Problem 1

Consider the transversal waves of an elastic rod. The cross-section parameter of the rod is Θ, its linear mass density is $A\rho$, and its Young's modulus is E. The Lagrangian of the system reads as

$$\mathcal{L} = \frac{1}{2}\rho A(\partial_t u)^2 - \frac{\Theta E}{2}(\partial_z^2 u)^2 \tag{1}$$

The two ends are fixed horizontally in two walls, therefore at the ends both the displacement and its z-derivative is zero.

(a) Write down the action for the system.

(b) Using the principle of least action derive the equations of motion for the system.

(c) Search the solution in the separated form: $u(z,t) = U(z)\varphi(t)$. Write down the appropriate equations for $U(z)$ and $\varphi(t)$.

(d) Write down the equation that determines the free oscillation frequencies of the system. Qualitatively solve the equation, graphically.

Solution:

(a) The action as usual is the time and space integral of the Lagrange density from some initial time t_i until some final, t_f, but which we drop in the remaining subexercises for sake of simplicity

$$S = \int_{t_i}^{t_f} \mathrm{d}t \int_{-L/2}^{L/2} \mathrm{d}z\, \frac{1}{2}a\left(\partial_t u\right)^2 - \frac{1}{2}b\left(\partial_z^2 u\right)^2 \tag{2}$$

with $a = \rho A$ and $b = \Theta E$, $\Theta = \frac{\pi R^4}{4}$.

(b) Boundary conditions: the field and its derivative disappear at the ends of the rod, $u(\pm L/2, t) = u'(\pm L/2, t) = 0$. With this the variation of the action simply yields

$$\begin{aligned}\delta S = \int \mathrm{d}t \int_{-L/2}^{L/2} \mathrm{d}z\, a\partial_t u \partial_t \delta u - b\partial_z^2 u \partial_z^2 \delta u = \delta S = \int \mathrm{d}t \int_{-L/2}^{L/2} \mathrm{d}z\, - a\partial_t^2 u\delta u - b\partial_z^4 u\delta u = 0 \\ \Rightarrow -a\partial_t^2 u - b\partial_z^4 u = 0\end{aligned} \tag{3}$$

where in the second term we performed two integrations by parts to arrive from $\partial_z^2 u\, \partial_z^2 \delta u$ to $\partial_z^4 u\, \delta u$. Now let us try to derive this resutls using the Euler-Lagrange field equations with $\frac{\partial \mathcal{L}}{\partial u} = 0$, $\partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t u)} = a\partial_t^2 u$, $\partial_z \frac{\partial \mathcal{L}}{\partial(\partial_z u)} = 0???$. Seemingly it can be interpreted as zero. But taking a closer look, the truth is that functional differentiation behaves differently as normal derivatives and there are two proper ways to handle them properly:

- As already discussed above, we massage the δS integral until only the variation of $\delta\xi$ appears, which allows us interpret the functional derivative of terms like $(\partial_z^n u)^2$ giving after n integration by parts $\frac{\delta \mathcal{L}}{\delta(\partial_z u)} = (-1)^n \partial_z^{2n} u$, in our case $\partial_z^4 u$.
- The other way is to introduce new, *independent* variables as $\partial_z^3 u, \partial_z^4 u, \ldots$ in the Lagrangian density, $\mathcal{L} = \mathcal{L}(u, \partial_t u, \partial_z u, \partial_z^2 u, \partial_z^3 u \partial_z^4 u, \ldots)$, and also incorporate the derviatives with respect to them into the Euler- Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \xi} = \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t \xi)} + \partial_z \frac{\partial \mathcal{L}}{\partial(\partial_z \xi)} + \partial_z^2 \frac{\partial \mathcal{L}}{\partial(\partial_z^2 \xi)} + \partial_z^3 \frac{\partial \mathcal{L}}{\partial(\partial_z^3 \xi)} + \ldots \tag{4}$$

Naturally both of the above methods give the same result.

