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1. PROBLEM

Two fermionic particles with spin $s = \frac{1}{2}$ are moving in a linear harmonic potential and they interact by a contact interaction described by the δ-function,

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2\left(x_1^2 + x_2^2\right) + \gamma\delta(x_1 - x_2)\,.$$

Let us refer the unperturbed system to the above Hamilton operator without the interaction term.

a) Give the ground state and the first excited state of the unperturbed system and the corresponding energies! Specify the spin part of these states?

b) Using first order perturbation theory find the energy corrections to the ground state and the first excited state!

Supporting information:

$$\varphi_0(x) = \frac{1}{\sqrt{x_0\sqrt{\pi}}}e^{-\frac{x^2}{2x_0^2}}\,,\ \varphi_1(x) = \frac{1}{\sqrt{x_0\sqrt{\pi}}}\frac{\sqrt{2}x}{x_0}e^{-\frac{x^2}{2x_0^2}}\ \left(x_0 = \sqrt{\frac{\hbar}{m\omega}}\right)$$

$$\int_{-\infty}^{\infty} e^{-z^2}dz = \sqrt{\pi}\,,\quad \int_{-\infty}^{\infty} z^2 e^{-z^2}dz = \frac{\sqrt{\pi}}{2}\,.$$

Solution:

Ground state

a) Since the spatial part of the ground state $\varphi_0(x_1)\varphi_0(x_2)$ is symmetric, the two-spin part must by antisymmetric, i.e. a singlet state,

$$\Phi_0(1,2) = \varphi_0(x_1)\varphi_0(x_2)\frac{1}{\sqrt{2}}\left(|1/2,1/2\rangle|1/2,-1/2\rangle - |1/2,-1/2\rangle|1/2,1/2\rangle\right)\,.$$

The ground state energy of the unperturbed system is then

$$H_0|\Phi_0\rangle = \left(H_{\rm osc}(1) + H_{\rm osc}(2)\right)|\Phi_0\rangle = 2\frac{\hbar\omega}{2}|\Phi_0\rangle \to E_0^{(0)} = \hbar\omega\,.$$

b) The first order correction to the ground state energy is given by

$$\begin{aligned}\delta E_0^{(1)} &= \langle\Phi_0|\gamma\delta(x_1 - x_2)|\Phi_0\rangle = \gamma\int\int dx_1 dx_2\,\varphi_0^*(x_1)\varphi_0^*(x_2)\delta(x_1 - x_2)\varphi_0(x_1)\varphi_0(x_2)\\ &= \gamma\int dx\,|\varphi_0(x)|^4 = \frac{\gamma}{x_0^2\pi}\int_{-\infty}^{\infty} e^{-\frac{2x^2}{x_0^2}}dx = \frac{\gamma}{\sqrt{2}x_0\pi}\underbrace{\int_{-\infty}^{\infty} e^{-z^2}dz}_{\sqrt{\pi}} = \frac{\gamma}{x_0\sqrt{2\pi}}\,.\end{aligned}$$

Excited state

a) Here the spatial part of the two-fermion wavefunction can be both symmetric and antisymmetric, $\frac{1}{\sqrt{2}}(\varphi_0(x_1)\varphi_1(x_2) \pm \varphi_0(x_2)\varphi_1(x_1))$. Consequently, the spin of the two fermions can be in the antisymmetric singlet state, $S = 0$

$$\Phi_{1,00}(1,2) = \frac{1}{\sqrt{2}}(\varphi_0(x_1)\varphi_1(x_2) + \varphi_0(x_2)\varphi_1(x_1))\,|0,0\rangle|$$

or in the symmetric triplet states, $S = 1$

$$\Phi_{1,1M}(1,2) = \frac{1}{\sqrt{2}}(\varphi_0(x_1)\varphi_1(x_2) - \varphi_0(x_2)\varphi_1(x_1))\,|1,M\rangle \quad (M = -1,0,1)\ .$$

