A16

The Hamiltonian of a system with one degree of freedom reads as

$$H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right) \tag{1}$$

Consider a canoncial transformation that is generated by a 2nd type generator function

$$W_1(q, Q) = \frac{Q}{q} \tag{2}$$

(a) Using the derivatives of the generator function determine the $p(q, Q)$ and $P(q, Q)$ relations.

(b) Using the results of a.) express the "old" variables in terms of the "new" ones, i.e. find the $q(Q, P)$ and $p(Q, P)$ functions.

(c) Determine the new form $K(Q, P)$ of the Hamiltonian.

(d) Starting from the new Hamiltonian determine the canonical equations for the new coordinate and momentum.

(e) Determine the solutions $Q(t)$ and $P(t)$.

A17

Consider the following transformation that rotates the coordinate axes of the phase-space (α is a real parameter):

$$Q = q\cos(\alpha) - p\sin(\alpha) \qquad P = q\sin(\alpha) + p\cos(\alpha) \tag{3}$$

(a) By calculating the Poisson bracket $\{Q, P\}$ show that the transformation is canonical.

(b) We would like to find a $W_1(q, Q)$ that generates the trasformation defined above. As a first step transform the relations above, and find the mixed $p(q, Q)$ and $P(q, Q)$ functions.

(c) Using the results of b.) determine the derivatives $\frac{\partial W_1}{\partial q}$ and $\frac{\partial W_1}{\partial Q}$.

(d) Solve the differential equations of c.), i.e. give an appropriate function $W_1(q, Q)$.

Solution:
In the Poisson brackets only terms with q and p give non-zero results

$$\{Q, P\} = \cos^2(\alpha)\{q, p\} - \sin^2(\alpha)\{p, q\} = \cos^2(\alpha) + \sin^2(\alpha) = 1\,. \tag{4}$$

For constructing the generator function let us express $p(q, Q)$ and $P(q, Q)$:

$$p = q\,\mathrm{ctg}(\alpha) - \frac{Q}{\sin(\alpha)},\ P = \frac{q}{\sin(\alpha)} - Q\,\mathrm{ctg}(\alpha) \tag{5}$$

As $p = \frac{\partial W_1}{\partial q} \to W_1 = \frac{q^2\,\mathrm{ctg}(\alpha)}{2} - \frac{qQ}{\sin(\alpha)} + f(Q)$ and $P = -\frac{\partial W_1}{\partial Q} \to W_1 = \frac{Q^2\,\mathrm{ctg}(\alpha)}{2} - \frac{qQ}{\sin(\alpha)} + q(q)$ by which we conlcude that with $f(Q) = \frac{Q^2}{2\,\mathrm{ctg}(\alpha)}$ and $g(q) = \frac{q^2}{2\,\mathrm{ctg}(\alpha)}$ the genreator takes the form

$$W_1(q, Q) = \frac{q^2}{2\,\mathrm{ctg}(\alpha)} - \frac{qQ}{\sin(\alpha)} + \frac{Q^2}{2\,\mathrm{ctg}(\alpha)}\,. \tag{6}$$

B13

The Hamiltonian of a one-dimensional Harmonic oscillator reads as

$$H = \frac{1}{2}q^2 + \frac{1}{2}p^2 \tag{7}$$

(We arrived to this special form ($m = 1$, $\omega = 1$) by rescaling time and energy units.)

(a) Consider the complex transformation

$$Q = \frac{x + ip}{\sqrt{2}} \qquad P = \frac{ix + p}{\sqrt{2}} \tag{8}$$

Using Poisson brackets show, that the transformation is canonical.

(b) Construct a 2nd type generator function that generates the above defined transformation.

(c) Determine the new form $K(Q, P)$ of the Hamiltonian. Write down and solve the canonical equations of motion.

(d) You can see, that the new Hamiltonian is complex valued, and the solutions of the canonical equations are also complex functions. However, the original p and x variables are real. Show that for real x and p the relation $P = iQ^*$ holds. Show that during the time evolution of Q and P this condition is conserved.

B14

Consider the following transformation,

$$Q = p^\alpha \cos(\beta q), \qquad P = p^\alpha \sin(\beta q) \tag{9}$$

where α and β are real parameters.

(a) Calculate the Poisson bracket $\{Q, P\}$ for generic α, β.

(b) What should be the relation between α and β to get a canonical transformation?

(c) Divide the two equations with each other, and determine the $P(q, Q)$ relation.

(d) Search for an appropriate first type $W_1(q, Q)$ generator function. Use the result of c.).

Solution:

$$\begin{aligned}\{Q, P\} =& \{p^\alpha \cos(\beta p), p^\alpha \sin(\beta q)\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = -\beta\alpha p^{2\alpha-1}\sin^2(\beta p) - \beta\alpha p^{2\alpha-1}\cos^2(\beta p) = 1 \\ &\leftrightarrow 2\alpha - 1 = 0 \rightarrow \alpha = 1/2\,,\end{aligned} \tag{10}$$

then $\beta\alpha = -1$ needs to be satisfied, so $\beta = -2$.
The $P(q, Q)$ function can then be expressed as

$$P/Q = -\tan(2q) \rightarrow P = -Q\tan(2q), \;\; p = (Q/\cos(2q))^2 \tag{11}$$

So for determining the gerneator function we consider the generating relations

$$P = -\frac{\partial W_1}{\partial Q} \rightarrow \frac{Q^2}{2}\tan(2q) + f(q), \;\; p = \frac{\partial W_1}{\partial q} \rightarrow \frac{Q^2}{2}\tan(2q) + g(Q) \tag{12}$$

So we conclude that $W_1 = \frac{Q^2}{2}\tan(2q)$.

B15

The canoncial coordinate and momentum of a system with one degree of freedom is q and p. We would like to transform to

$$Q = \alpha\frac{p}{q}, \qquad P = \beta q^2 \tag{13}$$

where α and β are unknown.

(a) Determine the corresponding Jacobi matrix M like in the lecture.

(b) Compute the matrix MJM^T, where J is the symplectic matrix.

(c) What relation must hold between α and β for the transformation to be canonical?

B16

Two coupled oscillators are described by the Hamiltonian

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_1 p_2) + \frac{m\omega^2}{2}(q_1^2 + q_2^2 - q_1 q_2) \tag{14}$$

We would like to find a canonical transformation that transforms into the normal coordinates of the system.

(a) Look for the new canonical momenta in the form

$$P_1 = a(p_1 - p_2), \qquad P_2 = b(p_1 + p_2) \tag{15}$$

Determine the values of the parameters a and b in such a way that the kinetic energy part reads as

$$\frac{1}{2}(P_1^2 + P_2^2) \tag{16}$$

(b) Look for the canonical coordinates in a form

$$Q_1 = c(q_1 - q_2), \qquad Q_2 = d(q_1 + q_2) \tag{17}$$

Determine the parameters c and d to make the transformation canonical.

(c) Express the new form of the Hamiltonian using the new variables $\{P_1, P_2, Q_1, Q_2\}$. If you computed everything correctly, then the final Hamiltonian is separated in the normal coordinates (i.e. there is no term including $Q_1 Q_2$, etc) What are the oscillation frequencies?

(d) Show that the following quantity is a constant of motion:

$$D = Q_1 P_2 - Q_2 P_1 \tag{18}$$