(c) Looking for the solution in form of $u(z,t) = U(z)\varphi(t)$ we get

$$-aU(z)\partial_t^2\varphi(t) - b\partial_z^4 U(z)\varphi(t) = 0 \Rightarrow \frac{\partial_t^2 \varphi}{\varphi} = -\frac{b}{a}\frac{\partial_z^2 U(z)}{U(z)} \tag{5}$$

as the two sides depend on different variables they can only be constants giving

$$\partial_t^2 \varphi = -\omega^2 \varphi \Rightarrow \varphi = e^{\pm i\omega t},\ \partial_z^4 U = \frac{a\omega^2}{b} U \Rightarrow U = Ae^{kz} + Be^{-kz} + Ce^{ikz} + De^{-ikz} \tag{6}$$

with $\frac{a}{b}\omega^2 = k^4$. Now for determining the possible values of k we exploit the boundary conditions $U(\pm L/2) = U'(\pm L/2) = 0$.

(d) Instead of solving the system of equations we search for symmetric and antisymmetric solutions:

$$U = \alpha\cosh(kz) + \beta\cos(kz),\ \alpha\cosh(kL/2) + \beta\cos(kL/2) = 0,\ \alpha\sinh(kL/2) - \beta\sin(kL/2) = 0 \Rightarrow \frac{\cos(kL/2)}{\cosh(kL/2)} = -\frac{\sin(kL/2)}{\sinh(kL/2)} \Leftrightarrow \tan(kL/2) = -\tanh(kL/2) \tag{7}$$

and

$$U = \alpha\sinh(kz) + \beta\sin(kz),\ \alpha\sinh(kL/2) + \beta\sin(kL/2) = 0,\ \alpha\cosh(kL/2) + \beta\cos(kL/2) = 0 \Rightarrow \frac{\cos(kL/2)}{\cosh(kL/2)} = \frac{\sin(kL/2)}{\sinh(kL/2)} \Leftrightarrow \tan(kL/2) = \tanh(kL/2) \tag{8}$$

Problem 2

A body of mass m is fixed to the end of an elastic rod. The cross-section of the rod is A, its Young's modulus is E, its mass-density is ρ, and the length of the rod is L.

The longitudinal displacement of the points of the rod is described by the field $\xi(z,t)$, the transversal displacement of the rod is negligible in our case. The position of the body is described by $u(t)$. The action of the system is

$$S = \int dt \left\{ \frac{1}{2} m(\dot{u})^2 + A \int dx \left[\frac{\rho}{2}(\partial_t \xi)^2 - \frac{E}{2}(\partial_x \xi)^2 \right] \right\} \tag{9}$$

As we see, if one naiively derives the equations of motion of this action, one gets a trivially wrong result: the body will not be fixed to the end of the rod. We have to include the constraint $\xi(L,t) = u(t)$ explicitely in the calculations.

(a) The constraint can be taken into account using a (time-dependent) Lagrange-multiplicator. Write down the action modified by the Lagrange multiplicator.

(b) Write down the variation of the action.

(c) Following the usual way, using integrations by part transform the action in a form where only the variations are present, while their derivatives are not. In the case of $\delta\xi$, be careful at the boundary $z = L$.

(d) Using the principle of least action derive the equations of motion for the system.

(e) Search for the solutions in the following wave-form:

$$\xi(x,t) = X_0 \sin(\omega t)\sin(kx) \tag{10}$$

Determine the connection between k and ω.

(f) Starting from the solution of e.) write down the equation of motion for the body. You should arrive to a transcendent equation for the possible k values. Don't solve the equation.

(g) Discuss the limit, when the mass of the body is negligible. What are the free oscillation frequencies of the system in that case?

(h) Discuss the limit, when the mass of the rod is negligible. What is the smallest free oscillation frequency in that case?

(i) Determine the energy density and energy current in the rod. Show that the energy current that "flows out" from the rod at the body is exactly the time derivative of the body's kinetic enegy.

Solution:

(a) Without the constraint term we would get

$$\delta S = \int dt \left\{ m(\dot{u})(\delta\dot{u}) + A \int dx \left[\rho(\partial_t\xi)(\partial_t\delta\xi) - E(\partial_x\xi)(\partial_x\delta\xi)\right] \right\} \Rightarrow m\ddot{u} = 0,\ \rho A\partial_t^2\xi = EA\partial_x^2\xi \tag{11}$$

seemingly wrong result, it describes just an elastic rod being completely independent of the brick... To take into account the constraint we apply the Lagrange multiplicator method, but with an auxiliary variable different for all time instants, $\mu \equiv \mu(t)$ enforcing that $u(t) = \xi(L,t)$, i.e.: the rod's end at L is connected to the body. This can also be interpreted as we have infinitely many independent constraint and $\mu(t)$ is thought of as a vector with infinitely many components:

$$\tilde{S} = \int dt \left\{ \frac{1}{2}m(\dot{u})^2 + \mu(t)\left[u(t) - \xi(L,t))\right] + A \int dx \left[\frac{\rho}{2}(\partial_t\xi)^2 - \frac{E}{2}(\partial_x\xi)^2\right]\right\} \tag{12}$$