The above states form a four-dimensional eigenspace of $H_0\,(1,2) = H_{\rm osc}(1)+H_{\rm osc}(2)$ with the energy

$$E_1^{(0)} = 2\hbar\omega\,.$$

Since the perturbation is independent of the spin and the unperturbed wavefunctions contain orthonormal two-spin vectors, the 4×4 matrix of the perturbation is diagonal. The first order correction to the energy for the state $\Phi_{1,00}$ can be calculated as

$$\begin{aligned}\delta E^{(1)}_{1,00} &= \langle\Phi_{1,00}|\gamma\delta(x_1-x_2)|\Phi_{1,00}\rangle = 2\gamma\int dx\,|\varphi_0(x)\varphi_1(x)|^2 \\ &= \frac{2\gamma}{x_0^2\pi}\int_{-\infty}^{\infty}\left(\frac{\sqrt{2}x}{x_0}\right)^2 e^{-\frac{2x^2}{x_0^2}}\,dx = \frac{\gamma\sqrt{2}}{x_0\pi}\underbrace{\int_{-\infty}^{\infty} z^2e^{-z^2}dz}_{\sqrt{\pi}/2} = \frac{\gamma}{x_0\sqrt{2\pi}}\end{aligned}$$

while for $\Phi_{1,1M}$

$$\begin{aligned}\delta E^{(1)}_{1,1M} &= \langle\Phi_{1,1M}|\gamma\delta(x_1-x_2)|\Phi_{1M}\rangle \\ &= \frac{\gamma}{2}\int\!\!\int dx_1dx_2\ (\varphi_0(x_1)\varphi_1(x_2)-\varphi_0(x_2)\varphi_1(x_1))^*\,\delta(x_1-x_2)\,(\varphi_0(x_1)\varphi_1(x_2)-\varphi_0(x_2)\varphi_1(x_1)) \\ &= \frac{\gamma}{2}\int dx\ |\varphi_0(x)\varphi_1(x)-\varphi_0(x)\varphi_1(x)|^2 = 0\,.\end{aligned}$$

2. PROBLEM 25 points

Two electrons occupy p-like orbitals in a Hydrogen-like atom: $\ell_1 = 1$, $\ell_2 = 1$.

a) What values can take the total angular momentum L?

b) Give the expectation value of the $\hat{\ell}_{1z}$ operator in the $|L,M\rangle = |1,1\rangle$ state!

Supporting information:

$$\left(\hat{\ell}_1+\hat{\ell}_2\right)^2|L,M\rangle = \hbar^2L\,(L+1)\,|L,M\rangle\,,\,\left(\hat{\ell}_{1z}+\hat{\ell}_{2z}\right)|L,M\rangle = \hbar M\,|L,M\rangle$$

Solution:

a) The possible values are according to the rules derived in the lecture:

$$l_1 + l_2 = 2 \Rightarrow L = 2,\ 1,\ 0$$

b) First we give the state with the highest $L = 2, M = 2$ quantum number:

$$|2,2\rangle = |1,1\rangle_1|1,1\rangle_2$$

Then acting on it with the lowering operator:

$$\begin{aligned}&L_-|2,2\rangle = \hbar\sqrt{6-2}|2,1\rangle = (l_{1,-}+l_{2,-})\,1,1\rangle_1|1,1\rangle_2 = \hbar\sqrt{2}\,[|1,1\rangle_1|1,0\rangle_2 + |1,1\rangle_1|1,0\rangle_2] \\ &\Rightarrow |2,1\rangle = \frac{1}{\sqrt{2}}\,[|1,1\rangle_1|1,0\rangle_2 + |1,0\rangle_1|1,1\rangle_2]\end{aligned}$$

Now we get the $|1,1\rangle$ state in question by orthogonalization, that is looking for it in form of $|1,1\rangle = c_1|1,1\rangle_1|1,0\rangle_2 + c_2|1,0\rangle_1|1,1\rangle_2$ with real c_1, c_2 coefficients and satisfying the normality condition, $c_1^2 + c_2^2 = 1$, which is orthogonal to $|2,1\rangle$:

$$\langle 2,1|1,1\rangle = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2 = \frac{1}{\sqrt{2}},$$

yielding the result

$$|2,1\rangle = \frac{1}{\sqrt{2}}\left[|1,1\rangle_1|1,0\rangle_2 - |1,0\rangle_1|1,1\rangle_2\right]$$

