Problem 1

Consider the longitudinal waves traveling in a thin, elastic rod. The longitudinal displacement of the points of the rod is described by the field $\xi(x, t)$. The Young's modulus of the rod is E, its mass density is ρ, and its cross-secion is A.

(a) Write down the (linear-) density of the kinetic energy as a function of the time-derivative of the field.

(b) Write down the (linear-) density of the elastic energy as a function of the x-derivative of the field.

(c) Write down the Lagrangian of the system.

(d) Using the principle of least action determine the equations of motion for the system.

(e) From the action, determine the expression for the (total) energy density in the system.

(f) Write down the energy of in a finite piece of the system. Determine its time-derivative.

(g) Determine the expression for the energy-current in the system. Derive the continuity equation for the energy.

Soution:

(a) We need to express the Langrangian density as the difference of the kinetic and potential energy density. We start with the kinetic energy density as the kinetic energy of the infinitesimal constituents with mass $\mathrm{d}m$ of the rod:

$$\mathrm{d}m = \rho \mathrm{d}V = \rho A \mathrm{d}x \Rightarrow \mathrm{d}K = \frac{1}{2}\rho A \mathrm{d}x(\partial_t \xi)^2 \equiv \mathcal{K}\mathrm{d}x \tag{1}$$

(b) The field describing the points of the rod is meant to relate its initial positions to its actual ones at time t, $\xi(x, 0) = x$. The potential or elastic energy density is expressed with the help of the Young modulus, in particular consider a $\mathrm{d}x$ part and elongate its length to $\mathrm{d}x'$, which is described by the field as $\mathrm{d}x' = \mathrm{d}x + \xi(x + \mathrm{d}x, t) - \xi(x, t) \to \frac{\mathrm{d}x' - \mathrm{d}x}{\mathrm{d}x} = \partial_x \xi(x, t)$. The energy denisty is then defined as $\mathcal{V} = \frac{1}{2}E(\text{relative elongation})^2$, giving

$$\mathcal{V} = \frac{1}{2}EA(\partial_x \xi)^2. \tag{2}$$

(c) From here the Lagrangian takes the form of

$$\mathcal{L} = \mathcal{K} - \mathcal{V} = \frac{1}{2}\rho A(\partial_t \xi)^2 - \frac{1}{2}EA(\partial_x \xi)^2. \tag{3}$$

(d) The Euler-Lagrange equations then read

$$S = \int\int \mathrm{d}t\,\mathrm{d}x\,\mathcal{L}(\xi, \partial_t\xi, \partial_x\xi, t) \text{ is minimal} \Rightarrow \frac{\partial \mathcal{L}}{\partial \xi} - \partial_t \frac{\partial \mathcal{L}}{\partial \partial_t \xi} - \partial_x \frac{\partial \mathcal{L}}{\partial \partial_x \xi} = 0, \tag{4}$$

The derivation is performed by taking the varaition of the action with respect to tits variables and expand the Lagrangian up to first order in the variation δs, which disappears at the boundaries $\delta\xi(x = 0, L; t) = \partial_x \delta\xi(x = 0, L; t) = \partial_x \delta\xi(x = 0, L; t) = 0$:

$$\begin{aligned} \delta S &= \int \mathrm{d}t\mathrm{d}x\,\mathcal{L}(\xi + \delta\xi, \partial_t\xi + \partial_t\delta\xi, \partial_x\xi + \partial_x\delta\xi) - \mathcal{L}(\xi, \partial_t\xi, \partial_x\xi) \\ &= \int \mathrm{d}t\mathrm{d}x \left[\frac{\partial \mathcal{L}}{\partial \xi} - \partial_t \frac{\partial \mathcal{L}}{\partial \partial_t \xi} - \partial_x \frac{\partial \mathcal{L}}{\partial \partial_x \xi}\right] \delta\xi \frac{\partial \mathcal{L}}{\partial \partial_t \xi} + \partial_x \delta\xi + \partial_t \delta\xi \frac{\partial \mathcal{L}}{\partial \partial_x \xi}\Big|_{x=1,L} \end{aligned} \tag{5}$$