(b) Now the variation for the modified action together with one integration by parts:

$$\delta\tilde{S} = -\int \mathrm{d}t \left\{ m\ddot{u}\delta u - EA\partial_x\xi(L,t)\delta\xi(L,t) + A\int \mathrm{d}x \left[\rho\partial_t^2\xi - E\partial_x^2\xi\right]\delta\xi + \mu(t)\left[\delta u(t) - \delta\xi(L,t)\right]\right\} \tag{13}$$

$$m\ddot{u} = \mu(t) \tag{14}$$
$$-\rho A\partial_t^2\xi + EA\partial_x^2\xi = 0 \tag{15}$$
$$EA\partial_x\xi(t,L) = -\mu(t) \tag{16}$$

where the last equation originates from the boundary terms after integrating by parts the $\partial_x\xi\partial_x\delta\xi$ term, as

$$\int_0^L \mathrm{d}x\,\partial_x\xi\partial_x\delta\xi = \partial_x\xi\delta\xi\Big|_0^L - \int_0^L \mathrm{d}x\,\partial^x\xi\delta\xi = \partial_x\xi(L,t)\delta\xi(L,t) - \int_0^L \mathrm{d}x\,\partial^x\xi\delta\xi \tag{17}$$

and because of the constraint the "system does not end,, at the fight end of the rod and because of the fact that the variations should disappear at the end of the system we only have that $\delta\xi(0,t) = 0$ and $u(t = t_i, t_f) = 0$. Expressing $\mu(t)$ we get

$$-\rho\partial_t^2\xi + E\partial_x^2\xi = 0 \tag{18}$$
$$m\ddot{u} = -EA\partial_x\xi(L,t) \tag{19}$$

(c) Now the first equation is the ,,usual" wave-equation for the solution of which we can search in terms of $\xi(x,t) = X_0\sin(\omega t)\sin(kx)$ we get:

$$\rho\omega^2\xi(x,t) - Ek^2\xi(x,t) = 0 \Rightarrow \omega = \sqrt{E/\rho} \tag{20}$$

as it should and from where by incorporating the constraint as well

$$\begin{aligned} m\ddot{u} = m\partial_t^2\xi(L,t) = -\omega^2 mX_0\sin(\omega t)\cos(kx) \equiv -EA\partial_x\xi(L,t) = -EAkX_0\sin(\omega t)\cos(kL) \\ \Rightarrow \tan(kL) = \frac{\rho A}{mk} \end{aligned} \tag{21}$$

being a transcendent equation.

(d) Let $y = kL$ and so $\tan(y) = \frac{\rho AL}{my} \equiv \frac{\gamma}{y}$. Now if $\gamma \ll 1$ that is the rod's mass is small compared to the brick then $\tan(y) \approx y$ giving the ,,small mass" solution of $y = \sqrt{\gamma}$, that is $k = \sqrt{\frac{\rho A}{mL}}$ and $\omega_0 = \sqrt{\frac{EA}{mL}} = \sqrt{\frac{D}{m}}$ with $D = \frac{EA}{L}$ being the spring constant! So we got back what we expected, a harmonic oscillator!
For other possible k solutions we have $kL \approx n\pi$ where the tangent is small!

(e) In case of the light brick limit, brick's mass is much smaller than the rod's mass, $\gamma \gg 1$ which means solutions being near to $kL = \frac{\pi}{2} + n\pi$. Implying that the brick will perform harmonic motions.

