Problem 1

A particle of resting mass m_0, and electric charge q is in a static homogeneous electric field E. The particle starts from rest. Solve the equations of motion for the particle. The particle is initially in the origin, and the electric field points in the x direction.

(a) First solve the equations in the nonrelativistic approximation.

(b) Write down the relativistic equations of motion.

(c) Solve the equation for the momentum of the particle.

(d) From the known momentum-time function $p(t)$, express the particles velocity $v(t)$.

(e) Draw the $v(t)$ function in a graph. Compare it with the nonrelativistic solution!

(f) Express the position $x(t)$ of the particle by integrating $v(t)$. Draw this function.

Solution:

(a) Without loss of generality let the electric field be parallel to the x axis and write up the equation of motion:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\mathbf{E}, \;\; \mathbf{p}_0 = 0 \Rightarrow \mathbf{p}(t) = qEt\,\hat{\mathbf{x}}, \tag{1}$$

from where $\mathbf{v}(t) = \frac{qEt}{m}\,\hat{\mathbf{x}}$ and $x(t) = \frac{qE}{2m}t^2 + x_0$.

(b) Realtivisitically we write similarly that

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\mathbf{E}, \;\; \mathbf{p}_0 = 0 \Rightarrow \mathbf{p}(t) = qEt\,\hat{\mathbf{x}}. \tag{2}$$

But now it is expressed with the velocity in a different way, $p = \frac{m_0 v}{\sqrt{1-v^2/c^2}}$, from where the expression for the velocity gets a bit more complicated:

$$\frac{v}{\sqrt{1-v^2/c^2}} = \frac{qEt}{m} \to \frac{v^2}{1-v^2/c^2} = \frac{q^2E^2t^2}{m^2} \to v^2 + \frac{q^2E^2t^2}{m^2c^2}v^2 = \frac{q^2E^2t^2}{m^2} \Rightarrow v = \frac{\frac{qE}{m}t}{\sqrt{1+\frac{q^2E^2}{m^2c^2}t^2}}. \tag{3}$$

So the lesson is that the Newtonian equation remains true with the time derivative taken in the rest frame, however velocities time dependence will drastically change in the relativistic case.

(c) Velocity starts for momenta, $Eqt \ll mc$ linearly according to the classical result, $v \approx \frac{Eq}{m}t$, but for large t, it saturates to the ultrarelativisitc result, $p \to mc$ and the velocity reaches the speed of light, $v \to c$!

(d) Now the coordinate, as the integral of the velocity

$$x(t) = \int \mathrm{d}t\, \frac{\frac{qE}{m}t}{\sqrt{1+\frac{q^2E^2}{m^2c^2}t^2}} = \frac{mc^2}{qE}\sqrt{1+\frac{q^2E^2}{m^2c^2}t^2}. \tag{4}$$

Ok that was easy... But, now what would have been the situation if we had had initially the particle moving in direction y with initial velocity v_0. For this case the equation of motion and its solution is almost the same albeit for the constant y component, initially equalling $p_0 = \frac{mv_0}{\sqrt{1-v_0^2/c^2}}$ and being preserved due to the equation of motion:

$$\mathbf{p} = \begin{pmatrix} Eqt \\ p_0 \end{pmatrix} \equiv \frac{m}{\sqrt{1-\frac{v_x^2+v_y^2}{c^2}}}\begin{pmatrix} v_x \\ v_y \end{pmatrix}. \tag{5}$$

So the point is that although we have only acceleration in the x direction the quantity, only the y component of the momentum, $mv_0/\sqrt{1-v_0^2/c^2}$ remains invariant, not the y component of the velocity, that is we have the equality for some unknown but time-dependent v_y

$$\frac{mv_y}{\sqrt{1-\frac{v_x^2+v_y^2}{c^2}}} = \frac{mv_0}{\sqrt{1-\frac{v_0^2}{c^2}}}. \tag{6}$$

Expressing v_y from the second one, $v_y^2 = (1 - \frac{v_c^2}{c^2})v_0^2$ and writing into the first one arrive at:

$$\frac{v_x^2}{c^2 - v_x^2 - v_y^2} = \left(\frac{Eqt}{mc}\right)^2 \Rightarrow v_x = \frac{\sqrt{1 - v_0^2/c^2}}{\sqrt{(1 - v_0^2/c^2)\left(\frac{qEt}{mc}\right)^2 + 1}} \frac{qEt}{m}, \tag{7}$$

note that for $v_0 = 0$ we get back the previous expression of the "trivial" case. Writing back the expression of v_x we get for v_y:

$$v_y = \frac{v_0}{\sqrt{(1 - v_0^2/c^2)\left(\frac{Eqt}{mc}\right)^2 + 1}}. \tag{8}$$

While the x component converges for large times to the speed of light again, while the y component goes to zero, as it should! Illustratively this means that the infinitely long acceleration along the x direction costs the reduction of velocity along all other direction, as the magnitude of velocity cannot excess the speed of light.