where the boundary terms drop because of the definition of $\delta\xi$, now δS can only be zero if its integrand is zeros for all values of x, t giving the Euler-Lagrange equations.
Now $\frac{\partial \mathcal{L}}{\partial \xi} = 0$, $\partial_x \frac{\partial \mathcal{L}}{\partial \partial_x \xi} = EA\partial_x^2 \xi$, $\partial_x \frac{\partial \mathcal{L}}{\partial \partial_t \xi} = \rho A \partial_t^2 \xi$

$$\rho \partial_t^2 \xi - E\partial_x^2 \xi = 0 \tag{6}$$

giving the well-known wave-equation with speed of sound $c = \sqrt{E/\rho}$.
Now we show that this is the result also for directly calcualting the variation of the action of the given LAgrangian of the rod:

$$\begin{aligned}
\delta S &= \int \mathrm{d}t\mathrm{d}x\, \frac{1}{2}\rho A(\partial_t\xi + \partial_t\delta\xi)^2 - \frac{1}{2}EA(\partial_x\xi + \partial_x\delta\xi)^2 - \frac{1}{2}\rho A\partial_t\xi^2 - \frac{1}{2}EA\partial_x\xi^2 \\
&= \int \mathrm{d}t\mathrm{d}x\, \frac{1}{2}\rho A(\partial_t\xi + \partial_t\delta\xi)^2 - \frac{1}{2}EA(\partial_x\xi + \partial_x\delta\xi)^2 - \frac{1}{2}\rho A\partial_t\xi^2 + \frac{1}{2}EA\partial_x\xi^2 \\
&= \int \mathrm{d}t\mathrm{d}x\, \rho A\partial_t\xi\partial_t\delta\xi - EA\partial_x\xi\partial_x\delta\xi = -\int \mathrm{d}t\mathrm{d}x\, \left(\rho A\partial_t^2\xi - EA\partial_x^2\xi\right)\delta\xi + \rho A\partial_t\xi\partial_t\delta\xi - EA\partial_x\xi\partial_x\delta\xi\Big|_{x=0,L} \\
&= \int \mathrm{d}t\mathrm{d}x\, \left(\rho A\partial_t^2\xi - EA\partial_x^2\xi\right)\delta\xi = 0
\end{aligned} \tag{7}$$

where again we integrated by parts and used the fact again that at the boundaries $\delta\xi(x = 0, L, t) = 0$.

(e) Energy density in the same way as the Hamilton function is calculated from the Lagrangian, that is, via a Leggendre-transformation:

$$\mathcal{H} = \partial_t\xi\frac{\partial\mathcal{L}}{\partial\partial_t\xi} - \mathcal{L} = \frac{1}{2}\rho A(\partial_t\xi)^2 + \frac{1}{2}EA(\partial_x\xi)^2 \tag{8}$$

(f) Then the energy of a finite segment of the system, say from x_0 to x_1 is simply given by the integral of the Hamiltonian density:

$$E = \int_{x_0}^{x_1} \mathrm{d}x\,\mathcal{H} \tag{9}$$

from which the energy current of th system is simply given as

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{x_0}^{x_1} \mathrm{d}x\,\mathcal{H} \\
&= -\int_{x_0}^{x_1} \mathrm{d}x\,\partial_t^2\xi\frac{\partial\mathcal{L}}{\partial\partial_t\xi} + \partial_t\xi\partial_t\frac{\partial\mathcal{L}}{\partial\partial_t\xi} - \partial_x\partial_t\xi\frac{\partial\mathcal{L}}{\partial\partial_x\xi} - \partial_t^2\xi\frac{\partial\mathcal{L}}{\partial\partial_t\xi} - \partial_t\xi\frac{\partial\mathcal{L}}{\partial\xi} - \partial_t\mathcal{L} \\
&= -\int_{x_0}^{x_1} \mathrm{d}x\, -\partial_x\partial_t\xi\frac{\partial\mathcal{L}}{\partial\partial_x\xi} - \partial_t\xi\partial_x\frac{\partial\mathcal{L}}{\partial\partial_x\xi} - \partial_t\mathcal{L} = -\int_{x_0}^{x_1}\mathrm{d}x\,\partial_x\left(\partial_t\xi\frac{\partial\mathcal{L}}{\partial\partial_x\xi}\right) = -\partial_t\xi\frac{\partial\mathcal{L}}{\partial\partial_x\xi}\Big|_{x_0}^{x_1}
\end{aligned} \tag{10}$$