(f) Energy density:

$$\begin{aligned}\mathcal{H} &= \partial_t \xi \frac{\partial \mathcal{L}}{\partial(\partial_t \xi)} - \mathcal{L} = A\rho(\partial_t \xi)^2 - A\left[\frac{\rho}{2}(\partial_t \xi)^2 - \frac{E}{2}(\partial_x \xi)^2\right] \\ &= \frac{1}{2}\rho A(\partial_t \xi)^2 + \frac{1}{2}EA(\partial_x \xi)^2 = -\rho A^2 X_0^2 \sin(\omega t)\sin(kx)\end{aligned} \tag{22}$$

Energy current density:

$$J_E = -EAX_0^2 k\omega \sin(\omega t)\cos(\omega t)\sin(kx)\cos(kx) \tag{23}$$

$$K_b = \frac{1}{2}m(\dot{u})^2,\ \frac{\mathrm{d}K_b}{\mathrm{d}t} = m\dot{u}\ddot{u} = -EA\partial_x \xi(L,t)\partial_t \xi(L,t) \tag{24}$$

so it is what we expect, that is the energy density current at the end of the rod!

Problem 3

One of the most important non-quadratic field theories is the sine-Gordon model for a field $\varphi(x,t)$, described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_t \varphi)^2 - \frac{1}{2}(\partial_x \varphi)^2 + \cos(\varphi) - 1 \tag{25}$$

(a) Derive the Euler-Lagrange equations of motion for the model.

(b) Determine the expression of the energy density in the model.

(c) Search for constant (in time and space) solutions that solve the equations of motion. What is the energy density in these solutions? Which configurations of these have finite total energies?

(d) We would like to find such solutions that transfer from one of these configurations to the other. Show that the following time-independent configuration solves the equations

$$\varphi_1(x,t) = 4\arctan(e^x) \tag{26}$$

Note.: This solution is called a standing soliton. Hint:

$$\sin(4\arctan(y)) = \frac{4(y-y^3)}{(1+y^2)^2} \tag{27}$$

(e) Write down the function in the $x \to \pm\infty$. Sketch the function.

(f) Determine the energy density, and its integral (the total energy) for this solution.

(g) Show that the following time-dependent solution solves the equations.

$$\varphi_2(x,t) = 4\arctan\left(e^{\frac{x-vt}{\sqrt{1-v^2}}}\right) \tag{28}$$

Note.: This is called the moving soliton solution with velocity v.

(h) Determine the energy density, and its integral (the total energy) for the solution φ_2. Hint:

$$\cos(4\arctan(y)) = 1 - 8\frac{y^2}{(1+y^2)^2} \tag{29}$$

Solution:

(a) Euler-Lagrange equations

$$\partial_t \frac{\partial \mathcal{L}}{\partial \partial_t \varphi} + \partial_x \frac{\partial \mathcal{L}}{\partial \partial_x \varphi} + \frac{\partial \mathcal{L}}{\partial \varphi} = -\sin\varphi - \partial_t^2 \varphi + \partial_x^2 \varphi = 0 \tag{30}$$

Note that in this case Euler-Lagrange-equations could be applied as no higher order derivatives appeared in $\mathcal{L}$. In the language of the variation of the action one would obtain:

$$\delta S = \int \mathrm{d}t\, \mathrm{d}x \; -\partial_t^2 \varphi \delta\varphi + \partial_x^2 \varphi \delta\varphi - \sin(\varphi)\delta\varphi = 0 \tag{31}$$

where apart from the usual integration by parts we also took the variation of $\cos(\varphi)$ as $\delta\cos(\varphi) = -\sin(\varphi)\delta\varphi$.

(b) Energy density:

$$\mathcal{H} = \partial_t \varphi \frac{\partial \mathcal{L}}{\partial \partial_t \varphi} - \mathcal{L} = \frac{1}{2}(\partial_t\varphi)^2 + \frac{1}{2}(\partial_x\varphi)^2 - \cos\varphi + 1 \tag{32}$$

(c) Static solution, $\partial_t\varphi = 0$:

$$-\sin\varphi + \partial_x^2\varphi = 0 \tag{33}$$

if the solution is constant we have $\sin\varphi_0 = 0 \Rightarrow \varphi_0 = n\pi$ for which the energy densities read

$$\mathcal{H} = -\cos(2k\pi)+1 = 0,\ n = 2k,\ \ \mathcal{H} = -\cos((2k+1)\pi)+1 = 2,\ n = 2k+1,\ \ H = \int \mathrm{d}x\, \mathcal{H} = \infty \tag{34}$$