Now in the expectaion value of l_{1z}, we act only on the first brakets, which are eigenstates of $l_{1,z}$, as a consequence of which only the same brakets can give nonzero results in the scalar product:

$$\langle 2,1|l_{1,z}|2,1\rangle = \frac{1}{2}\left[{}_1\langle 1,1|l_{1,z}|1,1\rangle_1 + {}_1\langle 1,0|l_{1,z}|1,0\rangle_1\right] = \frac{\hbar}{2} + 0 = \frac{\hbar}{2}$$

3. PROBLEM 25 points

Consider the scattering of a particle on a Dirac delta shell potential. The radial part of the wave-function, $\frac{1}{r}\psi(r)$, can be determined from the radial Schrödinger equation,

$$\frac{d^2\psi(r)}{dr^2} - \frac{\ell(\ell+1)}{r^2}\psi(r) - \frac{2m}{\hbar^2}\gamma\delta(r-R)\psi(r) = -k^2\psi(r)\ ,$$

where $k^2 = \frac{2mE}{\hbar^2}$ $(E > 0)$.

a) Determine the phase shift for $\ell = 0$ (δ_0)!

b) In which cases does the phase shift δ_0 vanish?

c) Give the low energy limit of the total cross-section!

Supporting information:

$$\lim_{r\to R+0}\frac{d\psi(r)}{dr} - \lim_{r\to R-0}\frac{d\psi(r)}{dr} = \alpha\psi(R) \quad \text{where} \quad \alpha = \frac{2m\gamma}{\hbar^2}$$

Solution:

a) In case of $\ell = 0$ the radial Schrödinger equation takes the form,

$$-\frac{1}{r}\frac{d^2}{dr^2}(r\varphi) + \frac{2m}{\hbar^2}\gamma\delta(r-R)\varphi = k^2\varphi\ .$$

By multiplying with $-r$ and introducing $\psi(r) = r\varphi(r)$,we get

$$\frac{d^2\psi}{dr^2} - \frac{2m}{\hbar^2}\gamma\delta(r-R)\psi = -k^2\psi\ .$$

The solutions inside and outside the sphere are $\psi_{r<R} = \sin(kr)$ and $\psi_{r>R} = a\sin(kr) + b\cos(kr)$, respectively. The solution must be continuous and its derivative has a jump at $r = R$. These conditions results in the following equations,

$$\begin{aligned} \sin(kR) &= a\sin(kR) + b\cos(kR) \\ ak\cos(kR) - bk\sin(kR) - k\cos(kR) &= \alpha\sin(kR)\ , \end{aligned}$$

where $\alpha = \dfrac{2m}{\hbar^2}\gamma$. We can write these two equations in matrix form,

$$\begin{pmatrix} \sin(kR) & \cos(kR) \\ k\cos(kR) & -k\sin(kR) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sin(kR) \\ \alpha\sin(kR) + k\cos(kR) \end{pmatrix}.$$

Employing

$$\begin{pmatrix} \sin(kR) & \cos(kR) \\ k\cos(kR) & -k\sin(kR) \end{pmatrix}^{-1} = \frac{1}{k}\begin{pmatrix} k\sin(kR) & \cos(kR) \\ k\cos(kR) & -\sin(kR) \end{pmatrix},$$

the coefficients can be calculated from

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{k}\begin{pmatrix} k\sin(kR) & \cos(kR) \\ k\cos(kR) & -\sin(kR) \end{pmatrix}\begin{pmatrix} \sin(kR) \\ \alpha\sin(kR) + k\cos(kR) \end{pmatrix}$$

$$\Downarrow$$

$$\begin{aligned} ka &= k + \alpha\sin(kR)\cos(kR \\ kb &= -\alpha\sin^2(kR) \end{aligned}$$

$$\Downarrow$$

$$\tan\delta_0 = -\frac{b}{a} = \frac{\alpha\sin^2(kR)}{k + \alpha\sin(kR)\cos(kR)}$$

b) The condition for vanishing $\delta_0 = 0$! In this case, the nominator must vanish, that is $\tan\delta_0$ must be zero,

$$\alpha\sin^2(kR) = 0$$

$$\Downarrow$$

$$k = \frac{n\pi}{R}$$

c) The phase shift equals to zero, if $kR = n\pi$. This is the case when the wavefunction vanishes at radius R.