Problem 2

A particle with resting mass m, and electric charge q is in a static homogeneous magnetic field B. The particle moves in the plane $x - y$ that is perpendicular to the field, that points in the z direction. The (inital) velocity of the particle is v and points initially in the x direction.

(a) First solve the problem in the nonrelativistic approximation.

(b) Write down the relativistic equations of motion.

(c) Exploiting the fact that the Minkowski-lenght of the particle's 4-momentum is constant, show that the length of the (usual) velocity vector remains also invariant.

(d) By using the result c.), express the equations of motion for $d\vec{v}/dt$.

(e) Remark: the equations are no more complicated than the ones in a.). Let's solve them.

(f) The particles motion is a uniform circular motion. Express the radius of the orbital and the time period of the motion. Compare the results with the nonrelativistic ones.

Solution:

(a) In the nonrelativistic fomulation we have

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\mathbf{v} \times \mathbf{B}. \tag{9}$$

Writing out for the two components we have $\dot{p}_x = qv_yB,\ \dot{p}_y = -qv_xB$, which can be written in a matrix form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 & qB/m \\ -qB/m & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix}, \tag{10}$$

from where the solution

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = v\begin{pmatrix} \sin(\Omega t + \varphi) \\ \cos(\Omega t + \varphi) \end{pmatrix}, \quad \Omega = \frac{qB}{m}. \tag{11}$$

Another way would have been to differentiate again to obtain

$$\ddot{p}_x = -\frac{q^2B^2}{m}p_x, \quad \ddot{p}_y = -\frac{q^2B^2}{m^2}p_y. \tag{12}$$

(b) In the relativistic case again we have to work with the velocity dependent relativistic masses, where according to the form of the differential equation, which remains unchanged, velocity remains parallel to the momentum. Furthermore, we also know that the length of the momentum is also time independent, from which one can conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}p^\mu p_\mu = 0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{mc^2}{\sqrt{1 - v^2/c^2}} - \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}^2 = mc^2\frac{\mathrm{d}}{\mathrm{d}t}\frac{1 - v^2/c^2}{\sqrt{1 - v^2/c^2}} = mc^2\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{1 - v^2/c^2} = 0 \Rightarrow v^2 \equiv \text{const.} \tag{13}$$

(c) This has serious implications, as the derivatives of the momentum components then simply read:

$$\frac{\mathrm{d}}{\mathrm{d}t}p_x = \frac{\mathrm{d}}{\mathrm{d}t}\frac{v_x}{\sqrt{1-v^2/c^2}} = \frac{\mathrm{d}v_x}{\mathrm{d}t}\frac{m}{\sqrt{1-v^2/c^2}}, \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}p_y = \frac{\mathrm{d}}{\mathrm{d}t}\frac{v_y}{\sqrt{1-v^2/c^2}} = \frac{\mathrm{d}v_y}{\mathrm{d}t}\frac{m}{\sqrt{1-v^2/c^2}}, \tag{15}$$

$$\tag{16}$$

which can again be cast into a matrix form

$$\frac{m}{\sqrt{1-v^2/c^2}}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 & qB \\ -qB & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 & \frac{qB\sqrt{1-v^2/c^2}}{m} \\ -\frac{qB\sqrt{1-v^2/c^2}}{m} & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} \tag{17}$$

being nothing else but the above nonrelativistic result but now with relativistic masses, giving the same result

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = v\begin{pmatrix} \sin(\tilde{\Omega}t+\varphi) \\ \cos(\tilde{\Omega}t+\varphi) \end{pmatrix} \tag{18}$$

with $\tilde{\Omega} = \frac{qB\sqrt{1-v^2/c^2}}{m}$.

(d) The position is also easily expressed as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{v}{\tilde{\Omega}}\begin{pmatrix} -\cos(\tilde{\Omega}t+\varphi) \\ \sin(\tilde{\Omega}t+\varphi) \end{pmatrix} + \text{const.} \tag{19}$$

with a radius of circular motion, $R = \frac{v}{\tilde{\Omega}} = \frac{vm}{qB\sqrt{1-v^2/c^2}}$ which got longer by the contraction factor.

