1. PROBLEM

Consider a charged particle moving in the potential of a three-dimensional isotropic harmonic oscillator and in a homogeneous magnetic field in the z direction:

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + \frac{1}{2}m\omega^2 r^2 \ .$$

Treating the magnetic field as perturbation, determine the first-order correction of the energy in the ground state of the oscillator!

Supporting information: Use the symmetric gauge $\vec{A} = \frac{1}{2}(-By, Bx, 0)$! Furthermore, $x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^+)$, $[a, a^+] = 1$

Solution:
In symmetric gauge the Hamiltonian reads

$$H = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2 r^2 + \frac{1}{2}m\left(\frac{\omega_L}{2}\right)^2\left(x^2+y^2\right) - \frac{\omega_L}{2}L_z = H_0 + W,\ \ W = \frac{1}{2}m\left(\frac{\omega_L}{2}\right)^2\left(x^2+y^2\right) - \frac{\omega_L}{2}L_z$$

Ground state energy of the oscillator $E_0 = \frac{3}{2}\hbar\omega$, the correction reads:

$$E_0^{(1)} = \left\langle 000 \left| \frac{1}{2}m\left(\frac{\omega_L}{2}\right)^2\left(x^2+y^2\right) - \frac{\omega_L}{2}L_z \right| 000 \right\rangle$$

Expressing the perturbation opertors with the ladder operators of the harmonic oscillator we get

$$-\frac{\omega_L}{2}L_z = -i\hbar\frac{\omega_L}{2}\left(a_x^+ a_y - a_y^+ a_x\right) \tag{1}$$

$$\frac{1}{2}m\left(\frac{\omega}{2}\right)^2\frac{x_0^2}{2}\left(a_x^+a_x^+ + a_y^+a_y^+ + a_xa_x + a_ya_y + a_xa_x^+ + a_x^+a_x + a_ya_y^+ + a_y^+a_y\right) \tag{2}$$

Putting these into the braket only terms with lowering operators on the right, with double raising or lowering operators vanish and with different orientations vanish. Only the $a_x a_x^+$ and $a_y a_y^+$ terms give nonzero contribtions, giving the correction:

$$E_0^{(1)} = \frac{1}{2}m\left(\frac{\omega_L}{2}\right)^2\frac{x_0^2}{2}\left\langle 000\left| a_xa_x^+ + a_ya_y^+\right| 000\right\rangle = \frac{\hbar\omega_L^2}{8\omega}$$

2. PROBLEM

Consider an electron moving in the potential of a three-dimensional isotropic harmonic oscillator and in a homogeneous magnetic field in the z direction:.

$$H = \frac{1}{2m}\left(\vec{p} + e\vec{A}^{(z)}\right)^2 + \frac{1}{2}m\omega^2 r^2 - \frac{2\mu_B}{\hbar}S_z B \ ,$$

where $\mathrm{curl}\,\vec{A}^{(z)} = B\vec{e}_z$. The system is in ground state, when at $t = 0$ the magnetic field is suddenly turned to direction x. Give the time dependence of the expectation value of the spin operators $\langle\vec{S}\rangle$ for $t > 0$!

Hint: Let us work in the Heisenberg picture! After switching the direction of the magnetic field, the Hamiltonian that governs the time evolution is given by

$$H = \frac{1}{2m}\left(\vec{p} + e\vec{A}^{(x)}\right)^2 + \frac{1}{2}m\omega^2 r^2 - \frac{2\mu_B}{\hbar}S_x B \ ,$$

with $\operatorname{curl}\vec{A}^x = B\vec{e}_x$. At $t=0$ the system was in an eigenstate of the operator S_z, therefore only those terms in $e^{\frac{i}{\hbar}Ht}S_i e^{-\frac{i}{\hbar}Ht}$ containing S_z give nonvanishing contributions to the expectation value of $\langle S_i\rangle$.

Solution:
The starting state is $|1/2\rangle$, the ground state of $-\frac{2\mu_B B}{\hbar}S_z$. The time evolution of the spin components is simply

$$e^{\frac{i}{\hbar}Ht}S_i e^{-\frac{i}{\hbar}Ht} \equiv e^{-\frac{i}{\hbar}\frac{2\mu_B}{\hbar}S_x Bt}S_i e^{\frac{i}{\hbar}\frac{2\mu_B}{\hbar}S_x Bt}$$

by the fact that S_x commutes with all the other operators int the Hamiltonian.
For $i=x$ i simply yields S_x, giving $\langle S_x\rangle = 0$
For $i=y$ we need to use the Haussdorf expansion, namely

$$S_y(t) = S_y - i\frac{2\mu_B B}{\hbar^2}\left[S_x, S_y\right]t - \frac{1}{2!}\left(\frac{2\mu_B B}{\hbar^2}\right)^2\left[S_x,\left[S_x,S_y\right]\right]t^2 + i\frac{1}{3!}\left(\frac{2\mu_B B}{\hbar^2}\right)^3\left[S_x,\left[S_x,\left[S_x,S_y\right]\right]\right]t^3 + \dots$$