d) The low energy limit of the phase shift:

$$\tan\delta_0 = \lim_{k\to 0}\frac{\alpha(kR)^2}{k + \alpha kR} = k\frac{\alpha R^2}{1+\alpha R}.$$

At low energies, only δ_0 has contribution to the total cross-section :

$$\sigma_{tot} = \frac{4\pi}{k^2}\delta_0^2 = \frac{4\pi}{k^2}k^2\frac{\alpha^2 R^4}{(1+\alpha R)^2} = 4\pi R^2\frac{(\alpha R)^2}{(1+\alpha R)^2}$$

4. PROBLEM 25 points

For $t < 0$ a charged harmonic oscillator is placed in a uniform electric field, which is turned off at $t = 0$:

$$H = \begin{cases} \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - \mathcal{E}qx & \text{if} \quad t < 0 \\ \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 & \text{if} \quad t \geq 0 \end{cases}.$$

For $t < 0$ the system is found in the corresponding ground state. Determine the expectation value of the position operator for $t \geq 0$!

Supporting remarks: Using the ladder operator $a = \frac{1}{\sqrt{2}x_0}\left(x + \frac{i}{m\omega}p\right)$, find the ladder operator b such that for $t < 0$ $H = \hbar\omega\left(b^+b + \frac{1}{2}\right) - E_0$, where E_0 is an appropriate constant. For $t < 0$, the ground state of the system is then given by $b\,|0_b\rangle = 0$. In Heisenberg picture, the ladder operators are given as $a_H(t) = ae^{-i\omega t}$ and $a_H^+(t) = a^+e^{i\omega t}$. Calculate $\langle 0_b|\, x_H(t)\,|0_b\rangle$!

Solution:

Change the x coordinate for $t<0$

$$\frac{1}{2}m\omega^2x^2-\mathcal{E}qx=\frac{1}{2}m\omega^2\left(x-\frac{\mathcal{E}q}{m\omega^2}\right)^2-\frac{\mathcal{E}^2q^2}{2m\omega^2}$$

$$\Downarrow$$

$$H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2\left(x-d\right)^2-E_0$$

$$d=\frac{\mathcal{E}q}{m\omega^2}\quad E_0=\frac{1}{2}m\omega^2d^2=\frac{\mathcal{E}^2q^2}{2m\omega^2}$$

Hamiltonians with ladder operators

$$H=\begin{cases}\hbar\omega\left(b^+b+\frac{1}{2}\right)-E_0 & \text{if} \quad t<0\\ \hbar\omega\left(a^+a+\frac{1}{2}\right) & \text{if} \quad t\geq 0\end{cases}.$$

$$b=\frac{1}{\sqrt{2}x_0}\left(\left(x-d\right)+\frac{i}{m\omega}p\right)=a-\frac{d}{\sqrt{2}x_0}$$

Ground state for $t<0$

$$b\left|0\right\rangle_b=0\rightarrow a\left|0_b\right\rangle=\frac{d}{\sqrt{2}x_0}\left|0_b\right\rangle$$

Heisenberg picture

$$a_H\left(t\right)=ae^{-i\omega t}\quad a_H^+\left(t\right)=a^+e^{i\omega t}$$

$$x_H\left(t\right)=\frac{x_0}{\sqrt{2}}\left(a_H\left(t\right)+a_H^+\left(t\right)\right)=\frac{x_0}{\sqrt{2}}\left(ae^{-i\omega t}+a^+e^{i\omega t}\right)$$

Expectation value of the position

$$\left\langle x\left(t\right)\right\rangle=\left\langle 0_b\right|x_H\left(t\right)\left|0_b\right\rangle=\frac{x_0}{\sqrt{2}}\left\langle 0_b\right|ae^{-i\omega t}+a^+e^{i\omega t}\left|0_b\right\rangle$$

$$=\frac{x_0}{\sqrt{2}}\frac{d}{\sqrt{2}x_0}\left(e^{-i\omega t}+e^{i\omega t}\right)=d\cos\omega t$$