Problem 3

Show that the four Maxwell equations can be expressed with the help of the electromagnetic tensor as

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \tag{20}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{21}$$

where $F^{\mu\nu} = \partial^\mu\mathcal{A}^\nu - \partial^\nu\mathcal{A}^\mu$ and $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}F_{\sigma\rho}$ denoting the dual of the electromagnetic tensor, where the lower case tensor is simply defined as $F_{\sigma\rho} = g_{\sigma\mu}g_{\rho\nu}F^{\mu\nu}$ and the generalized Levi-Civita symbol $\varepsilon^{0123} = 1$ with four indices and being antisymmetric for all of its indices.

(a) Express the four vector potential and identify its components!

(b) Express the four current in terms of the charge density and charge current density!

(c) Express the electromagnetic tensor and its dual tensor in matrix form!

(d) Show that the above equations really match the Maxwell equations!

As the four current contains both the charge density and the charge current density and transforms according to the Lorentz transformation, charge density and current density are not invariant when measured from frames moving with differetn veloctities.

(e) To illustrate this consider a one-diemnsional motion along axis x with a uniformly charged sphere with total charge Q. Apply a boost and describe its current vector from a moving frame of velocity $v = 0.8c$.

(f) As an additional exercise let us discuss the relativistic modification of the Doppler effect!

Solution:

(a) Scalar and vector potential:

$$-\nabla\Phi = \mathbf{E}, \ \ \nabla\times\mathbf{A} = \mathbf{B}. \tag{22}$$

Contravariant and covariant four vector potential:

$$\mathcal{A}^\mu = (\Phi/c, \mathbf{A})\,, \ \mathcal{A}_\mu = (\Phi/c, -\mathbf{A})\,. \tag{23}$$

Covariant and contravariant differentiation:

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right), \ \partial_\mu \equiv \frac{\partial}{\partial x_\mu} \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right). \tag{24}$$

Covariant divergence of a four vector:

$$\partial^\mu A_\mu = \frac{1}{c}\partial_t A_0 + \partial_i A_i = \frac{1}{c}\partial_t A_0 + \nabla\mathbf{A}. \tag{25}$$

(b) The four current is defined in a similar way as

$$j^\mu = (\rho c, \mathbf{j})\,, \ j_\mu = (\rho c, -\mathbf{j})\,, \tag{26}$$

from where the continuity equation reads:

$$\partial^\mu j_\mu = \partial_t\rho + \nabla\mathbf{j} = 0. \tag{27}$$

(c) Spatial parts of the electromagnetic tensor for $i \neq j$ and the zeroth row and column

$$F^{ij} = \partial^j\mathcal{A}^i - \partial^i\mathcal{A}^j = (\delta_{jl}\delta_{im} - \delta_{il}\delta_{jm})\,\partial_l A_m = -\varepsilon_{ijk}\varepsilon_{klm}\partial_l A_m = -\varepsilon_{ijk}B_k \tag{28}$$

$$F^{i0} = \partial^0\mathcal{A}^i - \partial^i\mathcal{A}^0 = \partial_i\Phi/c = -E_i/c = -F^{0i} \tag{29}$$

giving together with the lower case tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}, \ F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{30}$$

Without proof we give the formula for the dual tensor

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix} \tag{31}$$

with similar realtions

$$\tilde{F}^{ij} = \varepsilon_{ijk}E_k, \ \ \tilde{F}^{0i} = -B_i, \ \ \tilde{F}^{i0} = B_i \tag{32}$$

(d) Now the Maxwell equations:

$$\mu_0 j^0 = \frac{\rho}{\varepsilon_0 c^2} = \partial_i F^{i0} = \frac{1}{c^2}\partial_i E^i \Rightarrow \nabla\mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{33}$$

$$\mu_0 j^i = \frac{1}{c}\partial_t F^{0i} + \partial_k F^{ki} = -\frac{\partial_t E^i}{c^2} + \varepsilon_{ikl}\partial_k B_l = -\mu_0\varepsilon_0\partial E^i + (\nabla\times\mathbf{B})_i \Rightarrow \nabla\times\mathbf{B} = \mu_0\mathbf{j} + \mu_0\varepsilon_0\partial_t\mathbf{E} \tag{34}$$

$$\partial_\mu\tilde{F}^{\mu 0} = \partial_i B_i/c = 0 \Rightarrow \nabla\mathbf{B} = 0 \tag{35}$$

$$\partial_\mu\tilde{F}^{\mu j} = \frac{1}{c}\partial_t B_j + \partial_i\varepsilon_{ijk}E_k/c \Rightarrow \nabla\times\mathbf{E} = -\partial_t\mathbf{B} \tag{36}$$

