being constants of motion. In other words for systems with f dimensional $\mathbf{p}$ and $\mathbf{q}$, this canonical transformation maps to the motion of conserved quantities, which are also genreators of symmetries. In total we face an $f+1$ dimensional motion on the surface of a torus, where each indpendnet direction is generated by one of the indpenendent conserved quantities, e.g. time translation is generated by H, another independent directional motion, rotation is generated by L_z for central potentials. Now consider a free particle with Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}\,. \tag{41}$$

As a cosnequence of the separable structure of the Hamiltonian we can look for the action as $S = S_x(x) + S_y(y) + S_z(z) - Et$ giving in the differential equation

$$\frac{1}{2m}\left(\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S_y}{\partial y}\right)^2 + \left(\frac{\partial S_z}{\partial z}\right)^2\right) + \frac{\partial S}{\partial t} \tag{42}$$

Now let $S_x = \alpha_x x$, $S_y = \alpha_y y$, $S_z = \alpha_z z$ giving

$$\boldsymbol{\alpha}\mathbf{r} - \frac{\boldsymbol{\alpha}^2}{2m}t \tag{43}$$

Now the momenta $p_x = \frac{\partial S}{\partial x} = \alpha_x$, $p_y = \frac{\partial S}{\partial y} = \alpha_y p_x = \frac{\partial S}{\partial z} = \alpha_z$, being all constants, as expected fro mthe form of H! Now as by construction $\mathbf{P}$ is a constant fo motion it can only be equal to $\boldsymbol{\alpha}$ up to constant shifts. Now the not so trivial part, the new coordinates, that are constants of motion:

$$Q_x = \frac{\partial S}{\partial \alpha_x} = \beta_x = x - \frac{\alpha_x}{m}t,\ Q_y = \frac{\partial S}{\partial \alpha_y} = \beta_y = y - \frac{\alpha_y}{m}t,\ Q_z = \frac{\partial S}{\partial \alpha_z} = \beta_z = z - \frac{\alpha_z}{m}t \tag{44}$$

So position is some constant plus the linear time-evolution, that is initial position plus linear time-evolution, $\mathbf{x} = \frac{\boldsymbol{\alpha}}{m}t + \boldsymbol{\beta}$, with $\dot{\boldsymbol{\beta}} = 0$ and $\mathbf{x}(0) = \boldsymbol{\beta}$!