To pass the test, you have to gain minimum 20 points.

Problem 1 (12.5 points)

A beam of particles in a laboratory on Earth (that is the rest frame) is observed to travel with $v = 0.6c$ along the x direction. Measurements on the particles are also performed in a spacecraft travelling along the y axis with velocity $V = 0.6c$. Our goal is to predict from Earth what kind of motion of the particles will be observed from the spacecraft.

(a) Determine the 4-velocity of the particles in the reference frame of Earth. **(2.5 points)**

(b) Determine the Lorentz transformation that transforms to the frame of the moving spacecraft. **(2.5 points)**

(c) Express the 4−velocity of the particles in the frame of the spacecraft. **(2.5 points)**

(d) Determine the particles' "usual" 3− velocity (i.e.: spatial part of the 4−velocity), give its magnitude and its direction by the angle between its x and y components! **(5 points)**

Solutions are only briefly summarized
Solution:

(a) Easier to consider it only in $2 + 1$ dimensions $u^\mu = 5/4c\,(1, 3/5, 0)$.

(b) $\sqrt{1 - v^2/c^2} = 4/5$, $v/c = 3/5$:

$$\Lambda = \begin{bmatrix} 5/4 & 0 & -3/4 \\ 0 & 1 & 0 \\ -3/4 & 0 & 5/4 \end{bmatrix} \tag{1}$$

(c) $(u')^\mu = \Lambda u^\mu = c\,(25/16, 3/4, -15/16) = \frac{25}{16}c\,(1, 12/25, -3/5)$

(d) $\mathbf{v} = 12/25c\,(1, -5/4)$.

(e) $|\mathbf{v}| = 288/625 \approx 0.77\,c$, $\varphi = -\text{arctg}(5/4) = -0.896$

Problem 2 (12.5 points)

We would like to slow down a beam of muons moving in a homogenous electric field. The electric field's magnitude is E, charge and mass of the muons are $-q$ and m, respectively. Initially they move with velocity $v = 0.8c$ along the $+x$ direction. The electric field also points in the direction of $+x$, so it is slowing the motion of the muons with negative charge! The electric field is turned on at $t = 0$.

(a) Write down the relativistic equations of moiton! **(2 points)**

(b) Express $p_x(t)$ as a function of time (t is the time in the rest frame!) **(1.5 points)**

(c) Give the final time, t_f, when the muons stop moving! **(1.5 points)**

(d) Calculate, based on the form of $p_x(t)$, the time dependence of $v_x(t)$ as well! **(2.5 points)**

(e) Relate the infinitesimal changes in $\mathrm{d}t$ with the changes in the muons' proper time, $\mathrm{d}\tau$, via time dilation and express τ_f as an integral with respect to the rest frame time, t. **(2.5 points)**

(f) Compute the integral and determine the duration of the process, τ_f, in the muons' frame. **(2.5 points)**

Supplementary information:

$$(\text{asinh}(x))' = \frac{1}{\sqrt{1 + x^2}},\ (\text{acosh}(x))' = \frac{1}{\sqrt{x^2 - 1}} \tag{2}$$

Solution:

(a) $\frac{\mathrm{d}p}{\mathrm{d}t} = -Eq$, $p(0) = mv_0/\sqrt{1 - v_0^2/c^2} = 4/3\, mc$

(b) $p(t) = 4/3q, mc - qEt$

(c) t_f: $qEt_f = 4/3mc \Rightarrow t_f = \frac{4mc}{3qE}$

(d) $p(t) = mv/\sqrt{1 - v^2/c^2} \Rightarrow v = \frac{4/3 - qEt/mc}{\sqrt{1 + (4/3 - qEt/mc)^2}}\, c$

(e) $\mathrm{d}\tau = \sqrt{1 - v^2/c^2}\mathrm{d}t \Rightarrow \tau_f = \int_0^{t_f} \mathrm{d}t \sqrt{1 - v^2/c^2}$

(f) $\sqrt{1 - v^2/c^2} = \frac{1}{\sqrt{1 + (4/3 - qEt/mc)^2}} \Rightarrow \int_0^{t_f} \mathrm{d}t \frac{1}{\sqrt{1 + (4/3 - qEt/mc)^2}} = -\frac{mc}{qE} \mathrm{asinh}\,(4/3 - qEt/mc)\Big|_0^{t_f} = \frac{mc}{qE} \mathrm{asinh}(4/3) \equiv \mathrm{asinh}(4/3)t_f$

Problem 3 (12.5 points)

Consider a thin elastic rod of unit cross section and length L described by the Lagrangian density

$$\mathcal{L} = \frac{\rho}{2}(\partial_t \xi)^2 - \frac{E}{2}(\partial_x \xi)^2 - \frac{F}{12}(\partial_x \xi)^4. \tag{3}$$

(a) Using the Euler-Lagrange equations determine the equation of motion! **(2 points)**

(b) Looking for the solution in terms of plane waves, $\xi = \sin(\omega t - kx)$, give the relation between the frequency, ω, and the wavenumber, k, and determine the possible values of k knowing that at $t = 0$ $\xi(t = 0, x = 0, L) = 0$ (the displacement field disappears at the ends of the rod)! What is the requirement for the sign of F? **(3 points)**

(c) Express the energy density of the system! **(2 points)**

(d) Calculate the energy density current! **(2.5 points)**

(e) Calculate the change of energy inside the rod by integrating the energy density along the rod! **(3 points)**

(f) * **Bonus for (+2.5 points)** Show that it is the same as taking the difference of the energy density current at the ends of the rod!