(e) Consider in $1+1$ dimensions the four current of a rod with uniform ρ charge density in rest frame:

$$j^\mu = (\rho c, 0) \tag{37}$$

We apply the Lorentz transformation coresponding to a motion along the spatial dimension with velocity v, that is we act with the matrix $\Lambda = \frac{1}{\sqrt{1-v^2/c^2}}\begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix}$ on j^μ:

$$(j')^\mu = \frac{1}{\sqrt{1-v^2/c^2}}(\rho c, -v\rho) \tag{38}$$

from where we get that $\rho' = \frac{\rho}{\sqrt{1-v^2/c^2}}$ corresponding to the fact that amount of charge is unchanged while due to Lorentz contraction the size of the system got shorter resulting in a higher density, also mirorred in the second/spatial component as it is nothing else but the contracted charged rod moving with velocity v in the $-x$ direction! Note that the Minkowski length of the four current remains unchanged as

$$j_\mu j^\mu = \rho^2 c^2 = (j')_\mu (j')^\mu = \frac{1}{1-v^2/c^2}(\rho^2 - v^2\rho^2) = c^2\rho^2 \tag{39}$$

In case of $v = 0.8c$ we have $(j')^\mu = \frac{10\rho}{6}(c, -v)$.

(f) Doppler effect classically $f' = \frac{c}{c-v}f$, so the frequency of the sound gets higher if coming towards us and lower when going away. Now in case of electromagnetic fields we write

$$\tag{40}$$

$$\mathbf{E} = E_0\hat{\mathbf{y}}e^{-i\omega(t-x/c)},\ \mathbf{B} = \frac{1}{c}\hat{\mathbf{x}}\times\mathbf{E} = E_0\hat{\mathbf{y}}e^{-i\omega(t-x/c)} = \frac{E_0}{c}\hat{\mathbf{z}}e^{-i\omega(t-x/c)} \tag{41}$$

Now Lorentz transformation of a boost along the x axis gives of the electric part gives

$$\mathbf{E}'(x') = \frac{1}{\sqrt{1-v^2/c^2}}\left(\mathbf{E}(x) - v\mathbf{B}(x)\right) \tag{42}$$

$$\mathbf{B}'(x') = \frac{1}{\sqrt{1-v^2/c^2}}\left(\mathbf{B}(x) - \frac{v}{c^2}\mathbf{E}(x)\right) \tag{43}$$

with $x' = \Lambda x$ and note that transforming a four field, also depending on the coordinates involves also the transformation of its arguments, $\Lambda\mathbf{E}(x) = \mathbf{E}'(x')$. Writing back and express the new fields with the old ones we get

$$E'_y(x) = \sqrt{\frac{1-v/c}{1+v/c}}E_0 e^{-i\omega(\bar{t}-\bar{x}/c)} \tag{44}$$

$$B'_z(x) = \sqrt{\frac{1-v/c}{1+v/c}}\frac{E_0}{c}e^{-i\omega(\bar{t}-\bar{x}/c)} \tag{45}$$

where we transformed back the arguments to reach at

$$\bar{t} = \Lambda^{-1}t = \frac{1}{\sqrt{1-v^2/c^2}}\left(t + \frac{xv}{c^2}\right),\ \bar{x} = \frac{1}{\sqrt{1-v^2/c^2}}(x+vt) \Rightarrow \omega\left(\bar{t} - \bar{x}/c\right) = \sqrt{\frac{1-v/c}{1+v/c}}\omega\,(t - x/c) \tag{46}$$

validating our expectation that also in the moving frame we experience a plane wave but with modified amplitudes and frequency as

$$E'_0 = \sqrt{\frac{1-v/c}{1+v/c}}E_0,\ \ \omega' = \sqrt{\frac{1-v/c}{1+v/c}}\omega \tag{47}$$

Interestingly for $v = c$ we would get that $E'_0 = 0$, $\omega' = 0$!

Problem 4

Solve the problems 2.) and 3.) using the covariant form of the equations of motion.

