# A11

In this problem you will show that the basic equation of electrostatics (the Poisson-equation) can be deduced using a variational principle. Consider the stationary (time-independent) charge density $\rho(r)$, whose electrostatic energy is described by the following functional

$$U = \int d^3r \left(-\frac{\varepsilon_0}{2}(\nabla\phi)^2 + \rho\phi\right) \tag{1}$$

where $\phi(r)$ denotes the electrostatic potential from which the electric field can be determined by $E = -\nabla\phi$.

The variational principle states that the electricstatic field minimizes the electrostatic energy. The boundary conditions usually are described by $\phi(|r| \to \infty) = 0$, i.e. the potential is zero in the infinity.

Assuming that we change the potential by the infinitesimal variation $\delta\phi(r)$ write down the variation $\delta U$ of the energy functional.

After integrating by parts transform the variation $\delta U$ in a form

$$\delta U = \int d^3r M(r) d\phi(r) \tag{2}$$

i.e. the derivatives of the variation $d\phi(r)$ are not present anymore. Determine $M(r)$ as a function of $\rho(r)$ and $\phi(r)$. Throw out the boundary terms of the partial integration. (It can be shown that if $\rho(r)$ is confined in a finite volume then the boundary terms are exactly zero.)

The energy is minimal, if $\delta U$ is zero for any variation. It is equivalent to the equation $M(r) = 0$. Show that this equation is exactly the Poisson-equation of electrostatics.

# A12

Consider the bending waves in an elastic rod. As it has been discussed in class, the bending energy is proportional to the square of the curvature of the rod, therefore the Lagrangian of the system is

$$L = \frac{\lambda}{2}(\partial_t u(x,t))^2 - \frac{E\Theta}{2}(\partial_x^2 u(x,t))^2, \tag{3}$$

where $\lambda$ is the linear mass density (mass per unit length), $E$ is the Young's modulus of the rod, and $\Theta$ is the cross-section parameter. The field $u(x,t)$ describes the small transversal displacement of the rod.

(a) Determine the Euler-Lagrange equation of motion for the system. Be careful with the second-derivative term.

(b) Starting from the Lagrangian, determine the energy density in the model.

(c) Use the symmetric result obtained for the field at class with general $\omega$ and $k$ satisfying that $\rho A\omega^2 = \Theta E k^4$, $u(x,t) = u_0 \sin(\omega t)(\cosh(kx) + \cos(kx))$ and determine the total energy of the system, i.e.: calculate the integral $E(t) = \int_0^L \mathrm{d}x\, \mathcal{H}$.

# B8

A soap-membrane is streched on a square shaped frame of size $L$. In this problem we consider the standing wave modes of the membrane. The surface mass density of the membrane is $\lambda$, the surface tension is $\sigma$. The Lagrangian of the membrane reads as

$$\mathcal{L} = \frac{1}{2}\lambda(\partial_t u)^2 - 2\sigma\sqrt{1 + (\partial_x u)^2 + (\partial_y u)^2}, \tag{4}$$

where $u(x,y,t)$ is the vertical displacement of the membrane. The first term is the surface density of kinetic energy, while the second term stands for the surface energy density of the membrane.

(a) Approximate the square-root term in the Lagrangian up to orders of $(\partial_{x,y}u)^2$

(b) Derive the equations of motion (from the approximated Lagrangian).

Because of the boundary conditions, at the frame the displacement is zero, $u(0,y,t) = u(x,0,t) = u(L,y,t) = u(x,L,t) = 0$.

(c) Look for the standing-wave solutions in the form

$$u = A\sin(k_x x)\sin(k_y y)\sin(\omega t) \tag{5}$$

What are the possible values of $k_x$ and $k_y$ that are allowed by the boundary conditions?

(d) Determine the frequency $\omega$ as a function of the wave numbers $k_x$ and $k_y$.

# B9

The Lagrangian of a one dimensional system reads

$$L = \frac{1}{2}\partial_t\Psi\partial_x\Psi + \frac{\alpha}{6}(\partial_x\Psi)^3 - \frac{\nu}{2}(\partial_x^2\Psi)^2 \tag{6}$$

where the field $\Psi(x,t)$ describes some continous medium.

(a) Write down the action of the system.

(b) Determine down the variation $\delta S$.

(c) After integrating by parts transform the variation in the form

$$\delta S = \int dt \int dx M(x,t)\delta\Psi(x,t) \tag{7}$$

Determine M(x,t) as a function of $\Psi$ and its derivatives. Throw out the boundary terms.

(d) Determine the expression for the energy density

$$\varepsilon = \partial_t\Psi\frac{\partial L}{\partial(\partial_t\Psi)} - L \tag{8}$$