Solution:

(a)

$$\frac{\partial \mathcal{L}}{\partial \xi} = 0,\ \frac{\partial \mathcal{L}}{\partial \partial_x \xi} = -E\partial_x \xi - \frac{F}{3}(\partial_x \xi)^3,\ \frac{\partial \mathcal{L}}{\partial \partial_x \xi} = \rho \partial_t \xi \tag{4}$$

$$\rho \partial_t^2 \xi = E\partial_x^2 \xi + \frac{F}{3}\partial_x(\partial_x \xi)^3 \tag{5}$$

(b) $\partial_t^2 \leftrightarrow -\omega^2 \sin(\omega t - kx)$, $\partial_x^2 \leftrightarrow -k^2 \sin(\omega t - kx)$, $\partial_x(\partial_x \xi)^3 \leftrightarrow -3k^4 \sin(\omega t - kx)\cos^2(\omega t - kx)$, in addition $\sin(kL) = 0 \to k = \frac{n\pi}{L}$:
Investigation at $k = n\pi/L$, as an approximation:

$$\rho\omega^2 = Ek^2 + F\omega^4 \cos^2(n\pi/L) \to \omega^2 = \frac{E}{\rho}k^2 + \frac{F}{\rho}k^4 \tag{6}$$

(c) $\mathcal{H} = \partial_t \xi \frac{\partial \mathcal{L}}{\partial \partial_t \xi} - \mathcal{L} = \frac{\rho}{2}(\partial_t \xi)^2 + \frac{E}{2}(\partial_x \xi)^2 + \frac{F}{12}(\partial_x \xi)^4$

(d) $J_E = \partial_t \xi \frac{\partial \mathcal{L}}{\partial \partial_x \xi} = -E\partial_t \xi \partial_x \xi - \frac{F}{3}\partial_t \xi (\partial_x \xi)^3$

(e) $E = \int_0^L \frac{\rho\omega^2}{2}\cos^2(\omega t - kx) + \frac{Ek^2}{2}\cos^2(\omega t - kx) + \frac{Fk^4}{12}\cos^4(\omega t - kx)\mathrm{d}x = L\left(\frac{\rho\omega^2}{4} + \frac{Ek^2}{4} + \frac{Fk^4}{24}\right)$

Problem 4 (12.5 points)

The Lagrangian of a one dimensional system along the x axis in the interval $[0, L]$

$$\mathcal{L} = \frac{1}{2}\partial_t\xi\partial_x\xi - \frac{\nu}{2}(\partial_x^2\xi)^2 \tag{7}$$

where the field $\xi(x,t)$ describes some continuous medium.

(a) Write down the action of the system! **(2 points)**

(b) Based on the principle of least action give the variation of the action and determine the equation of motion! **(3 points)**

(c) Determine the $\mathcal{H}$ energy density of the system! **(2.5 points)**

(d) Determine the J_E energy density current of the system! **(2 points)**

(e) Show explicitly for this system that the time derivative of the energy density equals the minus space derivative of the energy density current, $\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = -\partial_x J_E$! **(3 points)**

Supplementary information:

$$\mathcal{H} = \partial_t\xi\frac{\partial\mathcal{L}}{\partial(\partial_t\xi)} - \mathcal{L},\ J_E = \partial_t\xi\frac{\partial\mathcal{L}}{\partial(\partial_x\xi)} \tag{8}$$

Solution:

(a) $S = \int \mathrm{d}t\,\mathrm{d}x\, \frac{1}{2}\partial_t\xi\partial_x\xi - \frac{\nu}{2}(\partial_x^2\xi)^2$

(b) $\delta S = \int \mathrm{d}t\,\mathrm{d}x\, \frac{1}{2}\partial_t\delta\xi\partial_x\xi + \frac{1}{2}\partial_t\xi\partial_x\delta\xi - \nu\partial_x^2\xi\partial_x^2\delta\xi = \int \mathrm{d}t\,\mathrm{d}x\,(-\partial_t\partial_x\xi - \nu\partial_x^4\xi)\delta\xi = 0 \Rightarrow \partial_t\partial_x\xi = -\nu\partial_x^4\xi$, integrating by parts two times.

(c) $\mathcal{H} = \partial_t\xi\frac{\partial\mathcal{L}}{\partial\partial_t\xi} - \mathcal{L} = \frac{\nu}{2}(\partial_x^2\xi)^2$

(d) $J_E = \partial_t\xi\frac{\partial\mathcal{L}}{\partial\partial_x\xi} = \frac{1}{2}(\partial_t\xi)^2 + \nu\partial_t\xi\partial_x^3\xi$, again using that the derivative according to $\partial_x\xi$ should be understood as a functional derivative.

(e) $-\partial_x J_E = -\partial_t\xi\partial_t\partial_x\xi - \nu\partial_x\partial_t\xi\partial_x^3\xi - \nu\partial_t\xi\partial_x^4\xi = \partial_t\xi\partial_t\partial_x\xi$

$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \nu\partial_x^4\xi\partial_t\xi = \partial_t\xi\partial_t\partial_x\xi$, again in a funcional derivative sense.