Using the commutation relations that $[S_x, S_y] = i\hbar S_z$ and $[S_x, S_z] = -i\hbar S_y$ we get

$$S_y(t) = S_y + \frac{2\mu_B B}{\hbar}S_z t - \frac{1}{2!}\left(\frac{2\mu_B B}{\hbar}\right)^2 S_y t^2 - \frac{1}{3!}\left(\frac{2\mu_B B}{\hbar}\right)^3 S_z t^3 + \dots = S_z\sin\left(\frac{2\mu_B B}{\hbar}t\right) + S_x\cos\left(\frac{2\mu_B B}{\hbar}t\right)$$

The non-vanishing contribution can only come from the S_z term, giving

$$\langle S_z(t)\rangle = -\frac{1}{2}\sin\left(\frac{2\mu_B B}{\hbar}t\right)$$

For the $S_z(t)$ component in a similar way odd power terms will give according to the commutaion relation $[S_z, S_x] = i\hbar S_y$ which in the end gives zero in the expectation value. The even power terms give, however, in an analogous way $S_z\cos\left(\frac{2\mu_B B}{\hbar}\right)$, which results in the expectation value

$$\langle S_z(t)\rangle = -\frac{1}{2}\cos\left(\frac{2\mu_B B}{\hbar}t\right)$$

3. PROBLEM

In Minkowski space, the time reversal transformation is given by

$$x^{0\prime} = -x^0 \quad x^{i\prime} = x^i \quad (i = 1, 2, 3)\ .$$

a) Prove that the solution $\psi(x)$ of the Dirac equation, $(\gamma^\mu(i\hbar\partial_\mu - qA_\mu(x)) - mc)\psi(x) = 0$ transforms under time reversal as $\psi'(x') = (T\psi(x))^*$, where $*$ denotes complex conjugation, and the matrix T satisfies $T^{-1}(\gamma^0)^* T = \gamma^0$ and $T^{-1}(\gamma^i)^* T = -\gamma^i$! (15 points)

Supporting information: The four potential transforms as an axial vector under time reversial, $A'_\mu(x') = (A_0(x), -A_i(x))$.

b) Using the standard representation of the matrices γ^μ, show that $T = c\,\Sigma_y$, where $|c| = 1$! (10 points)

c) Prove that the current density tranforms as an axial vector, $j'^\mu(x') = \left(j^0(x), -\vec{j}(x)\right)$, where $j^\mu(x) = c\psi(x)^\dagger\gamma^0\gamma^\mu\psi(x)$! (5 points)

Solution:

a) The Dirac equation written out in temproal and spatial coordinates

$$\left(\gamma^0(i\hbar\partial_0 - qA_0(\mathsf{x})) + \gamma^i(i\hbar\partial_i - qA_i(\mathsf{x})) - mc\right)\psi(\mathsf{x}) = 0 \tag{3}$$

Effect of time reversal:

$$\partial_0' = \frac{\partial}{\partial x'^0} = -\frac{\partial}{\partial x^0}, \partial_i' = \frac{\partial}{\partial x'^i} = \frac{\partial}{\partial x^i} = \partial_i \tag{4}$$

$$A_0'(\mathsf{x}') = \frac{\phi'(\mathsf{x}')}{c} = \frac{\phi'(\mathsf{x}')}{c} = A_0(\mathsf{x}) \ , \ A_i'(\mathsf{x}') = -A_i(\mathsf{x}) \tag{5}$$

$$\Downarrow$$

$$\left(\gamma^0\left(-i\hbar\partial_0 - qA_0(\mathsf{x})\right) - \gamma^i\left(i\hbar\partial_i + qA_i(\mathsf{x})\right) - mc\right)\psi'(\mathsf{x}') = 0 \tag{6}$$

$$\Downarrow$$

$$\left(\left(\gamma^0\right)^*\left(i\hbar\partial_0 - qA_0(\mathsf{x})\right) + \left(\gamma^i\right)^*\left(i\hbar\partial_i - qA_i(\mathsf{x})\right) - mc\right)\psi'(\mathsf{x}')^* = 0 \tag{7}$$