(a) Express the equations in the following form:

$$\frac{dp^\mu}{d\tau} = F^{\mu}_{.\nu} u^\nu \tag{48}$$

where $F^{\mu}_{.\nu}$ is the electromagnetic 4-tensor that contains the electric and magnetic fields together. What are the components of that tensor? Write down the equations of motion for the case of homogeneus electric and magnetic fields.

(b) We should get a simple set of linear differential equations. Solve them! What are the initial conditions?

(c) Express the solution $u^\mu(\tau)$ that is compatible with the initial conditions.

(d) Integrate $u^\mu(\tau)$ to determine $x^\mu(\tau)$.

(e) The x^0 coordinate of the particle is simply the time of the coordinate system (multiplied by c). From $x^0(\tau)$ derive the $\tau(t)$ function.

(f) Using the result of e.), express the usual $x(t)$ position-time functions of the particle.

Solution:

(a) Using the covariant form of equations of motions:

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = F^{\mu}_{\nu} u^\nu. \tag{49}$$

The equations of motion according to our previous studies read as

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \tag{50}$$

which we will denote by p^i, refering to the spatial parts of the four momentum. Writing out for the components:

$$\frac{\mathrm{d}p^i}{\mathrm{d}t\sqrt{1-v^2/c^2}} = \frac{\mathrm{d}p^i}{\mathrm{d}\tau} = \frac{qE_i}{c}\frac{c}{\sqrt{1-v^2/c^2}} + q\varepsilon_{ijk}\frac{v_j B_k}{\sqrt{1-v^2/c^2}} = \frac{qE_i}{c}u^0 + q_i\varepsilon_{ijk}B_k u^j \tag{51}$$

with the usual four velocity, $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \begin{pmatrix} \frac{c}{\sqrt{1-v^2/c^2}} \\ \mathbf{v} \end{pmatrix}$. So indeed the covariant/Lorentz invariant form with the four velocity and the proper time gives back the well-konwon Lorentz and electric force. These 3 equations give back the ones with $\mu = 1, 2, 3$ of the covariant form. Turning to the 0th term we get (exploiting thefact that $P_0 = \frac{mc}{\sqrt{1-v^2/c^2}}$):

$$\frac{\mathrm{d}}{\mathrm{d}\tau}p^\mu p_\mu = 0 \Rightarrow p^0\frac{\mathrm{d}p^0}{\mathrm{d}\tau} = p^i\frac{\mathrm{d}p^i}{\mathrm{d}\tau} = p^i\frac{qE^i}{\sqrt{1-v^2/c^2}} \Rightarrow \frac{\mathrm{d}p^0}{\mathrm{d}\tau} = \frac{E_i}{c}u^i \tag{52}$$

giving back the first row of the covariant form with the electromagnetic tensor defiend as

$$F^{\mu}_{\nu} = \partial^\mu A_\nu - \partial^\nu A_\mu = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E^x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} \tag{53}$$

(b) Now simple example with homogenous electric field parallel to the x axis:

$$\frac{\mathrm{d}p^0}{\mathrm{d}\tau} = qE/cu^1 \Rightarrow \frac{\mathrm{d}u^0}{\mathrm{d}\tau} = qE/cu^1, \tag{54}$$

$$\frac{\mathrm{d}p^1}{\mathrm{d}\tau} = qE/cu^0 \Rightarrow \frac{\mathrm{d}u^1}{\mathrm{d}\tau} = qE/cu^0 \tag{55}$$

yielding for both of them

$$\frac{\mathrm{d}^2 u^{0,1}}{\mathrm{d}\tau^2} = q^2 \frac{E^2}{m^2c^2} u^{0,1} \tag{56}$$

with initail conditions $u^1(0) = 0$ and $u^0(0) = c$, the solution of which is

$$u^\mu(\tau) = \begin{pmatrix} c\cosh\left(\frac{E}{mc}\tau\right) \\ c\sinh\left(\frac{E}{mc}\tau\right) \\ 0 \\ 0 \end{pmatrix}. \tag{57}$$

(c) Now the coordiante as a function of the proper time

$$x^\mu(\tau) = \int \mathrm{d}\tau\, u^\mu(\tau) = \frac{mc^2}{qE} \begin{pmatrix} \sinh\left(\frac{qE}{mc}\tau\right) \\ \cosh\left(\frac{qE}{mc}\tau\right) \end{pmatrix}. \tag{58}$$

(d) Now the reference frame time as a function of the proper time:

$$x^0(\tau) = ct(\tau) \Rightarrow t(\tau) = \frac{mc}{qE} \sinh\left(\frac{qE}{mc}\tau\right) \tag{59}$$

(e) From here one can express the position in terms of the reference frame time:

$$x^1(t) = \frac{mc}{qE} \cosh\left(\operatorname{asinh}\left(\frac{Eq}{mc}\right)\right) = \frac{mc^2}{qE}\sqrt{1 + \left(\frac{Eq}{mc}t\right)^2} \tag{60}$$

being exactly the same at which wearrived using the Newtonian, not general

Problem 5

A particle of resting mass m, and electric charge q is in a static homogeneous electric and magnetic fields E and B that are perpendicular to each other. The initial velocity of the particle is zero. Determine the motion of the particle. The magnetic inducion points in the z direction while the electric field points in the y direction.