(d) First let us express the first derivative, $\partial_x \mathrm{arctan}(e^x) = \frac{e^x}{1+e^{2x}} = \frac{1}{2\sinh(x)}$:

$$4\partial_x^2 \mathrm{arctan}(e^x) = 2\partial_x \left(\frac{1}{\sinh(x)}\right) = -2\frac{\sinh(x)}{\cosh^2(x)} \tag{35}$$

together with $\sin(4\mathrm{arctan}(e^x)) = 4\frac{e^x - e^{3x}}{(1+e^{2x})^2} = -2\frac{\sinh(x)}{\cosh^2(x)}$ giving indeed the static solution for the trial function $\varphi(x) = 4\mathrm{arctan}(\mathrm{e}^{\mathrm{x}})$, as:

$$\partial_x^2\varphi = -2\frac{\sinh(x)}{\cosh^2(x)} \equiv \sin(4\mathrm{arctan}(e^x)) \tag{36}$$

(e) Limit in $x \to \infty$:

$$\varphi(x\to\infty) = 4\pi/2 = 2\pi, \tag{37}$$

while for $x \to -\infty$ we have $\varphi(x \to -\infty) = \pi$.

(f) Energy density (Hint: $\cos(4\mathrm{arctan}(y)) = 1 - 8\frac{y^2}{(1+y^2)^2}$), where $\partial_t\varphi\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)} = 0$ for the static solution, so in reality we have $\mathcal{H} = -\mathcal{L}$, in which we agaian express the two terms as:

$$\begin{aligned} -\cos(4\mathrm{arctan}(e^x)) + 1 &= 8\frac{e^{2x}}{(1+e^{2x})^2} = \frac{2}{\cosh^2(x)} \\ \frac{1}{2}(\partial_x\varphi)^2 &= 8(\partial_x \mathrm{arctan}(e^x))^2 = \frac{2}{\cosh^2(x)} \end{aligned} \tag{38}$$

$$\mathcal{H} = \frac{1}{2}(\partial_x\varphi)^2 - \cos(\varphi) = \frac{4}{\cosh^2(x)} \tag{39}$$

Total energy:

$$H = \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{e^{2x}}{(1+e^{2x})^2} = 16\left[-\frac{1}{2(1+e^{2x})}\right]_{-\infty}^{\infty} = 8. \tag{40}$$

(g) Time-dependent solution: $\varphi_2 = 4\mathrm{arctan}\left(e^{\frac{x-vt}{\sqrt{1-v^2}}}\right)$.

Space derivative just includes an additional $\frac{1}{1-v^2}$ factor, while the sinus part gives the same just with the modified argument, $\sim x' \equiv \frac{x-vt}{\sqrt{1-v^2}}$. The time derivative results in a similar expression as the space derivative, but with an extra v^2 factor originating from the modified argument, $\partial_t^2\varphi = -2\frac{\sinh(x')}{\cosh^2(x')}\frac{v^2}{1-v^2}$ just cancelling the v dependent part in the space derivative term leading to $-\partial_t^2\varphi + \partial_x^2\varphi = -2\frac{\sinh(x')}{\cosh^2(x')}$ which is the same as $\sin(4\mathrm{arctan}(e^{x'}))$ and so satisfying the equation!

(h) Energy density for the time-dependent solution. Now $\mathcal{H} = \partial_t \varphi_2 \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi_2)} - \mathcal{L} = \frac{1}{2}(\partial_t \varphi)^2 + \frac{1}{2}(\partial_t \varphi)^2 - \cos(\varphi) + 1$. The first derivatives are the same as for the time-independnet case, but with an extrafactor of $\frac{1}{1-v^2}$ and $\frac{v^2}{1-v^2}$ for the space and time derivatives, respectively, with the shorthand notation of $x' = x - vt$:

$$\frac{1}{2}(\partial_t \varphi)^2 + \frac{1}{2}(\partial_t \varphi)^2 = \frac{1+v^2}{1-v^2} \frac{2}{\cosh^2(x')} \tag{41}$$

$$1 - \cos\varphi = \frac{2}{\cosh^2(x')} \tag{42}$$

Now the energy is the again the time-integral of the above expression:

$$H = \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{4}{1-v^2} \tanh\left(\frac{x - vt}{\sqrt{1-v^2}}\right) = \frac{8}{\sqrt{1-v^2}}, \tag{43}$$

as the first term is an odd function of x, so the rsult is nothing else but the previous, time-independnet one, but with the ,,contraction factor” $\frac{1}{\sqrt{1-v^2}}$.