$$\Downarrow$$

$$\left(T^{-1}\left(\gamma^0\right)^* T\left(i\hbar\partial_0 - qA_0(\mathsf{x})\right) + T^{-1}\left(\gamma^i\right)^* T\left(i\hbar\partial_i - qA_i(\mathsf{x})\right) - mc\right)T\psi(\mathsf{x}) = 0 \tag{8}$$

$$\Downarrow$$

$$\left(T^{-1}\left(\gamma^0\right)^* T\left(i\hbar\partial_0 - qA_0(\mathsf{x})\right) + T^{-1}\left(\gamma^i\right)^* T\left(i\hbar\partial_i - qA_i(\mathsf{x})\right) - mc\right)\psi(\mathsf{x}) = 0 \tag{9}$$

$$\Downarrow$$

$$T^{-1}\left(\gamma^0\right)^* T = \gamma^0 \quad T^{-1}\left(\gamma^i\right)^* T = -\gamma^i \tag{10}$$

b) Standard representation

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{11}$$

Let

$$T = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \tag{12}$$

$$\Downarrow$$

$$\begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{13}$$

$$\begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} 0 & \left(\sigma^i\right)^* \\ -\left(\sigma^i\right)^* & 0 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 0 & t^{-1}\left(\sigma^i\right)^* t \\ -t^{-1}\left(\sigma^i\right)^* t & 0 \end{pmatrix} \tag{14}$$

$$\Downarrow$$

$$t^{-1}\left(\sigma^i\right)^* t = -\sigma^i \tag{15}$$

Furhter exploiting that

$$\left(\sigma^x\right)^* = \sigma^x \quad \left(\sigma^y\right)^* = -\sigma^y \quad \left(\sigma^z\right)^* = \sigma^z \tag{16}$$

$$\Downarrow$$

$$\sigma^y\sigma^x = -\sigma^x\sigma^y \Longrightarrow \sigma^y\sigma^x\sigma^y = -\sigma^x \tag{17}$$

$$\sigma^y\left(-\sigma^y\right)\sigma^y = -\sigma^y \tag{18}$$

$$\sigma^y\sigma^z = -\sigma^z\sigma^y \Longrightarrow \sigma^y\sigma^z\sigma^y = -\sigma^z \tag{19}$$

As $\Sigma_y^* = -\Sigma_y$, the transformation of the wave function can be written

$$\psi'(\mathsf{x}') = c\Sigma_y\psi(\mathsf{x})^* \tag{20}$$

c)

$$j^{\mu}\left(\mathsf{x}'\right)' = \psi'\left(\mathsf{x}'\right)^{\dagger}\gamma^{0}\gamma^{\mu}\psi'\left(\mathsf{x}'\right) = \left(\Sigma_{y}\psi\left(\mathsf{x}\right)^{*}\right)^{\dagger}\gamma^{0}\gamma^{\mu}\Sigma_{y}\psi\left(\mathsf{x}\right)^{*} \tag{21}$$

$$= \left(\psi\left(\mathsf{x}\right)^{*}\right)^{\dagger}\left(\Sigma_{y}\gamma^{0}\gamma^{\mu}\Sigma_{y}\right)\psi\left(\mathsf{x}\right)^{*} \tag{22}$$

$$= \psi\left(\mathsf{x}\right)^{\dagger}\left(\Sigma_{y}\gamma^{0}\gamma^{\mu}\Sigma_{y}\right)^{*}\psi\left(\mathsf{x}\right) \tag{23}$$

$$= \psi\left(\mathsf{x}\right)^{\dagger}\Sigma_{y}\gamma^{0}\left(\gamma^{\mu}\right)^{*}\Sigma_{y}\psi\left(\mathsf{x}\right) \tag{24}$$

$$= \psi\left(\mathsf{x}\right)^{\dagger}\gamma^{0}\Sigma_{y}\left(\gamma^{\mu}\right)^{*}\Sigma_{y}\psi\left(\mathsf{x}\right) \tag{25}$$

$$= \Big(\psi\left(\mathsf{x}\right)^{\dagger}\psi\left(\mathsf{x}\right), -\psi\left(\mathsf{x}\right)^{\dagger}\gamma^{0}\gamma^{\mu}\Sigma_{y}\psi\left(\mathsf{x}\right)\Big) \tag{26}$$

$$= \Big(j^{0}\left(\mathsf{x}\right), -\vec{j}\left(\mathsf{x}\right)\Big) \tag{27}$$

where we used that $j^{\mu}\left(\mathsf{x}\right)$ is real.