(a) Write down the relativistic equations of motion for the particle in the covariant form (like in Problem 4).

(b) We could solve simply the equations of a.) (as a practice you can do it.), but now it's worth to follow a different way. Our argument is the following: in a crossed electric and magnetic field one can figure out a uniform linear motion, where the magnetic Lorentz-force and the electric force cancel each other. If we boost to a frame that moves with the velocity of that motion, the electric field strength must be zero, because our particle is in rest in that frame. Here we can solve the much easier problem, where only a magnetic field is present, and finally we transform back to the original frame of reference, and get the solution of our problem.

(c) What is the velocity of the uniform linear motion? When is it physically meaningful?

(d) Transform the field-strength tensor to that frame!

(e) Solve the problem in the moving frame!

(f) Transform back to the original frame, and express $x(t)$. Sketch the trajectory of the particle.

(g) What happens if the velocity in b.) is not physically meaningful? How looks like the trajectory of the particle in that case?

Solution:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{61}$$

The invariant quantities are (if we think of them as vector but with more than four components, $\sim$ four tensors)

$$FF = F_{\mu\nu}F^{\mu\nu} = \vec{B}^2 - \vec{E}^2 \qquad F\tilde{F} = F^{\mu\nu}F^{\delta\kappa}\epsilon_{\mu\nu\delta\kappa} = 2\vec{E}\cdot\vec{B} \tag{62}$$

In our case we have

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & -E_y/c & 0 \\ 0 & 0 & -B_z & 0 \\ E_y/c & B_z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{63}$$

The condition for the vanishing force is

$$E_y - B_z v_x = 0. \tag{64}$$

The minus sign comes from the fact that we used covariant vector here!

This gives

$$v_x = \frac{E_y}{B_z} \tag{65}$$

Clearly it is meaningful only when $FF > 0$.

The Lorentz transformation we want is

$$\Lambda = \begin{pmatrix} \cosh(\theta) & -\sinh(\theta) & 0 & 0 \\ -\sinh(\theta) & \cosh(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{66}$$

with $\tanh(\theta) = \frac{v_x}{c}$.

As matrices

$$F' = \Lambda F \left(\overline{\Lambda}^T\right) = \Lambda F \left(\Lambda^{-1}\right), \;\; (F')^{\mu}_{\cdot\nu} = \Lambda^{\mu}_{\epsilon} F^{\epsilon}_{\cdot\rho} \Lambda^{\cdot\rho}_{\nu} \tag{67}$$

As a consequence of which we get that the vector

$$\begin{pmatrix} E_y \\ B_z \end{pmatrix} \tag{68}$$

transforms as a two-vector under the 2x2 transformation

$$\Lambda = \begin{pmatrix} \cosh(\theta) & -\sinh(\theta) \\ -\sinh(\theta) & \cosh(\theta) \end{pmatrix}. \tag{69}$$

In the new coordinates the initial condition is

$$x'^{\mu}(0) = 0, \qquad v'_x(0) = -v, \quad v_y(0) = 0, \quad v_z(0) = 0, \tag{70}$$

that is the particle is moving with the velocity annulating the total force exerted on it in the rest frame, it corresponds in the transformed dynamics to zero electric field and a modified magnetic field. But what is the new magnetic field? It is given by

$$B_z'^2 = \vec{B}^2 - \vec{E}^2 = B_z^2 - E_y^2. \tag{71}$$

Or in a more straightforward way we just perform the matrix multiplication from both sides of the tensor keeping in mind that only four of its elements are non-zero! This results in the F' having new $E'_y = \cosh\theta E_y - \sinh\theta\, cB_z$ and $B'_z = \sinh\theta\, E_y/c - \cosh\theta\, B_z$, by which we have zero electric field if