So the change of energy inside the rod is given by the energy density current flowing in at x_0 plus the nergy density current flowing out at the end of the rod at x_1:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = J_E(x_0, t) - J_E(x_1, t) \tag{11}$$

with $J_E(x,t) = \partial_t\xi\frac{\partial\mathcal{L}}{\partial\partial_x\xi} = \partial_t\xi\left(-EA\partial_x\xi\right) = -EA\partial_t\xi\partial_x\xi$

Problem 2

A thin elastic rod hangs from the ceiling. The Young's modulus of the rod is E, its mass density is ρ, and its cross-secion is A. The longitudinal displacement of the points of the rod is described by the field $\xi(x, t)$.

(a) Construct the Lagrangian of the system (you will need the kinetic, elastic, and gravitational energy densities).

(b) Determine the displacement field in equilibrium.

(c) At time $t = 0$ we cut the rod from the ceiling. Determine the equation of motion for the rods displacement.

(d) Try to find a solution of the equation. Show that the lower end remains in rest until the shockwave of the cut hits it.

(e) Based on the Lagrangian determine the energy density and energy current density of the system.

(f) Analize the energy density and current of the "shockwave" solution.

Solution:

(a) Kinetic and elastic energy denisities are simply expressed as

$$\mathcal{K} = \frac{1}{2}\rho A(\partial_t \xi)^2 \quad (12)$$

$$\mathcal{V} = \frac{1}{2}EA(\partial_x \xi)^2 - \rho A g \xi \quad (13)$$

the last term originating from gravitational potential.

(b) The Euler-Lagrangian equations of motion then take the form with the Lagrangian $\mathcal{L} = \mathcal{K} - \mathcal{V} = \frac{1}{2}\rho A(\partial_t \xi)^2 - \frac{1}{2}EA(\partial_x \xi)^2 + \rho A g \xi$, where $\frac{\partial \mathcal{L}}{\partial \xi} = -\rho g$, $\partial_x \frac{\partial \mathcal{L}}{\partial \partial_x \xi} = EA\partial_x^2 \xi$, $\partial_t \frac{\partial \mathcal{L}}{\partial \partial_t \xi} = \rho A \partial_t^2 \xi$

$$E\partial_x^2 \xi - \rho \partial_t^2 \xi + \rho g = 0 \quad (14)$$

which in equilibrium can be searched for as $\xi_0 = \xi_0(x)$:

$$\partial_x^2 \xi_0 = -\rho g/E \Rightarrow \xi_0(x) = -\frac{\rho g E}{2}(x-L)^2 + \frac{\rho g}{2E}L^2. \quad (15)$$

which satisfies the natural conditions, $\xi_0(0) = 0$, $\partial_x \xi_0|_{x=L} = 0$. Now we also show that direct calculation of δS gives the same result:

$$\begin{aligned}
\delta S = &\int \mathrm{d}t\mathrm{d}x\, \frac{1}{2}\rho A(\partial_t \xi + \partial_t \delta\xi)^2 - \frac{1}{2}EA(\partial_x \xi + \partial_x \delta\xi)^2 - \frac{1}{2}\rho A \partial_t \xi^2 - \frac{1}{2}EA\partial_x \xi^2 + \rho g(\xi + \delta\xi) - \rho g \xi \\
= &\int \mathrm{d}t\mathrm{d}x\, \frac{1}{2}\rho A(\partial_t \xi + \partial_t \delta\xi)^2 - \frac{1}{2}EA(\partial_x \xi + \partial_x \delta\xi)^2 - \frac{1}{2}\rho A \partial_t \xi^2 + \frac{1}{2}EA\partial_x \xi^2 + \rho g \delta\xi \\
= &\int \mathrm{d}t\, \mathrm{d}x\, -\rho A \partial_t \xi \partial_t \delta\xi + EA\partial_x \xi \partial_x \delta\xi + \rho g \delta\xi = -\int \mathrm{d}t\mathrm{d}x\, \left(-\rho A \partial_t^2 \xi + EA\partial_x^2 \xi + \rho g\right)\delta\xi \\
&+ \rho A \partial_t \xi \partial_t \delta\xi - EA\partial_x \xi \partial_x \delta\xi\Big|_{x=0,L} = \int \mathrm{d}t\, \mathrm{d}x\, \left(-\rho A \partial_t^2 \xi + EA\partial_x^2 \xi + \rho g\right)\delta\xi = 0
\end{aligned} \quad (16)$$

