A9

Consider a very long elastic rod with mass density ρ, Young's modulus E, and cross-section A. The rod lies on the x axis. One end of the rod is in the origin ($x = 0$), the other end is far away, practically in the infinity ($x \to \infty$). The longitudinal waves in the rod are described by the Lagrangian density

$$L = \frac{\rho A}{2}(\partial_t \xi(x,t))^2 - \frac{EA}{2}(\partial_x \xi(x,t))^2, \tag{1}$$

where the field $\xi(x,t)$ describes the longitudinal displacement of the points of the rod.

(a) Write down the action, S and determine the equation of motion based on the principle of least action, $\delta S = 0$.

(b) Show that the Euler-Lagrange equation of motion for the system gives the same result in (a).

(c) We excite the rod by moving its end ($x = 0$) as $\xi(0,t) = a\sin(\omega t)$. Show directly that following plane-wave solution solves the equations of motion: $\xi(x,t) = a\sin(\omega t - kx)$. Determine the value of ω as a function of k.

(d) Starting from the Lagrangian, use the formula $J_E(x,t) = \partial_t \xi \frac{\partial \mathcal{L}}{\partial \partial_x \xi}$ to calculate the energy density current inside the rod.

(e) Determine the average power of the generator that excites the rod. (Hint: calculate the total transmitted energy in one period $T = 2\pi/\omega$)

A10

Consider a very long elastic rod with mass density ρ, Young's modulus E, and cross-section A. The rod lies on the x axis. One end of the rod is in the origin ($x = 0$), the other end is far away, practically in the infinity ($x \to \infty$). The longitudinal waves in the rod are described by the Lagrangian density

$$L = \frac{\rho A}{2}(\partial_t \xi(x,t))^2 - \frac{EA}{2}(\partial_x \xi(x,t))^2, \tag{2}$$

(a) Write down the Euler-Lagrange equation of motion for the system.

(b) The rod is in rest for $t < 0$, but at $t > 0$ we start to pull the $x = 0$ end of the rod towards the $-x$ direction with constant velocity v. As a result a shockwave starts to propagate within the rod towards the $+x$ direction. Find this shockwave solution. (*Hint:* Look for the solution in form of $\xi(x,t) = \xi_0(x) + p(x,t)$ as in **Problem 2** at class and write result for the $p(x,t)$ as at class satisfying that $\xi(0,t)$ describbes motion with constant velocity towards the $-x$ direction.)

(c) Fromt the condition that $\xi(0,t)$ describes constant velocity motion along $-x$ determine the density of the pulling force that we need to apply to move the end by velocity v.

(d) Starting from the Lagrangian, use the formula $J_E(x,t) = \partial_t \xi \frac{\partial \mathcal{L}}{\partial \partial_x \xi}$ to calculate the energy density current inside the rod.

B7

Consider the following Lagrangian:

$$\mathcal{L} = -\frac{1}{2m}\partial_x \Psi^* \partial_x \Psi - V(x)\Psi^*\Psi + \frac{1}{2}i(\Psi^*\partial_t \Psi - \Psi \partial_t \Psi^*), \tag{3}$$

where $\Psi(x,t)$ is a complex valued field, and $\Psi^*(x,t)$ denotes its complex conjugate. There are many ways to handle complex fields. Now we follow the most pedestrian way: we describe the field as a combination of two independent real fields.

(a) Consider the complex field as a real field with two-components (the real and the imaginary part.) Here $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are standard real fields. Rewrite the Lagrangian in the terms of these two real fields.

(b) Show that the Lagrangian is real (no complex factors are present).

(c) Write down the action using the real form of the Lagrangian.

(d) Derive the equations of motion for the two fields $\Psi_{1,2}$.

(e) Show that the two equations are the real and imaginary parts of the usual Schrödinger equation, (we use $\hbar = 1$ units.).