$\tanh\theta = \frac{E_y}{cB_z} \Rightarrow v = \frac{E_y}{B_z}$.
We copy from earlier:

$$\omega = \frac{qB_z'\sqrt{1-v^2/c^2}}{m}, \qquad r = \frac{v}{\omega} = \frac{mv}{qB_z'\sqrt{1-v^2/c^2}}. \tag{72}$$

Here we can actually use

$$1 - v^2/c^2 = 1 - \frac{E_y^2}{B_z^2} = \frac{B_z^2 - E_y^2}{B_z^2} = \frac{(B_z')^2}{B_z^2}, \tag{73}$$

as the $B_z^2 - E_y^2$ quantity is Lorentz invariant! So it is better to put

$$\omega = \frac{q(B_z')^2}{B_z m}, \qquad r = \frac{v}{\omega} = \frac{mvB_z}{q(B_z')^2} = \frac{mE_y}{q(B_z')^2}. \tag{74}$$

Here the solution with the proper initial conditions will be

$$x'(t) = -r\sin(\omega t'), \qquad y(t') = r(1-\cos(\omega t')). \tag{75}$$

So the 4-vector is

$$x'^{\mu}(t') = \begin{pmatrix} t' \\ -r\sin(\omega t') \\ r(1-\cos(\omega t')) \\ 0 \end{pmatrix}. \tag{76}$$

Now we transform back with

$$\Lambda = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) & 0 & 0 \\ \sinh(\theta) & \cosh(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{77}$$

We can use

$$\cosh(\theta) = \frac{B_z}{B_z'}, \qquad \sinh(\theta) = \frac{E_y}{B_z'}. \tag{78}$$

This gives

$$\begin{pmatrix} \frac{1}{B_z'}(B_z t' - E_y r\sin(\omega t')) \\ \frac{1}{B_z'}(-B_z r\sin(\omega t') + E_y t') \\ r(1-\cos(\omega t')) \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{B_z'}(B_z t' - \frac{mE_y^2}{q(B_z')^2}\sin(\omega t')) \\ \frac{E_y}{B_z'}(-\frac{mB_z}{q(B_z')^2}\sin(\omega t') + t') \\ r(1-\cos(\omega t')) \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{B_z'}(B_z t' - \frac{mE_y^2}{q(B_z')^2}\sin(\omega t')) \\ \frac{E_y}{B_z'}(-\frac{\sin(\omega t')}{\omega} + t') \\ r(1-\cos(\omega t')) \\ 0 \end{pmatrix}. \tag{79}$$

Now what we still need to figure out the inverse transformation of time, which we express with the helpf of the boosted time t' and $x' = \frac{mE_y}{q(B_z')^2}\sin(\omega t')$

$$t = \frac{1}{B_z'}\left(B_z t' - \frac{mE_y^2}{q(B_z')^2}\sin(\omega t')\right), \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{mE_y}{q(B_z')^2}\begin{pmatrix} \frac{B_z}{B_z'}(-\sin(\omega t') + \omega t') \\ (1-\cos(\omega t')) \end{pmatrix}. \tag{80}$$

It is useful to check the small time limit, when everything is non-relativistic. Expanding the function $t(t')$ to first order in t' we obtain

$$t = t'\frac{B_z'}{B_z}. \tag{81}$$

Also, $x(t)$ starts with t^3, so we will not investigate this. Instead we will look at $y(t)$, which is of order t^2 for small t. Expanding the solution we get

$$y(t) \approx \frac{mE_y}{q(B_z')^2}\omega^2\frac{t'^2}{2} = \frac{mE_y}{q(B_z')^2}\left(\frac{q(B_z')^2}{B_z m}\right)^2\left(\frac{B_z^2}{(B_z')^2}\frac{t^2}{2}\right) = \frac{qE_y}{2m}t^2. \tag{82}$$

This is the expected result!

The case of $|B_z| < |E_y|$

In the case when B'_z becomes imaginary, so would t', and we have different functions. We put

$$t' = i\tilde{t}, \qquad B'_z = iE'_y = i\sqrt{E_y^2 - B_y^2}. \tag{83}$$

and

$$t = \frac{1}{E'_y}\left(B_z\tilde{t} + \frac{mE_y^2}{q(E'_y)^2}\sinh(\omega\tilde{t})\right), \qquad \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{mE_y}{q(E'_y)^2}\begin{pmatrix} \frac{B_z}{E'_y}(-\sinh(\omega\tilde{t}) + \omega\tilde{t}) \\ (1 - \cosh(\omega\tilde{t})) \end{pmatrix}. \tag{84}$$

In the large $\tilde{t}$ limit

$$t \approx \frac{mE_y^2}{q(E'_y)^3}\frac{e^{\omega\tilde{t}}}{2}, \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{mE_y}{q(E'_y)^2}\frac{e^{\omega\tilde{t}}}{2}\begin{pmatrix} \frac{B_z}{E'_y} \\ 1 \end{pmatrix}. \tag{85}$$

We can read off that the speed becomes 1, as it should be, and the direction depends on the magnetic and electric fields.