(c) By cutting we give an initial condition for the time-dependent solution $\xi_0(x) = \xi_0(x,0)$ and $\partial_x \xi_0(0) = 0$. Let us look for the solution in form $\xi(x,t) = \xi_0(x) + p(x,t)$ leading to the equation:

$$-\rho A \partial_t^2 p + EA\partial_x^2 p = 0 \quad (17)$$

corresponding to a simple wave-equation with boundary condtions according to the ones for $\xi(x,t)$ (be careful, the correspondence between ξ and ξ_0 is given by $\xi(x,0) = \xi_0(x)$):
$\partial_x \xi\Big|_{x=L} = 0 \Rightarrow \partial_x p\Big|_{x=L} = 0$, that is the motion stops at the end of the rod
and $\partial_x \xi\Big|_{x=0} = 0 = \partial_x \xi_0\Big|_{x=0} + \partial_x p\Big|_{x=0} = \frac{\rho g}{E} + \partial_x p\Big|_{x=0} \Rightarrow \partial_x p\Big|_{x=0} = -\frac{\rho g}{E}L$, implying no motion at the point where the rod is fastened to the ceiling. Do not be disturbed by the fact that $\xi_0(0) = 0$, this does not imply that at time t $\xi(0,t) = 0$.
Try the solution in form of $p(x,t) = \varphi(ct - x) = \frac{\rho g L}{E}(ct - x)\Theta(ct - x)$, which satisfies both conditions together with the differential equation, called the shockwave soltuion, implying a linear chock running through the rod downwards elongating the static solutions, with velocity of sound, $c = \sqrt{E/\rho}$.

(d) Now the Lagrangian and the Hamiltonian takes the form with the given solution:

$$\mathcal{L} = \frac{1}{2}\rho A \partial_t \xi^2 - \frac{1}{2}EA\partial_x \xi^2 + \rho A g \xi \Rightarrow \mathcal{H} = \partial_t \xi \frac{\partial \mathcal{L}}{\partial \partial_t \xi} - \mathcal{L} = \frac{1}{2}\rho A \partial_t \xi^2 + \frac{1}{2}EA\partial_x \xi^2 - \rho A g \xi \quad (18)$$

Now the energy current density

$$\partial_t \mathcal{H} = \rho A \partial_t \xi \partial_t \xi^2 + EA\partial_x \xi \partial_t \partial_x \xi - \rho A g \partial_t \xi = EA\partial_x^2 \xi \partial_t \xi + EA\partial_x \xi \partial_x \partial_t \xi = -\partial_x\left(-EA\partial_x \xi \partial_t \xi\right) \quad (19)$$

(e) In the "shockwave" solution we ahve the following results:

$$\mathcal{H} = \frac{1}{2}\rho A \partial_t p^2 + \frac{1}{2} EA(\partial_x \xi_0^2 + \partial_x p_0^2 + 2\partial_x \xi \partial_x p) - \rho A g(\xi_0 + p) \quad (20)$$

$$\partial_t \mathcal{H} = EA\partial_t p(\partial_x p + \partial_x \xi_0) \quad (21)$$

(f) The derivatives, $\partial_t p$ and $\partial_x p$ are just a step functions until $t < x/c$ but with opposite signs, while $\partial_x \xi_0$ is a linear function increasing from its maximal value of $\frac{\rho g}{E}$ to 0.

Problem 3

A thin and long elastic rod is bent. In this problem our goal is to construct an expression for the total elastic energy of the system.

(a) In a short (lenght dl) piece of the rod, the radius of curvature of the bent rod is R, that is quite large (weakly bent rod). What is the elastic energy of that piece. (We know the Young modulus (E) and the shape of the cross section.)

(b) The rod is bent weakly in the $x - z$ plane such that it's shape is described by the function $\xi(z)$ for which $\xi(z)$ and $\xi'(z)$ are small. What is the elastic energy of the rod?

Solution:

(a) Consider a small angle, φ to which a relative elongation at x of $\frac{x}{R(z)}$ belongs with $R(z)$ generally depending also on the z variable. Integrating this along z we obtain the elastic energy

$$\int \mathrm{d}z \, \frac{1}{2} E \frac{x^2}{R(z)^2} \mathrm{d}x \, \mathrm{d}y. \quad (22)$$

The x, y integral is carried out over the cross section with radius a, that is

$$\int_0^a \mathrm{d}r \, r \int_0^{2\pi} \mathrm{d}\varphi r^2 \cos^2 \varphi = \frac{\pi a^4}{4} \Rightarrow U = \int \mathrm{d}z \frac{1}{2} \frac{E}{R^2(z)} \frac{\pi a^4}{4} \equiv \int \mathrm{d}z \, \frac{E}{2R^2(z)} \Theta \quad (23)$$

(b) Supposing small and slowly changing field describing the rods shape, $\xi(z), \xi'(z) \ll 1$ we have for the curvature/radius, $\frac{1}{R(z)} = \xi''(z) \Rightarrow$

$$U = \int \mathrm{d}z \, \frac{E}{2R^2(z)} \Theta(\partial_z^2 \xi)^2 \quad (24)$$

Problem 4

A thin and heavy rod of length L is horizontally fasten in a wall (see figure). The other end of the rod is free. The axis of the (non bent) rod is the z axis while the vertical axis is the x axis. The Young's modulus of the rods material is E, the cross-section parameter is Θ, and the rod's linear mass density is ρ.

(a) Write down the total (elastic + gravitational) energy of the system, if the rod's shape is described by the function $\xi(z)$.

(b) By minimizing the total energy, derive the differential equation that describes the shape of the bent rod.

(c) Determine the shape of the rod. What is the prolapse of the free end of the rod?

Solution:

(a) Given the shape unction $\xi(z)$, and the linear desnity $\lambda = \rho A$ by which we can define the infinitesimal mass along z, $\mathrm{d}m = \lambda \mathrm{d}z$:

$$V = \int \mathrm{d}z \frac{E}{2} \Theta(\partial_z^2 \xi)^2 - \lambda \xi g \quad (25)$$

(b) Minimalization of the energy gives the equation of the rod:

$$\begin{aligned}\delta U = 0 \Rightarrow \int \mathrm{d}z \frac{E\pi R^4}{4}\partial_z^2\xi\partial_z^2\delta\xi - \lambda g\delta\xi = \frac{E}{2}\Theta\partial_z^2\xi\partial_z\delta\xi\Big|_0^L - \int \mathrm{d}z\frac{E}{2}\Theta\partial_z^3\xi\partial_z\delta\xi + \lambda g\delta\xi \\ = \frac{E}{2}\Theta\partial_z^3\xi\delta\xi\Big|_0^L - \int \mathrm{d}z\frac{E}{2}\Theta\partial_z^4\xi\delta\xi + \lambda g\delta\xi = 0 \Rightarrow \frac{E}{2}\Theta\partial_z^4\xi - \lambda g = 0\end{aligned} \tag{26}$$

where we used that at the boundaries the derivatives disappear and so boundary terms disappear, $\xi(0) = \xi'(0) = \xi''(L) = \xi'''(L) = 0$. With these conditions the solution is the following:

$$\xi(z) = \frac{\lambda g}{12E\Theta} + \frac{C_3}{6}z^3 + \frac{C_2}{2}z^2 + C_1 z + C_0 \tag{27}$$