Problem 6 , Extra exercise

A spaceraft, accelerated by a rocket, departs from an intergalactic space station. (The spacecraft is far from any source of gravitational force.) In the beginning, the total resting mass of the spacecraft and its fuel is M_0. The burned fuel goes out from the rocket with velocity u (relative to the rocket), that can be relativistically large.

(a) First focus on the moment, when the total resting mass of the spacecraft is M. Investigate the rocket from the inertial system, where its velocity is zero in that moment. In a very short time a small amount of the burned fuel goes out of the rocket. The resting mass of the exhausted fuel is dm. Write down the conservation of 4-momentum. Express the change dM of the spacecraft's resting mass and the velocity dv of the spacecraft after the process.

(b) On the last class we saw that in 1-dimensional motions (like the spacecrafts motion in our case) it's worth to use rapidities instead of velocities. Transform the dv velocity of the rocket into rapidity $d\theta$.

(c) Using the previous results determine the resting mass of the rocket, its rapidity, when the total resting mass of the exhausted fuel is m.

(d) What is then the velocity of the spacecraft?

(e) We see, that the decrease of the resting mass of the rocket is not m. Why?

Solution:

(a) In the moving frame before and after the instant when the mass is M, the corresponding four momenta are

$$p^\mu = \begin{pmatrix} Mc \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad P^\mu = \begin{pmatrix} \frac{(M+\mathrm{d}M)c}{\sqrt{1-\mathrm{d}v^2/c^2}} + \frac{\mathrm{d}mc}{\sqrt{1-u^2/c^2}} \\ \frac{(M+\mathrm{d}M)\mathrm{d}v}{\sqrt{1-\mathrm{d}v^2/c^2}} - \frac{\mathrm{d}mu}{\sqrt{1-u^2/c^2}} \\ 0 \\ 0 \end{pmatrix} \tag{86}$$

where $\mathrm{d}M$ is the change in the mass of the spaccraft, while $\mathrm{d}m$ is the mass of the emitted fuel. Solving the two equations obtained by four momentum conservation:

$$Mc = \frac{(M+\mathrm{d}M)c}{\sqrt{1-\mathrm{d}v^2/c^2}} + \frac{\mathrm{d}mc}{\sqrt{1-u^2/c^2}} \Rightarrow \mathrm{d}M = -\frac{\mathrm{d}m}{\sqrt{1-u^2/c^2}} \tag{87}$$

$$\frac{(M+\mathrm{d}M)\mathrm{d}v}{\sqrt{1-\mathrm{d}v^2/c^2}} = \frac{\mathrm{d}mu}{\sqrt{1-u^2/c^2}} \Rightarrow M\mathrm{d}v = \frac{\mathrm{d}mu}{\sqrt{1-u^2/c^2}} = -\mathrm{d}Mu \Rightarrow \mathrm{d}v = -\frac{\mathrm{d}M}{M}u \tag{88}$$

(b) Expressed with rapidities:

$$\mathrm{d}v = c\,\mathrm{d}\left(\tanh\theta\right) \approx c\,\mathrm{d}\theta \Rightarrow \mathrm{d}\theta = -\frac{\mathrm{d}M}{M}\frac{u}{c} \tag{89}$$

Integrating out both sides from M_0 to the current M we get

$$\theta = \int_{M_0}^{M} -\frac{\mathrm{d}M'}{M'}\frac{u}{c} = -\frac{u}{c}\ln\frac{M}{M_0} \tag{90}$$

which can be cast into the form of velocities

$$v = c\tanh\theta = c\tanh\left(\frac{u}{c}\ln\left(\frac{M_0}{M}\right)\right) \tag{91}$$

Note that now this velocity is measured in the original frame.

(c) The decrease of the rocket's resting mass is computed in its instantanous rest frame, in which its rest energy is decreased by $\mathrm{d}Mc^2$ which can only be equal to the total energy of the $\mathrm{d}m$ fuel, being proportional to the relativistic mass of it, with $\mathrm{d}m$.