trivially by the first two conditions $C_1 = C_0 = 0$, while the further two conditions give:

$$\frac{2\lambda gL}{E\Theta} + C_3 \tag{28}$$

$$-\frac{\lambda gL^2}{E\Theta} + C_2 = 0 \tag{29}$$

giving the final result

$$\xi(z) = \frac{\lambda g}{12E\Theta} - \frac{\lambda gL}{3E\Theta}z^3 - \frac{\lambda gL}{2E\Theta}z^2 \tag{30}$$

from which we an tell the total displacement at the end of the rod, that is

$$\xi(L) = \frac{\lambda gL^4}{E\Theta} \tag{31}$$

Problem 5 Extra exercise solution of the small test

A particle of resting mass m and electric charge q moves in a homogeneous electric field whose strength is E. The field points in the y direction while the particle's velocity is initially $v_0 = 0.8c$ and points in the x direction.

(a) Determine the initial (usual) momentum vector of the particle.

(b) Write down the relativistic equations of motion for the particle's momentum vector. Solve the equation, i.e. determine the particle's momentum as a function of time.

(c) Consider the moment when the x and y components of the particle's momentum are equal. Determine the particle's 4-momentum in that moment. Use the known Minkowski-length of the 4-momentum for a particle of resting mass m.

(d) What is the particle's velocity vector in that moment? What are the x and y coordinates of the velocity?

Solution:

(a) Initially the particle has only non-zero monetum along the x axis, $p_x(0) = p_0 = \frac{mv_0}{\sqrt{1-v^0/c^2}}$. Nevertheless we do not even need the its expression with the velocities yet. In a three vector form we have

$$\mathbf{p} = (p_0, 0, 0) \tag{32}$$

(b) Newtionian approach, differentiation is performed with respect to the laboratory time, t:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\mathbf{E} = qE\,\hat{\mathbf{y}} \tag{33}$$

This implies a linearly increasing y component of the momentum, $p_y = qEt$, while the x componenet remains invariant, $p_x(t) = p_0$. Now again this does not imply in any way that also the velocity along the x axis would remain unchanged! So

$$\mathbf{p}(t) = (p_0.qEt, 0) \tag{34}$$

(c) At the moment when its two non-zero spatial components are equal we can write for the four momentum:

$$p^\mu(t) = \left(p^0, p, p, 0\right), \quad p = p_0 \tag{35}$$

Knowing that the Minkowski length of p^μ is independent of the time and $p^\mu(t)p_\mu(t) = m^2c^2$ we can express the four momentum at time t as

$$p^\mu p_\mu = m^2c^2 = \left(p^0\right)^2 - 2p^2 \Rightarrow p^\mu(t) = \left(\sqrt{m^2c^2 + 2p_0^2}, p_0, p_0, 0\right) \tag{36}$$

(d) Let us write now the x and y components with the corresponding velocities, where the equality of the momenta also implies equal velocities $v_x = v_y \equiv v/\sqrt{2}$ giving for the memtnum $p_0 = \frac{mv/\sqrt{2}}{\sqrt{1-v^2/c^2}}$ and $p^0 = \sqrt{m^2c^2 + \frac{m^2v^2}{1-v^2/c^2}} = \frac{mc}{\sqrt{1-v^2/c^2}}$:

$$p^\mu = \left(\frac{mc}{\sqrt{1-v^2/c^2}}, \frac{mv/\sqrt{2}}{\sqrt{1-v^2/c^2}}, \frac{mv/\sqrt{2}}{\sqrt{1-v^2/c^2}}, 0\right) \tag{37}$$

Now again using the invariane of the Minkowski length we can express the velocity as

$$(mc)^2 = \frac{(mc)^2}{1-v^2/c^2} - 2\frac{(mv_0)^2}{1-v_0^2/c^2} \Rightarrow v = \sqrt{\frac{\frac{2v_0^2/c^2}{1-v_0^2/c^2}}{1+\frac{2v_0^2/c^2}{1-v_0^2/c^2}}}\,c = \frac{2v_0}{\sqrt{1+v_0^2/c^2}}. \tag{38}$$